Properties of classical Lie Groups

With emphasis on dimension, compactness, connectedness and relations among the groups.

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In this note we will focus on the fundamentals of the classical Lie Groups. There will be no proofs or argumentation, just at bunch of facts listed together for the reader to get a quick reference for the classical Lie Groups.

This note is mainly written because of the authors own inability to remember the dimensions of the classical Lie Groups, but hopefully this note will serve as an aid for others when reading mathematical literature.

Definitions of classical Lie groups

First of all we need the definitions of the classical Lie Groups that we are working with. In the following $\mathbb{F}$ is one of the two fields $\mathbb{R}$ or $\mathbb{C}$.

- $\mathrm{GL}(n,\mathbb{F}) = \big\{x \in \mathrm{Mat}_{n,n}(\mathbb{F}) : \det(x) \neq 0\big\}$
- $\mathrm{GL}^+(n,\mathbb{F}) = \big\{x \in \mathrm{Mat}_{n,n}(\mathbb{F}) : \det(x) > 0\big\}$
- $\mathrm{SL}(n,\mathbb{F}) = \big\{x \in \mathrm{Mat}_{n,n}(\mathbb{F}) : \det(x) = 1\big\}$
- $\mathrm{O}(n,\mathbb{F}) = \big\{x \in \mathrm{GL}(n,\mathbb{F}) : x^T x = \mathrm{Id}\big\}$
- $\mathrm{SO}(n,\mathbb{F}) = \big\{x \in \mathrm{O}(n,\mathbb{F}) : \det(x) = 1\big\}$
- $\mathrm{U}(n) = \big\{x \in \mathrm{GL}(n,\mathbb{C}) : x^* x = \mathrm{Id}\big\}$
- $\mathrm{SU}(n) = \big\{x \in \mathrm{GL}(n,\mathbb{C}) : x^* x = \mathrm{Id}, \det(x) = 1\big\} = \mathrm{U}(n) \cap \det^{-1}(1)$
- $\mathrm{Sp}(n,\mathbb{F}) = \big\{x \in \mathrm{GL}(n,\mathbb{F}) : x^T J x = J\big\}$ where $J = \begin{bmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix} \in \mathrm{GL}(n,\mathbb{F})$.
- $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n,\mathbb{C})$
- $\mathrm{Spin}(n)$ is the double cover of $\mathrm{SO}(n,\mathbb{R})$. For $n > 2$ it turns out that $\mathrm{Spin}(n)$ is simply connected, and in this case it coincides with the universal covering of $\mathrm{SO}(n,\mathbb{R})$.

As you see the definitions are straight forward, only the last definition of the compact symplectic group is a bit strange. Beware that there are other definitions around, but in the end they are all equivalent to this one.

The definition of the Spin-group is a bit involved, and needs a bit of explanation, which will not be given here. The definition cannot be written as a set of matricies satisfying a specific equation.

Fundamental properties of classical Lie Groups

The main part of this script is the following table. It contains all the quick facts that you need to know about a classical Lie Group.

Group	*Lie group*	*#connected comp*	*Compact*	*Dimension*	π_1	*Lie algebra*
$\mathrm{GL}(n,\mathbb{R})$	$\mathbb{R}$	2	no	n^2		$\mathfrak{gl}(n,\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$
$\mathrm{GL}(n,\mathbb{C})$	$\mathbb{C}$	1	no	n^2	$\mathbb{Z}$	$\mathfrak{gl}(n,\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$
$\mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}$		no	n^2-1	$\begin{cases} 0 & n=1 \\ \mathbb{Z} & n=2 \\ \mathbb{Z}_2 & n>2 \end{cases}$	$\mathfrak{sl}(n,\mathbb{R}) = \left\{x \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{Tr}(x)=0\right\}$
$\mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}$	1		n^2-1	0	$\mathfrak{sl}(n,\mathbb{C}) = \left\{x \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{Tr}(x)=0\right\}$
$\mathrm{O}(n,\mathbb{R})$	$\mathbb{R}$	2		$\frac{1}{2}n(n-1)$		
$\mathrm{O}(n,\mathbb{C})$	$\mathbb{C}$		no, $n>1$	$\frac{1}{2}n(n-1)$		
$\mathrm{SO}(n,\mathbb{R})$	$\mathbb{R}$	1		$\frac{1}{2}n(n-1)$		$\mathfrak{so}(n,\mathbb{R})$ = skew symmetric matricies in $\mathfrak{gl}(n,\mathbb{R})$
$\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}$		no, $n>1$	$\frac{1}{2}n(n-1)$	$\begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z}_2 & n>2 \end{cases}$	$\mathfrak{so}(n,\mathbb{C})$ = skew symmetric matricies in $\mathfrak{gl}(n,\mathbb{C})$
$\mathrm{U}(n)$	$\mathbb{R}$	1	yes	n^2	$\mathbb{Z}$	$\mathfrak{u}(n)$ = skew hermitian matricies in $\mathfrak{gl}(n,\mathbb{C})$
$\mathrm{SU}(n)$	$\mathbb{R}$	1	yes	n^2-1	0	$\mathfrak{su}(n)$ = tracless skew hermitian matricies in $\mathfrak{gl}(n,\mathbb{C})$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathbb{R}$	1	no	$n(2n+1)$	$\mathbb{Z}$	$\mathfrak{sp}(2n,\mathbb{R}) = \{x \in End(\mathbb{R}^{2n}) : Jx + x^T J = 0\}$
$\mathrm{Sp}(2n,\mathbb{C})$	$\mathbb{C}$	1	no	$n(2n+1)$	0	$\mathfrak{sp}(2n,\mathbb{C}) = \{x \in End(\mathbb{C}^{2n}) : Jx + x^T J = 0\}$
$\mathrm{Sp}(n)$	$\mathbb{R}$	1	yes	$n(2n+1)$	0	$\mathfrak{sp}(n) = \left\{x \in \mathrm{End}(\mathbb{H}^n) : x + \bar{x}^t = 0\right\}$
$\mathrm{Spin}(n)$	$\mathbb{R}$	1	yes	$\frac{1}{2}n(n-1)$	0, $n \geq 3$	$\mathfrak{so}(n)$

Identities between classical groups

There are a lot of peculiar identities between the classical Lie Group – mainly between the low dimensional, though. In the following we list the most useful. The low dimensional identities are also called the accidental isomorphisms. These isomorphisms are a consequence of low dimensional isomorphisms between root systems of different families of simple Lie algebras.

- $\mathrm{GL}(n,\mathbb{C}) \cap \mathrm{SO}(2n) = \mathrm{U}(n)$
- $\mathrm{GL}(n,\mathbb{R}) \cap \mathrm{U}(n) = \mathrm{O}(n)$
- $\mathrm{GL}(n,\mathbb{R}) \simeq \mathrm{O}(n) \times \mathbb{R}^{\frac{1}{2}n(n+1)}$
- $\mathrm{GL}(n,\mathbb{C}) \simeq \mathrm{U}(n) \times \mathbb{R}^{n\cdots n}$
- $\mathrm{SL}(n,\mathbb{R}) \simeq \mathrm{SO}(n) \times \mathbb{R}^{\frac{1}{2}n(n+1)-1}$
- $\mathrm{O}(2n+1) \simeq \mathrm{SO}(2n+1) \times \mathbb{Z}_2$
- $\mathrm{O}(2n) \simeq \mathrm{SO}(2n) \times \mathbb{Z}_2$
- $\mathrm{U}(n) \simeq \mathrm{SU}(n) \times S^1$
- $\mathrm{Spin}(1) = \mathrm{O}(1)$
- $\mathrm{Spin}(2) = \mathrm{U}(1) = \mathrm{SO}(2)$
- $\mathrm{Spin}(3) = \mathrm{Sp}(1) = \mathrm{SU}(2) = S^3$
- $\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$
- $\mathrm{Spin}(5) = \mathrm{Sp}(2)$
- $\mathrm{Spin}(6) = \mathrm{SU}(4)$
- $\mathrm{SO}(3) = \mathbb{R}\mathrm{P}^3$
- $\mathrm{U}(n) = \mathrm{SU}(n) \rtimes \mathrm{U}(1)$

Principal bundles

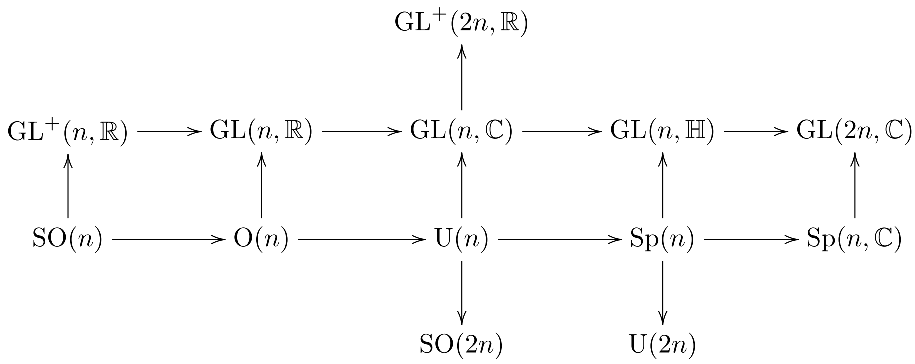
The classical groups provide us with the above isomorphisms, but they also give rise to the following principal bundles.

In the following $V_k(\mathbb{R}^n)$ is the Stiefel manifold. The points in the Stiefel manifold are k-tuples of orthonormal vectors in $\mathbb{R}^n$. Furthermore $G_k(\mathbb{R}^n)$ is the Grassman manifold.

Group	*Bundle*
$\mathrm{SO}(n-1)$	$\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to S^{n-1}$
$\mathrm{O}(n-1)$	$\mathrm{O}(n-1) \to \mathrm{O}(n) \to S^{n-1}$
$\mathrm{U}(n-1)$	$\mathrm{U}(n-1) \to \mathrm{U}(n) \to S^{2n-1}$
$\mathrm{SU}(n-1)$	$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}$
$\mathrm{Sp}(n-1)$	$\mathrm{Sp}(n-1) \to \mathrm{Sp}(n) \to S^{4n-1}$
$\mathrm{U}(n-1)$	$\mathrm{U}(n-1) \to \mathrm{SU}(n) \to \mathbb{CP}^{n-1}$
$\mathrm{O}(n-1)$	$\mathrm{O}(n-1) \to \mathrm{SO}(n) \to \mathbb{RP}^{n-1}$
$\mathrm{O}(n-k)$	$\mathrm{O}(n-k) \to \mathrm{O}(n) \to \mathrm{V}_k(\mathbb{R}^k)$
$\mathrm{O}(k) \times \mathrm{O}(n-k)$	$\mathrm{O}(n) \times \mathrm{O}(n-k) \to \mathrm{O}(n) \to G_k(\mathbb{R}^k)$
$\mathrm{U}(k) \times \mathrm{U}(n-k)$	$\mathrm{U}(k) \times \mathrm{U}(n-k) \to \mathrm{U}(n) \to G_k(\mathbb{C}^k)$
$\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$	$\mathrm{Sp}(k) \times \mathrm{Sp}(n-k) \to \mathrm{Sp}(n) \to G_k(\mathbb{H}^n)$

Relationship between classical Lie groups

It is obvious that the classical Lie groups are connected in some way. Above we have seen some identities that relate the different groups. But what about inclusions? The following diagram shows how all the classical groups are contained in each other, and give an overview of how each group is related to the other groups.



Short exact sequences

The last thing we look at is certain exact sequences that relate some of the classical Lie groups. The first equality is just the definition of the Spin-group, while the other short exact sequence is a useful description of $\mathrm{U}(n)$ in terms of $\mathrm{SU}(n)$ and $U(1)$.

$$\begin{array}{ccccccccc} \{e\} & \to & \mathbb{Z}_2 & \to & \mathrm{Spin}(n) & \to & \mathrm{SO}(n) & \to & \{e\} \\ 1 & \to & \mathrm{SU}(n) & \to & \mathrm{U}(n) & \to & \mathrm{U}(1) & \to & 1 \end{array}$$