# Assessment in Derived Categories of Coherent Sheaves, MT11

Jakob Blaavand

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**Abstract**

This assessment is a review of aspects of derived categories of coherent sheaves. We will review the definitions of derived categories and of coherent sheaves on a smooth projective variety. Following this we will discuss properties such as Serre duality and applications of Serre duality. Lastly we will discuss the Bondal–Orlov Reconstruction theorem. These notes are mainly based on [2], [3] and [4].

## 1 Derived categories

Derived categories are complicated objects. We will try go give a motivation for them by giving an example from algebraic topology. In algebraic topology we want an invariant of simplicial complexes, that allow us to decide when topological realizations of the simplicial complexes are homotopy equivalent. Our first choice is to study homology. But it turns out that there exists topological spaces $X, Y$ such that $H_i(X) \simeq H_i(Y)$ for all $i$, where $X$ and $Y$ are not homotopy equivalent – taking homology in some sense throws away too much information. Homology is defined as the homology of a chain complex, and actually, if we recall the construction of functors such as sheaf cohomology, Tor, Ext etc., they are all defined by homology of injective or locally free resolutions. In stead of just remembering e.g. sheaf cohomology group $H^i(X, \mathcal{F})$ we should go with the whole complex $\Gamma^\cdot(X, \mathcal{I}^\cdot)$, where $\mathcal{I}^\cdot$ is an injective resolution of $\mathcal{F}$. Whitehead's theorem illustrates the power of this idea:

**Theorem 1.1** (Whitehead)**.** *Simplicial complexes $X$ and $Y$ have homotopy equivalent geometric realizations if, and only if, there exists a simplicial complex $Z$ and simplicial maps $f : Z \to X$ and $g : Z \to Y$ such that the maps $f_* : C.(Z) \to C.(X)$ and $g_* : C.(Z) \to C.(Y)$ induce maps on homology that are isomorphisms for all $i$.*

So if we want to obtain a complete homotopy invariant, we need to consider not only homology of a chain complex, but really the whole complex itself. Derived categories is a general setup, where we can formulate Whitehead's theorem.

## 1.1 Definition of the derived category

Start with an abelian category $\mathcal{A}$. Objects of the new category $C(\mathcal{A})$ are complexes of objects of $\mathcal{A}$ and maps are chain maps. That is objects

$$\cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots$$

with $A^i \in \mathrm{Obj}(\mathcal{A})$ and $d^{i+1} \circ d^i = 0$ for all $i$. A morphism between $A^\cdot$ and $B^\cdot$ is a chain map $f : A^\cdot \to B^\cdot$, that is a collection of maps $f^i : A^i \to B^i$ such that the appropriate squares commute. We use cohomological notation rather than homological, but by declaring $A^i = A_{-i}$ we could have used homological as well.

The problem we faced in algebraic topology was maps inducing isomorphisms on homology, that were not homotopy equivalences them selves. A *quasi-isomorphism* is a morphism, $f$, in $C(\mathcal{A})$, where the induced maps on cohomology, $H^i(f^\cdot)$, are isomorphisms for all $i \in \mathbb{Z}$. Complexes $A^\cdot$ and $B^\cdot$ are quasi-isomorphic if they are related by a chain of quasi-isomorphisms. It should be noted that quasi-isomorphisms are often not invertible.

**Definition 1.2.** The *derived category*, $D(\mathcal{A})$, of an abelian category, $\mathcal{A}$, and the functor $Q : C(\mathcal{A}) \to D(\mathcal{A})$ are universal with the property that if $f^\cdot$ is a quasi-isomorphism, then $Q(f^\cdot)$ is an isomorphism.

So the objects of $D(\mathcal{A})$ are the same as the objects of $C(\mathcal{A})$, and the morphisms now also include inverses of quasi-isomorphisms.

Remark that a category as defined above actually exists, and is called *the* derived category, since the universal property ensures uniqueness up to isomorphism.

We understand the objects of $D(\mathcal{A})$, but how should we understand morphisms? Can we draw an analogy to Whitehead's theorem?

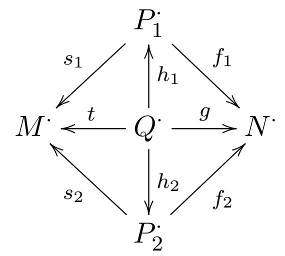
**Theorem 1.3.** *Let $\mathcal{A}$ be an abelian category. Then*

1. *$Q : C(\mathcal{A}) \to D(\mathcal{A})$ factorizes through* $\mathrm{Ho}(\mathcal{A})$*, the homotopy category of $\mathcal{A}$, so $Q$ identifies homotopic morphisms.*

2. *Morphisms $\alpha : A^\cdot \to B^\cdot$ in $D(\mathcal{A})$ are of the form of a roof $\alpha = Q(f) \circ Q(s)^{-1}$ with $s$ a quasi-isomorphism*

$$\begin{array}{ccc} & P^\cdot & \\ {\scriptstyle Q(s)}\swarrow & & \searrow{\scriptstyle Q(f)} \\ A^\cdot & & B^\cdot \end{array} \qquad \textit{represented in } \mathrm{Ho}(\mathcal{A}) \textit{ by} \qquad \begin{array}{ccc} & P^\cdot & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\ A^\cdot & & B^\cdot \end{array}$$

3. *Two morphisms* $A^{\cdot} \xleftarrow{s_i} P_i^{\cdot} \xrightarrow{f_i} N^{\cdot}$ , $i = 1, 2$ *give the same morphism if, and only if, there exists a complex* $Q^{\cdot}$ *and maps* $h_1, h_2, t, f$ *such that the following diagram commutes in* $\mathrm{Ho}(\mathcal{A})$.



Much more can be said about general derived categories but we will go on to study the derived category of a particular abelian category.

## 2 Coherent sheaves

Let $X$ be a smooth projective variety. Coherent sheaves can be defined in the more general setting with $(X, \mathcal{O}_X)$ being a locally ringed space, but we will not be concerned with that level of generality.

We view $X$ as a locally ringed space $(X, \mathcal{O}_X)$ with $\mathcal{O}_X$ the structure sheaf on $X$. An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian groups on $X$ together with maps $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ compatible with restriction maps and making $\mathcal{F}(U)$ a module over $\mathcal{O}_X(U)$, here is $U \subset X$ open. These sheaves form an abelian category $\mathrm{Mod}(\mathcal{O}_X)$.

**Example 2.1.** An $\mathcal{O}_X$-module, $\mathcal{F}$, is locally-free (of finite rank)[1] if for all $x \in X$ there exists a neighborhood $U \subset X$ of $x$ such that $\mathcal{F}|_U \simeq \mathcal{O}_X^{\oplus n}|_U$. Given a vector bundle $\pi : Y \to X$ the abelian groups

$$\mathcal{F}(U) = \{\text{sections of } \pi \text{ over } U\},$$

constitute a locally-free sheaf. This gives an equivalence between the set of vector bundles over $X$ and the set of locally-free $\mathcal{O}_X$-modules.

The locally-free $\mathcal{O}_X$-modules do not form an abelian category in general. We therefore need to include something more.

**Definition 2.2.** A sheaf $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ is *coherent*, if for all $x \in X$ exist a neighborhood $U \subset X$ of $x$ and an exact sequence

$$\mathcal{O}_X^{\oplus m}|_U \to \mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U \to 0.$$

Denote by $\mathrm{Coh}(X)$ the category of coherent sheaves on $X$.

[1] Our locally-free $\mathcal{O}_X$-modules will always be of finite rank

Note that $\mathrm{Coh}(X) \subset \mathrm{Mod}(\mathcal{O}_X)$ is an abelian category, and that if $X$ is quasi-projective, then it is the smallest abelian subcategory containing all locally-free $\mathcal{O}_X$-modules.

**Example 2.3.** Examples of coherent sheaves are

1. locally-free sheaves
2. sky-scraper sheaves
3. the structure sheaf $\mathcal{O}_X$ itself.

A neat way of thinking about coherent sheaves are as vector bundles with fibres of varying dimension. Let $\mathcal{F}$ be a coherent sheaf on $X$ and $x \in X$ a point. The vector space $\mathcal{F}_{(x)} := \mathcal{F}_x/(m \cdot \mathcal{F}_x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is interpreted as the fibre of $\mathcal{F}$ at $x$. Here $\mathcal{F}_x$ is the stalk at $x$, which is an $\mathcal{O}_{X,x}$-module so last expression where $k(x)$ is the residue field at $x$ makes sense. $m$ is the maximal ideal corresponding to $x$, i.e. the ideal of germs of functions vanishing at $x$. Now in our case $k(x) \simeq \mathbb{C}$, but the mentioned definition is also the recipe for the general situation of $X$ a Noetherian scheme. The rank at a point $x$ is $\dim_{k(x)} \mathcal{F}_x/(m \cdot \mathcal{F}_x)$. This notion of rank is consistent with the usual definition of rank, when $\mathcal{F}$ is locally-free (i.e. a vector bundle). Rank is not a constant function on $X$, but a corollary to the following theorem by Nakayama shows that rank is locally constant, and jumps only occur on closed subvarieties.

**Theorem 2.4** (Nakayama)**.** *For any $\mathcal{F} \in \mathrm{Coh}(X)$ coherent sheaf*

$$\mathcal{F}_{(x)} = 0 \iff \mathcal{F}_x = 0 \iff \text{there exists } x \in U \subset X \text{ such that } \mathcal{F}|_U = 0.$$

**Corollary 2.5.** *Let $\mathcal{F} \in \mathrm{Coh}(X)$, then*

1. *$\mathcal{F}$ is locally-free of rank $r$ if, and only if, $\dim_{\mathbb{C}} \mathcal{F}_{(x)} = r$ for all $x \in X$*
2. *For all $k \geq 0$ the set $\{x \in X \mid \dim_{\mathbb{C}} \mathcal{F}_{(x)} \leq k\} \subset X$ is open.*
3. *There exists a finite stratification $X = \sqcup X_i$, where $X_i \subset X$ are locally-closed subvarieties, such that $\mathcal{F}|_{X_i}$ are locally-free for all $i$.*

# 3 Derived category of coherent sheaves

Now we look at the gadget $D(X) := D(\mathrm{Coh}(X))$, the derived category of coherent sheaves on a smooth projective variety $X$. We denote by $D^*(X)$, where $* = b, +, -$ the versions where we take as objects $b =$ bounded complexes of coherent sheaves, $+ =$ complexes bounded from below, and $- =$ complexes bounded above.

## 3.1 Properties

We will now focus our attention to some properties of the derived category of coherent sheaves.

A very important player is the canonical bundle of $X$, $\omega_X$. Note that for any locally-free sheaf $\mathcal{E}$ the functor $\mathrm{Coh}(X) \to \mathrm{Coh}(X)$ given by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{E}$ is exact. In particular it descends to the derived categories $D^*(X) \to D^*(X)$. On a triangulated category the useful shift functor $[i] : D^*(X) \to D^*(X)$ for $i \in Z$ is exact.

Assume that $X$ is a smooth projective variety of dimension $n$. Then define a functor $S_X$ as the composition

$$D^*(X) \xrightarrow{\omega_X \otimes ()} D^*(X) \xrightarrow{[n]} D^*(X) \ ,$$

where $* = b, +, -$.

One of the most useful tools in dealing with derived categories of coherent sheaves is Serre duality.

**Theorem 3.1** (Serre duality)**.** *Let $X$ be a smooth projective variety over a field $k$ with $k$-dimension $n$. Then $S_X : D^b(X) \to D^b(X)$ is a $k$-linear equivalence, such that for any two objects $\mathcal{F}, \mathcal{G} \in D^b(X)$ there exists an isomorphism*

$$\eta_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}_{D(X)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{D(X)}(\mathcal{G}, S_X(\mathcal{F}))^* = \mathrm{Hom}_{D(X)}(\mathcal{G}, \mathcal{F} \otimes \omega_X[n])^*$$

*of $k$-vector spaces, which is functorial in $\mathcal{F}$ and $\mathcal{G}$.*

Note that since we define $\mathrm{Ext}^i(\mathcal{F}^\cdot, \mathcal{G}^\cdot) := \mathrm{Hom}_{D(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot[i])$, Serre duality can be stated as isomorphisms for any $i \in \mathbb{Z}$

$$\mathrm{Ext}^i(\mathcal{F}^\cdot, \mathcal{G}^\cdot) \simeq \mathrm{Ext}^{n-i}(\mathcal{G}^\cdot, \mathcal{F}^\cdot \otimes \omega_X)^*.$$

Now it is very easy to prove the following proposition.

**Proposition 3.2.** *Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $X$, with $X$ as above. Then* $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ *for $i > n$.*

*Proof.* This follows directly from Serre duality, since

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes \omega_X)^* = 0$$

for negative $n - i$. □

Serre duality is used in almost any kind of study of derived categories. By using the general notion of a Serre functor in the setting of a triangulated category (since $D(X)$ is a triangulated category, we define a Serre functor to be a functor between triangulated categories that satisfies the conclusion in Theorem 3.1), Serre duality can be proven relatively easily – at least in the case of a projective morphism between smooth varieties. We will not do this. In the remainder of this note, we will focus on a key ingredient in the Serre functor above: the canonical bundle $\omega_X$.

The canonical bundle can help answer questions like, how much information about the variety is actually stored in the derived category. In light of the initial remarks about algebraic topology this is a natural question to ask. Furthermore it is a fact that for $X$, $Y$ projective varieties then if $\mathrm{Coh}(X) \simeq \mathrm{Coh}(Y)$, then $X \simeq Y$, so $\mathrm{Coh}(X)$ determines $X$. Further it is a fact that there are examples of smooth projective varieties, with isomorphic bounded derived categories (as triangulated categories), but where the varieties themselves are not isomorphic. So in some sense the derived category of coherent sheaves is a weaker invariant than the category of coherent sheaves. But how much weaker? It contains more information than cohomology, since we still have the complexes.

**Proposition 3.3.** *Let $X$, $Y$ be smooth projective varieties over a field $k$. If there exists an exact equivalence $D^b(X) \simeq D^b(Y)$ between the derived categories, then $\dim(X) = \dim(Y)$. Moreover their canonical bundles have the same degree.*

The degree of a bundle is the smallest positive power of the bundle such that it's trivial. Orders can be infinite.

This proposition tells us that the derived category at least carries information about the dimension of the variety. To get more information out we must consider the canonical bundle. It seems that the further away $\omega_X$ is of being trivial, the more information is stored in $D^b(X)$. The precise statement is the Bondal–Orlov Reconstruction theorem.

**Theorem 3.4** (Reconstruction)**.** *Let $X$ be a projective variety, such that $\omega_X$ is ample or anti-ample, and let $Y$ be any projective variety. If $D^b(X) \simeq D^b(Y)$ as triangulated categories, then $X \simeq Y$.*

This theorem can also be extended to describe autoequivalences of $D^b(X)$ when $\omega_X$ is ample or anti-ample.

**Theorem 3.5.** *Let $X$ be a smooth projective variety with ample or anti-ample canonical bundle. The group of autoequivalences of $D^b(X)$ is generated by automorphisms of $X$, shifts, and tensoring by line bundles. In other words*

$$\mathrm{Aut}(D^b(X)) \simeq \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)).$$

The Reconstruction theorem follows from studying what kind of objects of $D^b(X)$ are really shifts of line bundles on $X$. Using this knowledge one can show that the pluricanonical rings of $X$, $R(X) := \oplus_{k \geq 0} H^0(X, \omega_X^k)$, and $Y$ are isomorphic, $R(X) \simeq R(Y)$. By Proposition 3.3 we see that if $\omega_X$ is ample or anti-ample, then the same is true for $\omega_Y$. So finally we have

$$X \simeq \mathrm{Proj}\, R(X) \simeq \mathrm{Proj}\, R(Y) \simeq Y,$$

where the outer isomorphisms are given by ampleness. More details can be found in [3] or in the original paper by Bondal and Orlov [1].

As a conclusion on this review we give the proof of Proposition 3.3, since Serre duality is an important ingredient in the proof. In the proof we need a general fact about Serre functors.

**Lemma 3.6.** *Let $\mathcal{A}$, $\mathcal{B}$ be $k$-linear categories over a field $k$, and both have finite dimensional homomorphism groups. If $\mathcal{A}$ and $\mathcal{B}$ are endowed with a Serre functor (i.e. a functor satisfying the conclusion in Theorem 3.1) $S_{\mathcal{A}}$ resp. $S_{\mathcal{B}}$, then any linear equivalence $F : \mathcal{A} \to \mathcal{B}$ commutes with Serre duality, that is there exists an isomorphism*

$$F \circ S_{\mathcal{A}} \simeq S_{\mathcal{B}} \circ F.$$

*Proof.* Since $F$ is an equivalence it is fully faithful, and so for any two objects $A, B \in \mathrm{Obj}(\mathcal{A})$ we have

$$\begin{aligned} \mathrm{Hom}(A, S_{\mathcal{A}}(B)) &\simeq \mathrm{Hom}(F(A), F(S_{\mathcal{A}}(B))) \\ \mathrm{Hom}(B, A) &\simeq \mathrm{Hom}(F(B), F(A)). \end{aligned}$$

Together with the isomorphisms from being Serre functors

$$\begin{aligned} \mathrm{Hom}(A, S_{\mathcal{A}}(B)) &\simeq \mathrm{Hom}(B, A)^* \\ \mathrm{Hom}(F(B), F(A)) &\simeq \mathrm{Hom}(F(A), S_{\mathcal{B}}(F(B)))^* \end{aligned}$$

we get a functorial isomorphism

$$\mathrm{Hom}(F(A), F(S_{\mathcal{A}}(B))) \simeq \mathrm{Hom}(F(A), S_{\mathcal{B}}(F(B))).$$

Now since $F$ is an equivalence, any object in $\mathcal{B}$ is isomorphic to some $F(A)$. Yoneda's Lemma give that $A \mapsto \mathrm{Hom}(A, \ )$ is fully faithful, and so there exists a functorial isomorphism $F \circ S_{\mathcal{A}} \simeq S_{\mathcal{B}} \circ F$. □

*Proof of Proposition 3.3.* $X$ and $Y$ are smooth projective varieties, and from Theorem 3.1 we have Serre duality available in our derived category toolbox. The Serre functor is defined as $S_X(\mathcal{F}^\cdot) = \mathcal{F}^\cdot \otimes \omega_X[\dim X]$ for any $\mathcal{F}^\cdot \in D^b(X)$. From Lemma 3.6 we know that any equivalence $F : D^b(X) \to D^b(Y)$ commutes with $S_X$ and $S_Y$.

Now fix a closed point $x \in X$. Let $k_x$ be the skyscraper sheaf associated to the point $x$. Then

$$k_x \simeq k_x \otimes \omega_X \simeq S_X(k_x)[-\dim X],$$

and so

$$\begin{aligned} F(k_x) &\simeq F(k_x \otimes \omega_X) \simeq F(S_X(k_x)[-\dim X]) \\ &\simeq F(S_X(k_x))[-\dim X] \simeq S_Y(F(k_x))[-\dim X] \\ &= F(k_x) \otimes \omega_Y[\dim Y - \dim X], \end{aligned}$$

where third isomorphism is exactness of $F$, and fourth isomorphism is commutativity of $F$ with $S_X$ and $S_Y$.

$F$ is an equivalence, so $F(k_x)$ is a bounded non-trivial complex. If $i$ is the maximal integer such that $H^i(F(k_x)) \neq 0$, then we also find that

$$0 \neq H^i(F(k_x)) \simeq H^i(F(k_x) \otimes \omega_Y[\dim Y - \dim X]) \simeq H^{i+\dim Y - \dim X}(F(k_x)) \otimes \omega_Y,$$

since tensoring with a line bundle commutes with cohomology. So if $\dim Y - \dim X > 0$ this computation contradicts maximality. We could do the same with $i$ the minimal number such that $H^i(F(k_x)) \neq 0$ and would get that $\dim Y - \dim X < 0$ give a contradiction. Thus $\dim X = \dim Y =: n$.

If $\omega_X^k \simeq \mathcal{O}_X$. Then $S_X^k[-kn] \simeq id$ and so

$$F^{-1} \circ S_Y^k[-kn] \circ F \simeq S_X^k[-kn] \simeq id.$$

Thus $S_Y^k[-kn] \simeq id$ and we see especially $\omega_Y^k \simeq \mathcal{O}_Y$. □

# 4 Conclusion

This was a very brief review of aspects of derived categories of coherent sheaves on smooth projective varieties. We discussed definitions of central ingredients, Serre duality, which is an extremely useful tool, and lastly we discussed how much data about $X$ is stored in $D^b(X)$, especially the Bondal–Orlov Reconstruction theorem. We also proved that the derived category, $D^b(X)$, knows the dimension of $X$.

# References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[4] D. Huybrechts *Fourier–Mukai transforms in algebraic geometry*, Oxford University Press, 2006.