3-manifold invariants derived from link invariants

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1 Introduction

These are notes for a talk given at University of California, Berkeley, November 5th 2010, at the weekly student seminar GRASP.

In the last 20 years, low dimensional topology has lived and thrived very well. One of the big motivations is a classification of 3-manifolds, which turn out to be surprisingly hard, compared to other dimensions. In the process of telling 3-manifolds apart, we construct invariants. There are numerous ways of doing this, but many have are created in the same way. Find a process of creating a 3-manifold from some simpler object, could be a link, tetrahedron or spine, and then define your invariants in terms of these simpler objects. Now show that your gadget, gives the same value, for any two ways of constructing the manifold in question. Often the two constructions are related by a sequence of certain moves. Then you 'just' have to show invariance under these moves.

In these notes we concentrate on creating 3-manifolds by surgery on a link in S^3. An example of another type of construction, could be the Turaev-Viro invariant, [?], based on a triangulation of the manifold.

The outline for the notes, is first of all a basic introduction to knot theory, and a motivation to why we need knot invariants. There is a big zoo of different invariants, each with its special features and limitations. We will concentrate on the Jones polynomial, and define it via the Kauffman bracket. Following this, we will discuss the surgery connection between links and 3-manifolds. This connection will be ysed to transfer our link invariants, to invariants of 3-manifolds. Last but not least we will use this construction to discuss the invariants constructed in [BHMV1].

2 Basic knot theory

2.1 Definitions

Definition 2.1. A *knot* is a smooth embedding of S^1 into S^3, $K : S^1 \hookrightarrow S^3$. A *link* is a collection of finitely many disjoint knots in S^3. Two links, L_1, L_2 are equivalent, if there is an orientation preserving diffeomorphism $h : S^3 \to S^3$, such that $h(L_1) = L_2$.

Normally we draw diagrams of the knots and links, in $\mathbb{R}^2$. The pictures we draw, must contain the information about the crossings. Pictures like figure 1, are called *knot diagrams*.

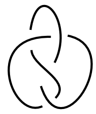


Figure 1: The Figure-8 Knot

One of the basic, and most importnant, theorems in knot theory, is a theorem of Reidemeister. The theorem reformulates the definition of link equivalence in terms of link diagrams.

Theorem 2.2 ([Rei])**.** *Two links, L_1, L_2 are equivalent, if and only if, any diagram of L_1 and L_2 can be related by a finite sequence of Reidemeister moves, and ambient isotopies.*

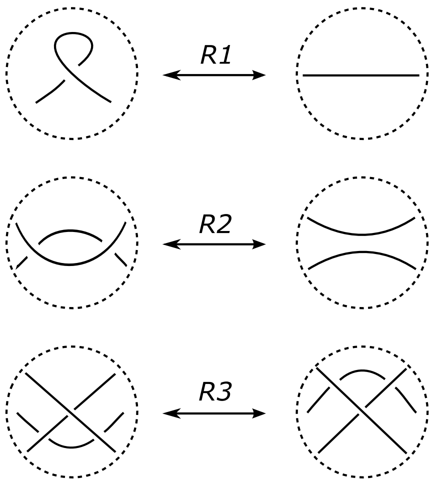


Figure 2: The three Reidemeister moves

Remark 2.3. (i) The Reidemeister moves are local. That is, the moves showed in figure 2 are equal outside the dashed circles. To limit yourself to a local picture, is a very common thing to do in knot theory. In that way you can say general things about all links.

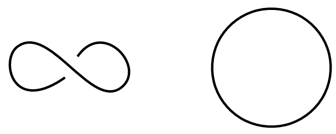
(ii) We should also have included all the mirror images of the Reidemeister moves.

Example 2.4. (i) It is easy to see that the two diagrams in figure **??** represent two equivalent knots.

(ii) Maybe it is not clear, why the two diagrams in figure 4, represent equivalent diagrams. But they do. Figure 5 illustrates a series of Reidemeister moves taking one diagram to the other.

Invariants

The most fundamental problem in knot theory, is to determine wether two given knots are equivalent. It is a really simple question to ask, and as figure 4 and 5 shows, it



(a) Twisted unknot (b) Unknot

Figure 3: Two diagrams of the unknot

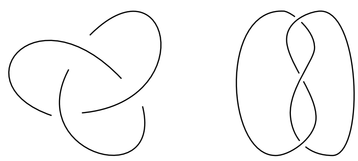


Figure 4: Two diagrams of the trefoil

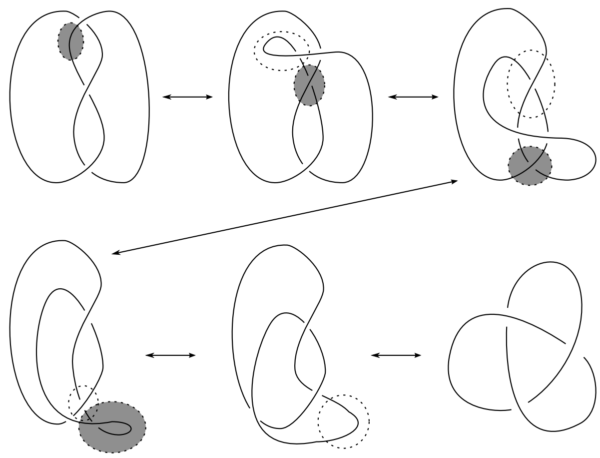


Figure 5: Six Reidemeister moves proves equivalence of diagrams in figure 4

can be quite hard to answer. We need something more than just the Reidemeister moves to work with.

Definition 2.5. Let M be a set. Then a function

$$F : \{\text{diagrams of links}\}/\sim\ \longrightarrow M$$

where the equivalence relation is Reidemeister moves, is called a *link invariant*.

There are a lot of link invariants, and each one of them has its good and bad properties. But until 1984 no effective invariant, was able to distinguish the trefoil from its mirror image.[1] If you play with the Reidemeister moves – or create the trefoil on a string – you will soon realize that the trefoil, and its mirror image must be different. Just because you cannot find a series of Reidemeister moves from one to the other, does not give a mathematical proof, that the two are different. Maybe

[1]It was know that the trefoil and its mirror image were inequivalent, but it was a long and hard proof, which could not be generalized to more complicated knots.

someone smarter than you is able to do that. The best example of this is the Perko pair. For years everyone believed that the two diagrams in figure 6 were different. But in 1974 Kenneth Perko [Per] found a sequence of Reidemeister moves, proving that the two diagrams actually represented the same knot.

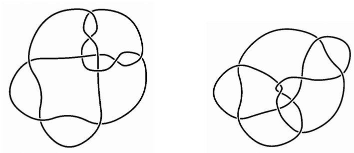


Figure 6: The perko Pair

In 1984 Vaughn Jones, from University of California, Berkeley, created his famous polynomial. This linkinvariant was able, in an easy way, to tell the trefoil apart from its mirror image. This replaced Dehn's old complicated proof by something much simpler.

2.2 The Jones polynomial

Definition 2.6. *The Jones polynomial* is a function, which to an oriented link diagram associates a Laurant polynomial.

$$V : \{\text{oriented link diagrams}\}/\sim \longrightarrow \mathbb{Z}[A, A^{-1}].$$

There are several ways to define the Jones polynomial, but the one given here, is not the one Vaughn Jones originally came up with. Jones' original definition is based on his work on von Neumann algebras [Jon], and is a lot more complicated than the simple definition discovered by Louis Kauffman [Kau].

Definition 2.7. The Kauffman bracket is a function, $\langle\ \rangle$, from unoriented link diagrams, modulo the second and third Reidemeister move, to Laurant polynomials, satisfying the following relations

(i) $\langle \emptyset \rangle = 1$

(ii) $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle$

(iii) $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

Smooth every crossing in all possible ways. Each of the 2^n ways to smooth out a link (n is the number of crossings) is called a state. Each state is a a disjoint union of circles. Each of these circles gives a factor of $(-A^{-2} - A^{2})$. Sum all these states. This will give the Kauffman bracket of the diagram.

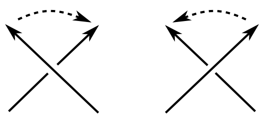
Remark 2.8. (i) The normalization chosen here, is not the usual one. Normally you require $\langle \bigcirc \rangle = 1$ and not $\langle \emptyset \rangle = 1$, but that will only affect the Jones polynomial with a factor of $-A^{-2} - A^{2}$, which we will take care of in the definition of the Jones polynomial. The reason for this alternative normalization, is, that it will be easier to work with this normalization later.

(ii) Kauffman bracket is *not* a link invariant.

$$\Big\langle \text{kink} \Big\rangle = -A^{3}\langle \curvearrowright \rangle, \qquad \Big\langle \text{kink} \Big\rangle = -A^{-3}\langle \curvearrowright \rangle$$

But it is an invariant of *framed link diagrams.* Framed link diagrams are diagrams, where two links are equivalent, if and only if, there is a series of Reidemeister 2 and 3 moves which relates diagrams of the links. Of course you can still use ambient isotopies as well. These kind of diagrams arises, if you look at your knots, as embedded annuli instead of embedded circles. Then a Reidemeister 1 move will produce a full twist of the annuli, and therefore cannot be used. When we discuss surgery we will only look at framed links.

Definition 2.9. The *writhe* is a function from oriented link diagrams, modulo Reidemeister 2 and 3, to the integers. Given a diagram, D, the writhe is the sum of the signs of the crossings in D, and is denoted by $\omega(D)$. The signs of a crossing is defined in figure 7.



-1 +1

Figure 7: Sign of a crossing

Remark 2.10. The writhe is *not* a link invariant:

$\omega($  $) = \omega($  $) - 1$

$\omega($  $) = \omega($  $) + 1$

Theorem 2.11 (The Jones polynomial)**.** *The combination* $V_L(A) = (-A)^{-3\omega(D)}(-A^{-2} - A^2)^{-1}\langle D\rangle$ *is a link invariant, and is called the* Jones polynomial*.*

Proof. If we accept that the writhe and the Kauffman bracket are well defined functions, which we have not shown, and will not do, the only nontrivial thing to check is that the above definition is invariant under Reidemeister 1 – but this follows immidiately from remarks 2.8 and 2.10.

To check that writhe and Kauffman bracket are well defined functions are two straight forward exercises, which are left to the reader. □

Example 2.12. Now it is easy to calculate the Jones polynomial of the trefoil, 3_1, and its mirror image, $\overline{3_1}$.[2]

$$V_{3_1}(A) = -A^{16} + A^{12} + A^4$$
$$V_{\overline{3_1}} = -A^{-16} + A^{-12} + A^{-4}$$

Remark 2.13. (i) It is not hard to prove that $\overline{V_L} = V_{\overline{L}}$, where $\overline{V_L}(A) = V_L(A^{-1})$, so for any link, where the Jones polynomial is not a symmetric polynomial, the link is different from its mirror image. The Jones polynomial of the Figure 8-knot is $A^8 - A^4 + 1 - A^{-4} + A^{-8}$, so the Jones polynomial cannot tell the Figure 8-knot from its mirror image, and that is not strange, since they are equivalent. It is a good exercise, in use of Reidemeister moves, to show this.

(ii) Normally we substitute A^{-4} with t to get rid of the large exponents. It can be shown, that for any link, with an odd number of components, the exponents will always be divisible by 4, and links with an even number of components will have exponent 2 mod 4.

[2] These are the standard names, given in the Rolfsen table of knots.

3 Historical outline

In 1984 Vaughn Jones created his famous polynomial [Jon]. It is the first link polynomial that was easy to calculate, and in the same time could separate the most links, and give a solution to some longstanding conjectures.

In 1988 Edward Witten interpreted the Jones polynomial, in terms of 3-dimensional physics, based on a heuristic use of Chern-Simons theory. His construction extends the Jones polynomial, to links in an arbitrary compact oriented 3-manifold, and could produce invariants of 3-manifolds, [Wit].

In 1990 and 1991, Reshetikhin and Turaev, produced the same invariants Witten had produced, but they did it with mathematical rigour. Their invariants are based on quantum groups at a root of unity, [RT].

In 1992 Blanchet, Habegger, Masbaum and Vogel derived invariants from the Kauffman bracket [BHMV1]. These invariants turned out to be the same invariants as Reshetikhin-Turaev's. In the following, it is the [BHMV1]-invariants, we will construct.

4 Surgery

Let $K \subset S^3$ be a knot and $N(K)$ a tubular neighborhood of K (just thicken the knot). $\partial N(K)$ is a torus, T^2, and cutting S^3 along $\partial N(K)$ gives us two manifolds: the knot exterior, $E(K) = \overline{S^3 \setminus N(K)}$, which is the closure of $S^3 \setminus N(K)$, and the solid torus $N(K)$.

We could use any orientation preserving homeomorphism $h : \partial D^2 \times S^1 \to \partial E(K)$ to glue $D^2 \times S^1$ back into $E(K)$, that is, any element of the mapping class group group of T^2, which is $SL(2, \mathbb{Z})$, could be used. The space we obtain is a closed orientable 3-manifold. This 3-manifold we say is obtained by surgery on S^3 along K with h.

$M = E(K) \cup_h N(K)$ depends on h, but we only have to specify the image on a meridian $\partial D^2 \times \{*\}$. If we know what h is on a meridian, we also know what h is on the a small cylinder of T^2, around the meridian. See figure 8 The rest of $D^2 \times S^1$ is D^3 glued along S^2 with a orientation preserving homeomorphism, i.e. an element of the mapping class group of S^2, $\Gamma(S^2)$. Since $\Gamma(S^2) = \{id\}$ any orientation preserving homeomorphism is isotopic to id.

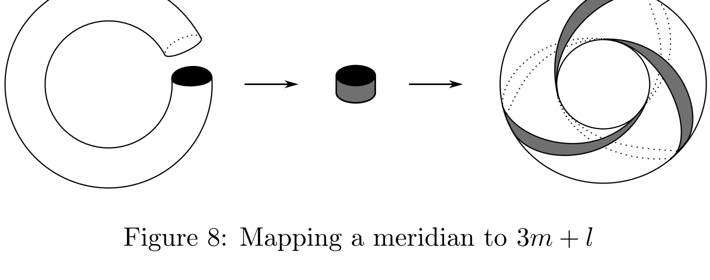


Figure 8: Mapping a meridian to $3m + l$

Since $H_1(E(K)) = \mathbb{Z}$, any meridian, m_1, of $N(K)$ pushed into $E(K)$, is a generator for $H_1(E(K))$, we will call it l_2, as it is a longitude. Besides that, there is an unique longitude, l_1, in $N(K)$, which, when pushed into $E(K)$, is homological trivial in $E(K)$ – we call this curve m_2, as it is a meridian.

(m_2, l_2) form a basis for $H_1(\partial E(K))$, and any simple closed curve on $\partial E(K)$, is isotopic to a curve of the form $c = qm_2 + pl_2$. Our homeomorphism is therfore determined by the integers (p, q). To prove our results, we only need to look at surgeries, where $q = 1$. This kind of surgery is called integral sugery.

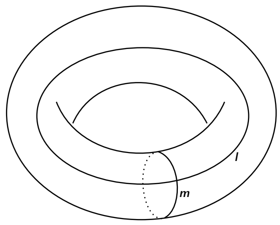


Figure 9: The curves m and l are generators of the homology

So far, the previous works for any link, but from this point on, we will concentrate on framed links. Eventhough they are embedded annuli, we still just draw them as 1-dimensional objects, where we do not use the first Reidemeister move.

Given a knot, how should we determine p? The answer is, that we should undo all possible twists, and count them, with signs, that number is p. So surgery along $\bigcirc$ sends a meridian, m_1, to $c = 0 \cdot l_2 + m_2 = m_2$, and if we choose the homeomorphism to be $l_1 \to l_2$ on the last generator, it is clear that we get $S^2 \times S^1$. To get S^3, we should have exchanged meridians and longitudes. In general $\bigcirc^{\,p}$ (here the super p indcates p twists should be added to the knot) produces a lens space $L(p, 1)$. Surgery along an unknot with a single twist, sends m_1 to $c = m_2 + l_2$, that is $L(1, 1) = S^3$. This last equality can be seen, by composing the homeomorphism with a Dehn twist along m_2. Such a Dehn twist does not effect m_2, but replaces l_2 with another longitude, such that $c = l_2$, and $L(1, 1) = L(1, 0) = S^3$.

The last statement might need som more explanation. Every homemorphism used in a surgery procedure is an element of $SL_2(\mathbb{Z})$, i.e. a matrix $\begin{pmatrix} q & r \\ p & s \end{pmatrix}$, with $qs - pr = 1$. As it is only the image of $m_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is important to us we can precompose this matrix with elements from $SL_2(\mathbb{Z})$, which is the identity on the first column, but are allowed to change the second. That is, elements of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, e.g. n Dehn twists around m_1. If we take $h(m_1)$ as the first basis vector for $H_1(\partial(E(K))) = \mathbb{Z}^2$, then we are also allowed to compose with an element, which does not change the image of $h(m_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $m_1 \mapsto m_2 + l_2$ and $l_1 \mapsto l_2$, then $h = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. To produce S^3 we want the image of m_1 to be l_2 and the image of l_1 to be $-m_2$ (othervise we cannot get an element of $SL_2(\mathbb{Z})$). The way we can do that is to precompose, and compose with a Dehn twist around $-m_2$, i.e. multiply h from right and left with $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

You can in some sense see this as nothing more, than a change of basis.

Theorem 4.1 ([Lic])**.** *Every closed, connected, orientable* 3*-manifold,* M*, can be obtained from* S^3 *by integral surgery on a framed link* $L \subset S^3$*.*

If M is obtained by integral surgery on L, we call L a surgery presentation of M. This theorem gives us a connection between framed links in S^3 and closed, connected, orientable 3-manifolds. The theorem do not say anything about, how you should find a surgery presentation for a given 3-manifold, it just gives the existence. As we shall see later, many different links can be surgery presentations for the same manifold.

Our goal is now to construct a way, to transfer our link invariants, to become 3-manifold invariants. The final ingredient we need to do this, is a theorem by Kirby.

Theorem 4.2 ([Kir])**.** *The closed, connected, oriented manifolds obtained by surgery on framed links* L, L'*, are homemorphic by an orientation preserving homemorphism, if and only if, the link* L' *can be obtained from the link* L*, by a sequence of Kirby moves:*

KI Add, or delete, an unlinked unknot with a positive, or negative, twist.

KII You are allowed to slide a component of a link over another component. See figure 10, where the unknot is slid over an unknot, with a single positive twist.

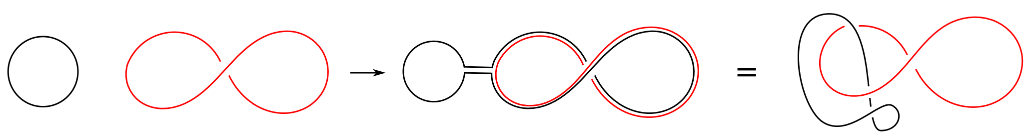


Figure 10: Slide unknot over twisted unknot. A Kirby 2 Move

That surgery is invariant under Kirby Move 1 is obvious, by the fact that surgery on $\bigcirc^{\pm 1}$ produces S^3, so if M is obtained by integral surgery on L, then integral surgery on $L \sqcup \bigcirc^{\pm 1}$ produces $M \# S^3 \simeq M$. Invariance under Kirby Move 2 is more involved to prove.

Example 4.3. From above, we know that doing surgery on S^3, along the framed $\bigcirc$, produces $S^2 \times S^1$. By the first Kirby move, we can add an unlinked unknot with a positive twist, and still get the same manifold, $S^2 \times S^1$. Our link is now the left side of figure 10. Doing a Kirby 2 move on this link, by sliding the black unknot over the red twisted unknot, we still produce the same manifold. This proves that the Hopf link with a positive twist on each component is a surgery presentation of $S^2 \times S^1$. This shows, that very different links can produce the same manifold, and also, that it can be hard to determine what manifold is produced from a given link.

5 Invariants of 3-manifolds

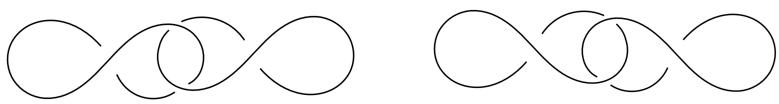
To create 3-manifold invariants, we take an invariant of framed links, and turn it into something, which is invariant under Kirby Move 1 and Kirby Move 2. Our invariant of framed link is of course the Kauffman bracket.

Definition 5.1. Given a n-component framed link $L \subset S^3$. The meta–bracket $\langle \ldots \rangle_L$ is a multilinear function $B^n \to \mathbb{Z}[A, A^{-1}]$, where $B = \mathbb{Z}[A, A^{-1}][z]$, and A is a primitive $2p$ root of unity, p is an integer. The meta–bracket, $\langle b_1, \ldots, b_n \rangle_L$, is defined as the Kauffman bracket of the framed link obtained from L by replacing the i'th component by n_i parallel copies if $b_i = z^{n_i}$. In particular if $b_i = 1$ the i'th component is removed, and if $b_i = z$ the i'th component remains unchanged.

Theorem 5.2 (BHMV)**.** *Let M be a connected, closed, orientable 3-manifold, obtaind from surgery on a framed link $L \subset S^3$. Suppose given $\Omega_p \in B$ satisfying the following relations for all $b \in B$*

$$\langle \Omega_p, b \rangle_{L_1} = \langle \Omega_p \rangle_{U_1} \langle b \rangle_U \quad \text{and} \quad \langle \Omega_p, b \rangle_{L_{-1}} = \langle \Omega_p \rangle_{U_{-1}} \langle b \rangle_U$$

where U_p is the unknot with framing p, and L_1, (L_{-1}) are Hopf links, with positive (resp. negative) twist on each component, the links are showed in figure 11. Furthermore, suppose that $\langle \Omega_p \rangle_{U_1}$ and $\langle \Omega_p \rangle_{U_{-1}}$ are invertible. Then $\theta_p(M) = \frac{\langle \Omega_p, \ldots, \Omega_p \rangle_L}{\langle \Omega_p \rangle_{U_1}^{b_+(L)} \langle \Omega_p \rangle_{U_{-1}}^{b_-(L)}}$ is an invariant of M.



(a) L_1 (b) L_{-1}

Figure 11: The links L_1 and L_{-1}

The invariants are nicer with the normalization $\langle \emptyset \rangle = 1$, but we could also have used $\langle \bigcirc \rangle = 1$.

Here $b_+(L)$ and $b_-(L)$ denote the number of positive and negative eigenvalues of the linking matrix of L. The ij'th entry in the linking matrix, is the linking number between component i and component j. The ii'th entry is the writhe of the i'th component. The linking number $lk(K, L)$ is the sum of crossing signs of crossings between K and L – times $\frac{1}{2}$.

Remark 5.3.

(i) It is not at all obvious, that the supposed element, $\Omega_p \in B$, exists. But it does. It is essentially a sum of Chebyshev polynomials. As an example, the first two are $\Omega_1 = 1$, $\Omega_2 = 1 + \frac{1}{2}z$. It is the existence of this element, which makes us able to prove invariance under Kirby move 2.

(ii) Given this Ω_p, the theorem is not hard to prove. Invariance under the first Kirby move is easy to see: To add an unknotted unknot with a single twist, gives an extra component in the numerator, which can be split off, since the component is unlinked. This extra factor should, hopefully, cancel with something in the denominator. The linking matrix get an extra row and coloumn, with ± 1 in the diagonal entry, and 0 elsewhere. This gives an extra positive, or negative, eigenvalue depending on the type of the twist. The contribution from this extra eigenvalue, cancels with the factor in the numerator.

Proposition 5.4. *The invariant θ_p has the following properties.*

(i) $\theta_p(S^3) = 1$ *for all* p

(ii) $\theta_p(-M) = \overline{\theta_p(M)}$ *where* $-M$ *denotes* M *with reversed orientation.*

(iii) $\theta_p(M \# N) = \theta_p(M)\theta_p(N)$

(iv) $\theta_p(S^2 \times S^1) = \begin{cases} p & p \leq 2 \\ \frac{-p}{(A^2 - A^{-2})^2} & p \geq 3 \end{cases}$

Remark 5.5. (i) The above proposition follow directly from the definition of θ_p and the surgery presentations of S^3, $S^2 \times S^1$, and that we obtain $-M$ from surgery on S^3 along $\overline{L}$, where L is a surgery presentation of M.

(ii) We would like these invariants to be associated to a TQFT, but the invariants are for connected closed orientable 3-manifolds, and according to Turaev [Tur, Chap III.4.1], we should be able to calculate the invariant of a disjoint union of 3-manifolds. It is possible to tweak the θ_p's to become the quantum invariants of a TQFT. This is done in [BHMV2].

5.1 Generalizations of the Jones polynomial

Let M be a connected orientable closed 3-manifold, and $L \subset M$ a framed link. We know, that M can be obtained by integral surgery on a framed link $K \subset S^3$. We can assume, up to isotopy, that the link L is contained in $S^3 \setminus K \subset M$.

We can extend the Kauffman bracket by immitating the construction in Theorem 5.2.

$$\theta_p(M, L) = \frac{\langle \Omega_p, \ldots, \Omega_p, z, \ldots, z \rangle_{K \cup L}}{\langle \Omega_p \rangle_{U_1}^{b_+(L)} \langle \Omega_p \rangle_{U_{-1}}^{b_-(L)}}$$

The only difference from Theorem 5.2, is that we include L in the meta-bracket in the numerator. Instead of inserting Ω_P on each of the components of L, we just insert z, which means implies that we should not cabel the components of L.

Conclusion

In this note we have developed a method ot construct 3-manifold invariants from link-invariants. This is a very general method, and there is no reason why [BHMV1]-invariants, should be the only example of such an invariant. [RT]-invariants are also based on surgery and Kirbys theorem. There might be more invariants defined in this way, which I am not aware of.

Bibliography

[BHMV1] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Three-manifold invariants derived from the kauffman bracket. *Topology*, 31(4):685–699, 1992.

[BHMV2] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the kauffman bracket. *Topology*, 34(4):883–927, 1995.

[Jon] V. Jones. A polynomial invariant of knots via von neumann algebras. *Bull. Amer. Math. Soc.*, 12:103–112, 1985.

[Kau] L. Kauffman. State models and the jones polynomial. *Topology*, 26:395–407, 1987.

[Kir] R. Kirby. A calcuclus of framed links in s^3. *Invent. Math.*, 45:35–56, 1978.

[Lic] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math.*, 76:531–540, 1962.

[Per] Kenneth Perko. On the classification of knots. *Proc. Amer. Math. Soc.*, 45:262–266, 1974.

[Rei] K. Reidemeister. *Knot Theory.* BSC Associates, Translated from German by Leo F. Boron, Charles O. Christenson and Bryan A. Smith, 1983.

[RT] N. Reshetikhin and Vladimir Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127:1–26, 1990.

[Tur] Vladimir Turaev. *Quantum invariants of knots and 3-manifolds.* De Gruyter, 2nd edition, 2010.

[Wit] E. Witten. Quantum field theory and the jones polynomial. *Comm. Math. Phys.*, 121, 1989.