Assessment in Symplectic Geometry, MT11

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Abstract

This assessment is a review of aspects of mirror symmetry with special attention to the predictions in [1] about the number of rational curves on a quintic threefold. The calculations in that paper is reviewed, and afterwards Gromov–Witten invariants for quintic threefolds are defined and calculations are discussed. The section about [1] depends heavily on the exposition of [1] given in [2].

1 String theory and Mirror Symmetry

Mirror symmetry is a child of string theorists incredible insights in how the world should work. To understand mirror symmetry - or at least just a small part of it - we have to start from string theory. In string theory spacetime is locally $M^{1,3} \times M^6$, where $M^{1,3}$ is the usual spacetime from special relativity and M^6 is a compact Calabi–Yau manifold. M^6 has a size of around the Planck length $(10^{-33}cm)$. There are therefore no chance that we should be able to view the extra 6 dimensions by observing on a macroscopic level, but the extra dimensions should have influence on quantum phenomena.

String theorists found out that two different Calabi–Yau manifolds, M, W, could give the spacetime the same physics – whatever this means. If that is the case M, W are called a mirror pair. To understand this definition we need to investigate a bit of quantum field theory.

In quantum field theory the probabilities for a given outcome of an observable used on a particle, is determined by all paths of the particle - not just the ones that minimize the action integral. The way these probabilities usually are computed is via path integrals, and perturbative methods of reducing the path integrals to Feynman diagrams. These diagrams can in some way be thought of as the motion of particles in spacetime. As an example is a particle splitting viewed as a vertex with three edges. In string theory these diagrams are replaced by surfaces. The particles are now 1-dimensional closed strings, and their propagation and interactions in spacetime sweeps out a surface with boundary components. The example about particle splitting is now represented by a three-holed sphere (also called 'a pair of pants'). This surface is called a *worldsheet*. These pair of pants surfaces are the basic building blocks for Riemann surfaces with boundary components, and they will become very important in this review.

If spacetime is fixed we can view the propagation of a particle through spacetime as a map $\varphi : \Sigma \to M$ from the worldsheet Σ to spacetime M. We're interested in what extra 6 dimensions of the Calabi–Yau manifold can give to the theory, so we assume that φ maps into the Calabi–Yau part, and from now on M will denote a Calabi–Yau manifold.

The probabilities of measuring a given outcome of an observable with a specified initial state, are determined by correlation functions. Correlation functions are also called Yukawa couplings. If $\mathcal{O}_P$ is an operator associated to an observable acting on a fixed Hilbert space (according to the axioms of quantization) correlation functions are denoted by $\langle \mathcal{O}_{P_1}, \ldots, \mathcal{O}_{P_k} \rangle$. P_i is a position of a point on the worldsheet - it is really a point on the boundary of the surface, but in the limit where the boundary has shrunk to a point. These correlation functions are complex valued functions, and are as such not probabilities, but they include also phase information.

There are special types of correlation functions, namely those that do not depend on the actual location of the point P on the worldsheet, but just on the fact that we have chosen a point. These are the topological correlation functions, and these are the ones we're interested in. Since the pair of pants surfaces are the smallest building blocks we're only interested in $\langle \mathcal{O}_{P_1}, \mathcal{O}_{P_2}, \mathcal{O}_{P_3} \rangle$. Even though this is a big reduction, in the definition of the correlation function we still need to integrate over $\mathrm{Maps}(\Sigma, M)$, where $\Sigma = \mathbb{C}P^1 \setminus \{P_1, P_2, P_3\}$, which is infinite dimensional.

As always in field theories we need a action-functional on the space of maps. Now fix the metric on Σ and M, and for $\varphi \in \mathrm{Maps}(\Sigma, M)$ sufficiently smooth we define

$$\mathcal{S}[\varphi] = \int_\Sigma \|d\varphi\|^2 d\mu,$$

where the metric defines the norm. Since M is Calabi–Yau it is especially Kähler and we have a complex structure, J, which is compatible with a symplectic structure ω. Now the action functional can be recast in a different shape

$$\mathcal{S}[\varphi] = \int_\Sigma \|\bar{\partial}_J \varphi\|^2 d\mu + \int_\Sigma \varphi^*(\omega).$$

In every homotopy class $\int_\Sigma \varphi^*(\omega)$ is a lower bound for $\mathcal{S}[\varphi]$. This lower bound is exactly obtained on J-holomorphic maps, $\bar{\partial}_J \varphi = 0$.

Now by using the stationary phase method we can rewrite the correlation function from an ill-defined integral over an infinite dimensional space to an infinite sum over homotopy classes, where the summands are integrals over finite dimensional moduli spaces of J-holomorphic curves with fixed homology class.

$$\langle \mathcal{O}_{P_1}, \ldots, \mathcal{O}_{P_k} \rangle = \int \mathcal{D}\varphi\, \mathcal{O}_{P_1} \ldots \mathcal{O}_{P_k} e^{-2\pi \mathcal{S}[\varphi]}$$
$$= \sum_{\text{homotopy classes}} e^{-2\pi \int_\Sigma \varphi^*(\omega)} \int_{\mathcal{M}} \mathcal{D}\varphi\, \mathcal{O}_{P_1} \ldots \mathcal{O}_{P_k}.$$

When M is a Calabi–Yau manifold the topological correlation functions fall into two distinct categories *A-models* and *B-models*.

1.1 A-model and B-models

In the A-model the operators $\mathcal{O}_{P_j}$ correspond to harmonic differential forms α_j on M and correlation functions $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ are infinite series, where the term corresponding to homotopically trivial maps $\Sigma \to M$ is $\int_M \alpha_1 \wedge \alpha_2 \wedge \alpha_3$. The other terms are related to certain counting problems for J-holomorphic curves in a Calabi–Yau manifold.

In the B-model we use the fact that M is a Calabi–Yau manifold. One of the equivalent definitions of a Calabi–Yau manifold is that there exists a non-vanishing holomorphic n-form Ω on M, where $\dim_{\mathbb{C}} M = n$. In the B-model the operators correspond to elements in $H^q(\wedge^p T^{(1,0)} M)$ where $T^{(1,0)} M$ is the holomorphic part of the tangent bundle - the splitting induced by the complex structure. Correlation functions are now defined as the composition of the following three maps.

There is natural map

$$H^{q_1}(\wedge^{p_1} T^{(0,1)} M) \times H^{q_2}(\wedge^{p_2} T^{(0,1)} M) \times H^{q_3}(\wedge^{p_3} T^{(0,1)} M) \to H^n(\wedge^n T^{(0,1)} M),$$

where $p_1 + p_2 + p_3 = n = q_1 + q_2 + q_3$. This map can be composed with interior multiplication by Ω, given a map from $H^n(\wedge^n T^{(0,1)} M)$ to $H^n(\mathcal{O}_M)$. By Serre duality $H^n(\mathcal{O}_M) \simeq (H^0(K_M))^*$ where K_M is the canonical bundle on M that in the Calabi–Yau case is trivial. Since the correlation functions should be complex valued we finally evaluate on $\Omega \in H^0(K_M)$. That is, if the operators $\mathcal{O}_{P_i}$ correspond to the elements θ_i in $H^{q_i}(\wedge^{p_i} T^{(0,1)} M)$ then the map described is

$$\int_M \Omega \wedge (\theta_{P_1} \cdot \cdots \cdot \theta_{P_k} \cdot \Omega).$$

This is really just the term of homotopically trivial maps in the definition of correlation function. By physical arguments that can be found in [3] and [6] all other terms in the B-model disappear such that

$$\langle \mathcal{O}_{P_1}, \ldots, \mathcal{O}_{P_k} \rangle = \int_M \Omega \wedge (\theta_{P_1} \cdot \cdots \cdot \theta_{P_k} \cdot \Omega). \tag{1}$$

The B-model correlation functions are to prefer over the A-model correlation functions, since we have methods to deal with this single integral.

1.2 Mirror symmetry

If M, W are two Calabi–Yau manifolds producing the same physics (meaning having the same superconformal field theory), where you just change the role of A- and B- correlation functions then M, W are mirror manifolds. If this is the case we should have isomorphisms

$$H^q(\wedge^p(T^{(1,0)}M)^*) \simeq H^q(\wedge^p(T^{(1,0)}W)),$$

and vice versa. Further than that there should be formulas relating the A- and B-correlation functions. In other words, there should be a relation between the number of rational curves in M and period integrals of Ω. The name mirror symmetry comes from the above isomorphism since this reflects that the Hodge diamond of M is the mirror of W.

In the case of two Calabi–Yau threefolds $h^{1,1}(M) = h^{1,2}(W)$ and since $h^{1,1}(M) = \dim T_{[\omega]}\mathcal{M}_{KC}(M)$ is the dimension of the tangent space of the moduli space of complexified Kähler forms on M, and $h^{1,2} = \dim T_{[J]}\mathcal{M}_{CS}(W)$ is the dimension of the tangent space of the moduli space of complex structures on W, one could hope that there was a isomorphism between the mentioned moduli spaces. This is however a bit too optimistic. The conjecture is that in the large complex structure limit the isomorphism should exist. Due to length limitations, we will not go into further discussion of this conjecture.

2 The first instance of mirror symmetry

One of the first instances of mirror symmetry was in [1], where the B-model correlation function of the mirror of a general quintic threefold is calculated. As discussed the B-model correlation function should be the same as the A-model correlation function of the general quintic threefold. It turns out that the key ingredients in the B-model correlation function is – as expected – related to the number of rational curves on the general quintic threefold. We will here review the calculation from [1], in the light of the presentation given in [2, Chap. 2].

Now $\mathbb{C}P^4(5)$ is the family of manifolds that can be represented as quintic hypersurfaces in $\mathbb{C}P^4$. Each of these hypersurfaces are Calabi–Yau and has $h^{1,1} = 1$ and $h^{1,2} = 101$. We will consider a family of hypersurfaces, M_ψ, given by the polynomials

$$p = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5.$$

These are the most general quintics invariant under the group of symmetries

$$G = \{(a_1, \ldots, a_5) \in \mathbb{Z}_5^5 \mid \sum_i a_i \equiv 0 \text{ mod } 5\}/\mathbb{Z}_5,$$

where $\mathbb{Z}_5$ is embedded diagonally, and $g = (a_1, \ldots, a_5) \in G$ acts on $\mathbb{C}P^4$ as

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (\alpha^{a_1} x_1, \ldots, \alpha^{a_5} x_5),$$

where $\alpha = e^{2\pi i/5}$ is a fifth root of unity.

This gives a family of hypersurfaces in the singular space $\mathbb{C}P/G$, and the family of hypersurfaces inherits these singularities. If $\psi \neq -5\alpha^k$, $0 \leq k \leq 4$, the hypersurfaces is a Calabi–Yau orbifold. The orbifold singularities can be resolved simultaneously for all ψ without destroying the Calabi–Yau property. This is the family of mirrors W_ψ. There is no canonical way of resolving the singularities, but all will have the same moduli space of complex structures. We therefore just choose one. W_ψ is the mirror of the Calabi–Yau M_ψ, and when we talk about a certain pair, we often forget the index.

If we consider the map $(x_1, \ldots, x_5) \mapsto (\alpha^{-1} x_1, x_2, \ldots, x_5)$ it corresponds to the map $W_\psi \simeq W_{\alpha\psi}$, and so ψ and $\alpha\psi$ give the same complex structure. Therefore the map ψ^5 is a well defined function on the moduli space of complex strutures. Since $h^{2,1}(W) = 1$, the moduli space of complex structures on W is 1-dimensional, and we choose $x = \psi^{-5}$ as a local coordinate. To understand the moduli space we need to know where the singularities of W are. It can be shown that the singularities are when $\psi = -5\alpha^k$, $0 \leq k \leq 4$ and $\psi = \infty$. Translated into the coordinate on the moduli space, the moduli space has boundary at the points $x = -5^{-5}$ and $x = 0$.

By Mirror Symmetry the moduli space of complex structures on W should correspond to the moduli space of complexified Kähler structures on M. If we let $H \in H^2(M, \mathbb{Z}) \simeq \mathbb{Z}$ be a Kähler class that generates the ring, then $\omega = tH$ is a complexified Kähler form when t is in the upper half plane of $\mathbb{C}$. It can be shown that $q = e^{2\pi i t}$ is a local parameter for the moduli space of complexified Kähler classes, with $q = 0$ the boundary point. For later sections it should be noted that if l is a line in M, then q can be written as

$$q = e^{2\pi i t} = e^{2\pi i \int_l \omega}.$$

The claim is now that $x = 0$ and $q = 0$ corresponds to the same boundary point. If this is true it would imply that x is the right choice of coordinate on the moduli space of complex structures of W. The claim follows from the fact that it is only around $x = 0$ that a certain 3-form has maximal unipotent monodromy. A deeper analysis of mirror symmetry tell that this is exactly what we're after. We will however not pursue this further.

Now we know that the local coordinates q and x describe corresponding boundary points. The local mirror map is however not given by $q = x$ - it is more subtle than that. We need to take so called 'quantum corrections' into account. What we need to do is therefore to describe x and q as functions of each other which will give us the mirror map. Then we'll compute the B-model correlation functions in terms of q and then give an expression for the A-model correlation function.

2.1 Mirror map

One way to study the Kähler moduli space is to find special cycles for its integral homology, and using the periods of these cycles to define the mirror map.

From [5] there exists a minimal integral vanishing cycle γ_0 near $x = 0$, so that γ_0 is invariant under monodromy. Furthermore there is a minimal integral cycle γ_1 that transforms under monodromy about $x = 0$ as $\gamma_1 \mapsto \gamma_1 + m\gamma_0$ for some $m \in \mathbb{Z}$. The special case of the quintic at hand, $m = 1$. So if Ω is a holomorphic threeform on W (a such exists since W is Calabi–Yau) then $\int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega$ transforms under monodromy about $x = 0$ to $\int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega + 1$, so $\exp(2\pi i \int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega)$ is a well-defined function in a neighborhood around $x = 0$. It can be shown that γ_0 is unique up to a sign, and γ_1 is defined up to addition of multiples of γ_0 – and the overall sign. Hence the quantity is canonically determined on the moduli space of W, and since W is a Calabi–Yau Ω is defined up to a scalar, so the quotient is independent of the particular choice of Ω. Therefore it is natural to assume that the mirror map is $t = \int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega$, and

$$q = e^{2\pi i \int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega}.$$

This assertion can be shown to be correct, but proving this would not keep this outline brief.

To express q as a function of x, we need to express the periods as functions of x. This is done by finding the periods as solutions to the Picard–Fuchs equation, which in this particular case looks like

$$y'''' + f_1 y''' + f_2 y'' + f_3 y' + f_4 y = 0,$$

where f_i depend on the coordinate x and the differentiation is with respect to x, and $y = \int_\gamma \Omega$ is a period of Ω and γ is three-cycle on W.

Now let's make a particular choice of Ω and find all the solutions to the differential equation. Let

$$\Omega = \mathrm{Res}\left(\frac{\psi}{p} \varepsilon_{\nu_1\nu_2\nu_3\nu_4\nu_5} x_{\nu_1} dx_{\nu_2} \wedge dx_{\nu_3} \wedge dx_{\nu_4} \wedge dx_{\nu_5} \right),$$

where $\varepsilon_{\nu_1\nu_2\nu_3\nu_4\nu_5}$ is the permutation symbol. The form before taking residues is well-defined on $\mathbb{C}P^4$ but has singularities at $p = 0$, i.e. on the hypersurfaces from the beginning of this section. Taking the residue produces a holomorphic threeform. With this choice the Picard–Fuchs equation is

$$\Big(x\frac{d}{dx}\Big)^4 y + \frac{2 \cdot 5^5 x}{1 + 5^5 x}\Big(x\frac{d}{dx}\Big)^3 y + \frac{7 \cdot 5^4 x}{1 + 5^5 x}\Big(x\frac{d}{dx}\Big)^2 y + \frac{2 \cdot 5^4 x}{1 + 5^5 x}\Big(x\frac{d}{dx}\Big) y + \frac{24 \cdot 5x}{1 + 5^5 x} y = 0.$$

The periods y_0, y_1 satisfy this equation. There are standard methods to solve generalized hypergeometric equations as the Picard–Fuchs equation, and applying these we get that y_0 and y_1 are uniquely determined up to $y_0 \mapsto b_1 y_0$ and $y_1 \mapsto b_1 y_1 + b_2 y_1$, where $b_1, b_2 \in \mathbb{C}$, $b_1 \neq 0$. Then $q = e^{2\pi i y_1/y_0}$ is determined up to a constant $c_1 = e^{2\pi i b_2/b_1}$. It can be shown that $c_1 = -1$, and furthermore that

$$q = -x + 770x^2 + \dots \quad \text{and} \quad x = -q + 770q^2 + \cdots .$$

The methods for solving generalized hypergeometric functions can give closed expressions for q in terms of x

$$q = -x \exp\left(\frac{5}{y_0(x)} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left[\sum_{j=n+1}^{5n} \frac{1}{j} \right] (-1)^n x^n \right),$$

where

$$y_0(x) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} (-1)^n x^n .$$

2.2 B-model correlation function

Next step is to calculate the B-model correlation function in terms of x. From (1) we know that it is defined by

$$\left\langle x\frac{d}{dx}, x\frac{d}{dx}, x\frac{d}{dx} \right\rangle = \int_W \Omega \wedge \Omega''' =: Y,$$

where the differentiation on Ω is $x\frac{d}{dx}$. Since neither Ω' or Ω'' have $(0,3)$ components it can be shown using the Picard–Fuchs equation that Y satisfies the differential equation

$$\left(x\frac{d}{dx} \right) Y = \frac{-5^5 x}{1 + 5^5 x} Y.$$

If this should be done rigorously the differentiations of Ω should have been covariant differentiation of Ω with respect to a connection in the line bundle defined by the scalar-ambiguity in Ω.

The solution to the above equation is $Y = \frac{c_2}{1+5^5 x}$, where c_2 is a constant. If we choose to normalize Ω such that $\int_{\gamma_0} \Omega = 1$ we have to replace Ω by $\frac{\Omega}{y_0(x)}$, and thus we get that $Y = \frac{c_2}{(1+5^5 x) y_0(x)^2}$.

The initial goal was to compute $\langle H, H, H \rangle$ where $H \in H^1(M, \Omega^1_M)$ is a holomorphic generator. $H^1(M, \Omega^1_M)$ is the tangent space to the Kähler moduli space. With t being a function on this moduli space tH is a vector field, and we identify it with $\frac{d}{dt}$. The local coordinate is $q = e^{2\pi i t}$, and thus $\frac{d}{dt} = 2\pi i q \frac{d}{dq}$ and since we established that under the mirror map q is a function of x, $\frac{d}{dq}$ is mapped to $\frac{dx}{dq}\frac{d}{dx}$. Now define $H := 2\pi i q \frac{d}{dq}$ that under the mirror map is $2\pi i q \frac{dx}{dq}\frac{d}{dx}$. Now

$$\begin{aligned}\langle H, H, H\rangle &= \left(2\pi i q \frac{dx}{dq}\right)^3 \left\langle \frac{d}{dx}, \frac{d}{dx}, \frac{d}{dx}\right\rangle = \left(2\pi i \frac{q}{x}\frac{dx}{dq}\right)^3 \left\langle x\frac{d}{dx}, x\frac{d}{dx}, x\frac{d}{dx}\right\rangle \\ &= \left(2\pi i \frac{q}{x}\frac{dq}{dx}\right)^3 Y = \left(2\pi i \frac{q}{x}\frac{dq}{dx}\right)^3 \frac{c_2}{(1+5^5x)y_0(x)^2} \\ &= \left(\frac{q}{x}\frac{dq}{dx}\right)^3 \frac{5}{(1+5^5x)y_0(x)^2},\end{aligned}$$

since it turns out that $c_2 = \frac{5}{(2\pi i)^3}$.

If we now insert x, $\frac{dq}{dx}$ and $y_0(x)$ as functions of q and write it as a power series in $\frac{q^d}{1-q^d}$ we get that

$$\begin{aligned}Y &= 5 + 2875\frac{q}{1-q} + 609250 \cdot 2^3 \frac{q^2}{1-q^2} + 317206375 \cdot 3^3 \frac{q^3}{1-q^3} + \dots \\ &= 5 + \sum_{d=1}^{\infty} \tilde{n}_d d^3 \frac{q^d}{1-q^d}. \end{aligned} \tag{2}$$

The first terms in the expansion are given in Table 1

Table 1: The coefficients in the expansion of the B-model correlation function in $\frac{q^d}{1-q^d}$

d	$\tilde{n}_d$
1	2875
2	609250
3	317206375
4	2424675300
5	229305888887625

According to Mirror Symmetry the expansion of the B-model correlation function Y in terms of q, should be the A-model correlation function for the quintic threefold M. From the introduction the coefficients in the A-model correlation function should be related to the number of rational curves on M. We recognize 2875 as the number of lines on the quintic threefold, so there is a chance that this could be true. In the next section we will discuss how to rigorously define the $\tilde{n}_d$ in terms of Gromov–Witten invariants counting J-holomorphic curves, and see that $\tilde{n}_d$ indeed is the number of degree d rational curves on M, at least when $d \leq 9$.

3 Gromov-Witten invariants

In this section we will define Gromov–Witten invariants, and tie it together with the A-model correlation function. We will especially look at the case of the quintic threefold discussed above.

There are several different ways to define Gromov–Witten invariants. The tow major approaches is a symplectic and an algebraic. Since we want to apply the theory to projective manifolds we will choose the algebraic version. It has been shown that the two approaches agree on their common domain of validity. The way we will go about defining the invariants is by setting up a list of axioms that the invariants should satisfy. This doesn't give existence of such invariants, but it provides a list of properties that is satisfied if they exists.

The common idea of the different approaches to Gromov–Witten invariants can be illustrated by considering a complex projective manifold M. This is both algebraic and symplectic, so both types of constructions apply to this. Lets write (M, ω, J) to illustrate the different structures on M. Fix $\beta \in H_2(M, \mathbb{Z})$ and cycles $Z_1, \ldots, Z_n$ on M. We are now interested in J-holomorphic maps $f : \Sigma \to M$ from a surface Σ with marked points $p_1, \ldots, p_n$ such that $f(p_i) \in Z_i$ and $f_*[\Sigma] = \beta$. We are interested in counting the number of these, and the Gromov–Witten invariants are basically this count. This is however a too naive approach, and we need to add extra adjectives to get nice moduli space of such maps.

A way of doing this is to restrict the attention to stable maps. If (M, ω) is a compact symplectic manifold, J is an almost-complex structure compatible with ω. Fix $g, m \geq 0$ and $\beta \in H_2(M, \mathbb{Z})$. We will consider triples $(\Sigma, \vec{p}, u)$ where $(\Sigma, \vec{p})$ is a prestable Riemann surface of genus g (possible singular) with marked points $\vec{p}$. $(\Sigma, \vec{p})$ is prestable if Σ is a compact connected complex 1-dimensional variety whose only singularities are ordinary double points. Furthermore $\vec{p} = (p_1, \ldots, p_m)$ are different smooth points on Σ and the automorphism group of $(\Sigma, \vec{p})$ is finite. $u : \Sigma \to M$ is a J-holomorphic map satisfying $u_*([\Sigma]) = \beta$ in $H_2(M, \mathbb{Z})$. An isomorphism between $(\Sigma, \vec{p}, u)$ and $(\Sigma', \vec{p}', u')$ is a biholomorphism $f : \Sigma \to \Sigma'$ with $f(p_i) = p'_i$. The surface $(\Sigma, \vec{p}, u)$ is stable if the automorphism group of $(\Sigma, \vec{p}, u)$ is finite.

Let $\overline{M}_{g,m}(M, J, \beta)$ be the set of isomorphism classes of stable triples $(\Sigma, \vec{p}, u)$ where Σ has genus g with m marked points and $u_*([\Sigma]) = \beta$ in $H_2(M, \mathbb{Z})$. Let $\overline{M}_{g,m}$ be the moduli space of prestable Riemann surfaces. Then $\overline{M}_{g,m} \simeq \overline{M}_{g,m}(\text{point}, J, 0)$.

Theorem. *$\overline{M}_{g,m}(M, J, \beta)$ is a compact Hausdorff topological space, and all evaluation maps $ev_i : \overline{M}_{g,m}(M, J, \beta) \to M$ given by $ev_i : [\Sigma, \vec{p}, u] \mapsto u(p_i)$ are continuous.*

If J is a generic almost complex structure, then the open subset of $[\Sigma, \vec{p}, u]$ in $\overline{M}_{g,m}(M, J, \beta)$ for which Σ is non-singular and $u : \Sigma \to M$ is an embed-

ding is a smooth oriented real manifald of dimension

$$2(c_1(M) \cdot \beta + (n-3)(1-g) + m),$$

where $\dim M = 2n$. *Also the* ev_i*'s are smooth on this open subset.*

$\overline{M}_{g,m}(M, J, \beta)$ *carries an oriented Kuranishi structure, without boundary and corners, and with virtual dimension as above.*

$\overline{M}_{g,m}$ *is a compact oriented orbifold of real dimension* $6g + 2n - 6$ *when* $n + 2g \geq 3$.

The rough idea of Gromov–Witten classes is that the maps $u : \Sigma \to M$ sending the marked points to prechosen cycles Z_i, defines a subset of $\overline{M}_{g,m}(M, J, \beta)$, which in turn gives a cohomology class in $H^*(\overline{M}_{g,m}(M, J, \beta), \mathbb{Q})$.

Let $\alpha_j \in H^*(M, \mathbb{Q})$ be cohomology classes dual to Z_i. The Gromov–Witten class $I_{g,m,\beta}(\alpha_1, \dots, \alpha_n) \in H^*(\overline{M}_{g,m}, \mathbb{Q})$ is intuitively supposed to be the cohomology class represented by the set of pointed curves $(\Sigma, p_1, \dots, p_n)$ from above. So the Gromov–Witten classes are a system of maps

$$I_{g,m,\beta} : H^*(M, \mathbb{Q})^{\otimes m} \to H^*(\overline{M}_{g,m}, \mathbb{Q}).$$

The Gromov–Witten *invariants* are now

$$\langle I_{g,m,\beta} \rangle (\alpha_1, \dots, \alpha_m) = \int_{\overline{M}_{g,m}} I_{g,m,\beta}(\alpha_1, \dots, \alpha_m).$$

If not $I_{g,m,\beta}(\alpha_1, \dots, \alpha_n)$ is a top-dimensional class then the Gromov–Witten invariants vanish. When there are only finitely many J-holomorphic stable curves $u : \Sigma \to M$, $I_{g,m,\beta}(\alpha_1, \dots, \alpha_m)$ should be a top dimensional class, and then $\langle I_{g,m,\beta} \rangle (\alpha_1, \dots, \alpha_n)$ should be the number of these curves.

To make this construction mathematically rigorous we have to discuss virtual fundamental classes and Kuranishi structures, but to keep down the length of this review, we omit this description. Rather we will give an axiomatic definition, and if one do the above heuristic approach the right way, it's an example of the following definition.

Definition 3.1. Let (M, ω, J) be a complex smooth projective variety, and $\beta \in H_2(M, \mathbb{Z})$. For $g, m \geq 0$ and $m + 2g \geq 3$ Gromov–Witten classes $I_{g,m,\beta}$, and Gromov–Witten invariants $\langle I_{g,m,\beta} \rangle$ are maps

$$\begin{aligned} I_{g,m,\beta} &: H^*(M, \mathbb{Q})^{\otimes m} \to H^*(\overline{M}_{g,m}, \mathbb{Q}) \\ \langle I_{g,m,\beta} \rangle &: H^*(M, \mathbb{Q})^{\otimes m} \to \mathbb{Q}, \end{aligned}$$

and are related by

$$\langle I_{g,m,\beta} \rangle (\alpha_1, \dots, \alpha_m) = \int_{\overline{M}_{g,m}} I_{g,m,\beta}(\alpha_1, \dots, \alpha_m).$$

$I_{g,m,\beta}$ should satisfy ten axioms. We only list the ones needed in the following.

Linearity $I_{g,m,\beta}$ is linear in each variable

Effectivity $I_{g,m,\beta} = 0$ if $[\omega] \cdot \beta < 0$

Fundamental class If $m+2g \geq 4$ then we get a natural map $\pi_m : \overline{M}_{g,m} \to \overline{M}_{g,m-1}$ by forgetting the last point. If $[M] \in H^0(M, \mathbb{Q})$ is the fundamental class of M, then the axiom asserts that

$$I_{g,m,\beta}(\alpha_1, \ldots, \alpha_{m-1}, [M]) = \pi_m^* I_{g,m-1,\beta}(\alpha_1, \ldots, \alpha_{m-1}).$$

Divisor If $m + 2g \geq 4$ then let $\pi_m : \overline{M}_{g,m} \to \overline{M}_{g,m-1}$ be as above. If $\alpha_m \in H^2(M, \mathbb{Q})$ then

$$\pi_{m*} I_{g,m,\beta}(\alpha_1, \ldots, \alpha_{m-1}, \alpha_m) = (\int_\beta \alpha_m) \cdot I_{g,m-1,\beta}(\alpha_1, \ldots, \alpha_{m-1}).$$

Point mapping When $\beta = 0$, $g = 0$ then

$$I_{0,m,0}(\alpha_1, \ldots, \alpha_m) = \begin{cases} (\int_M \alpha_1 \cup \cdots \cup \alpha_m)[\overline{M}_{g,m}] & \text{if } \sum_{i=1}^m \deg \alpha_i = 2 \dim M \\ 0 & \text{otherwise} \end{cases}$$

For Gromov–Witten invariants this is

$$\langle I_{0,m,0} \rangle (\alpha_1, \ldots, \alpha_m) = \begin{cases} \int_M \alpha_1 \cup \alpha_2 \cup \alpha_3 & \text{if } m = 3 \\ 0 & \text{otherwise} \end{cases}$$

An axiom not mentioned above is the splitting axiom that is very important to quantum cohomology, and basically is responsible for the associativity of the product in quantum cohomology.

3.1 Computations

In the introduction we discussed A- and B-model correlation functions. Above the B-model correlation functions for the mirror of a quintic threefold was calculated. This function should be the same as the A-model correlation function of the generic quintic threefold. We also saw the A-model correlation function should look like a sum over homotopy classes of integrals over moduli spaces of stable J-holomorphic curves. In this section we shall define this A-model correlation function, and relate it to a count of stable J-holomorphic curves.

If (M, ω, J) is a smooth projective Calabi–Yau threefold with complex structure J and ω symplectic form, and look at the definition of Gromov–Witten invariants in Definition 3.1, it turns out that the right way of defining the A-model correlation function is as

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \sum_\beta \langle I_{0,3,\beta} \rangle (\alpha_1, \alpha_2, \alpha_3) q^\beta,$$

where $q^\beta := e^{2\pi i \int_\beta \omega}$, $\alpha_i \in H^2(M, \mathbb{C})$. We're interested in M being a quintic threefold. Here $H_2(M, \mathbb{Z}) \simeq \mathbb{Z}$ and by choosing a line $l \subset M$ as a generator, then $\beta = d \cdot [l]$, and $q = e^{2\pi i \int_l \omega}$ so $q^\beta = e^{2\pi i \int_{d[l]} \omega} = e^{2\pi i d \int_{[l]} \omega} = q^d$, and the correlation function is

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \sum_{d \in \mathbb{N}} \langle I_{0,3,d} \rangle (\alpha_1, \alpha_2, \alpha_3) q^d,$$

since $\langle I_{0,3,d} \rangle (\alpha_1, \alpha_2, \alpha_3) = 0$ for $d < 0$ by the effectivity axiom (this is a reasonable axiom since $\beta \cdot [\omega]$ is the area of the J-holomorphic curve, and if this is negative such a curve should not exist). Now suppose $d > 0$ and $\beta = d[l]$ and $\alpha_1, \alpha_2, \alpha_3 \in H^2(M, \mathbb{C})$. Now there exists n_γ unique rational numbers such that

$$N_\beta = \langle I_{0,0,\beta} \rangle = \sum_{\beta = k\gamma} n_\gamma k^{-3}.$$

The sum is over all $k > 0$ in $\mathbb{Z}$ and $\gamma \in H_2(M, \mathbb{Z})$ with $\beta = k\gamma$, that is

$$N_d = \sum_{k|d} n_{\frac{d}{k}} k^{-3}.$$

By the above and the divisor axiom we have

$$\begin{aligned} \langle I_{0,3,d} \rangle (\alpha_1, \alpha_2, \alpha_3) &= N_d \int_\beta \alpha_1 \int_\beta \alpha_2 \int_\beta \alpha_3 = \left(\sum_{k|d} n_{\frac{d}{k}} k^{-3} \right) \int_\beta \alpha_1 \int_\beta \alpha_2 \int_\beta \alpha_3 \\ &= \sum_{k|d} n_{\frac{d}{k}} \int_{\frac{d}{k}[l]} \alpha_1 \int_{\frac{d}{k}[l]} \alpha_2 \int_{\frac{d}{k}[l]} \alpha_3. \end{aligned}$$

Now since $h^{1,1} = 1$, it's only necessary to know

$$\langle I_{0,3,d} \rangle (H, H, H) = \sum_{k|d} n_{\frac{d}{k}} \left(\int_{\frac{d}{k}[l]} H \right)^3 = \sum_{k|d} n_{\frac{d}{k}} \left(\frac{d}{k} \right)^3 \left(\int_{[l]} H \right)^3.$$

We can choose H such that $\int_{[l]} H = 1$, that is

$$\langle I_{0,3,d} \rangle (H, H, H) = \sum_{k|d} n_{\frac{d}{k}} \left(\frac{d}{k} \right)^3 = \sum_{k|d} n_k k^3,$$

and then

$$\langle H, H, H \rangle = \sum_{d \geq 1} \left(\sum_{k|d} n_k k^3 \right) q^d + \langle I_{0,3,0} \rangle (H, H, H) = 5 + \sum_{d \geq 1} n_d d^3 \frac{q^d}{1 - q^d}, \tag{3}$$

since $\langle I_{0,3,0} \rangle (H, H, H) = \int_M H \cup H \cup H = 5$ by the point mapping axiom. We see that (2) and (3) are of the same form – we just need the n_d's to be the same. We therefore hope that the numbers n_d are related to a count of rational curves.

For the quintic and $d \leq 9$ this relation between n_d and degree d rational curves are especially nice, since here n_d is exactly the number of rational curves of degree d in M. This follows from the fact that the n_d's are finite for $d \leq 9$ and a theorem about the contribution of multiple covers of a curve to the so called virtual fundamental class for the moduli space $\overline{M}_{0,0}(M, \beta)$.

If we know how to calculate n_d we have a way of checking if the mirror conjecture is true. It is however a tricky business to calculate n_d, but it has been done. In [4, Lecture 2] is given an overview of the calculation of n_d for $d = 1, 2, 3$. The calculations find exactly the same numbers as the power series expansion of the B-model correlation function in [1]. We therefore see an indication of why n_d and $\tilde{n}_d$ might be the same for all d, as claimed by the mirror symmetry conjecture.

For $d \geq 10$ the story is a bit more subtle. It is known that for $d = 10$, n_{10} is finite but is not exactly the number of degree 10 rational curves, basically the double covers of degree 5 nodal curves contribute 6 times as much to n_{10} as they should. That the n_d's are actually finite for higher d is a part of Clemens Conjecture.

4 Conclusion

In Section 1 we discussed the basics of String Theory and explained why J-holomorphic curves lie at the heart of the subject. We also discussed the A- and B-model correlation functions and discussed Mirror Symmetry.

In Section 2 we investigated [1], and outlined the calculation of the B-model correlation function of the mirror of a generic quintic threefold. The coefficients in the power series expansion was hoped to be related to a count of rational curves in the quintic threefold.

In Section 3 we defined Gromov–Witten classes and invariants for smooth projective varieties. We also discussed computations of the A-model correlation function, and it's definition using Gromov–Witten invariants for smooth projective Calabi–Yau threefolds. Finally we found that for $d \leq 9$ the numbers n_d appearing in the power series expansion of the B-model correlation function of the mirror of the quintic threefold, is exactly the number of degree d rational curves in the quintic threefold.

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