Deformation Quantization and Geometric Quantization of Abelian Varieties

Progress Report

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Preface

This progress report is the culmination of my three years of PhD-studies at Centre for Quantum Geometry of Moduli Spaces (QGM) and Department of Mathematical Sciences, Aarhus University.

First and foremost I would like to thank my advisor Professor Jørgen Ellegaard Andersen. In all of the three years he has had confidence in me and gave me inspiration in my mathematical work. I am very grateful that I have had the opportunity work with him and to be a part of the QGM centre. The vibrant international atmosphere of postdocs and visiting researchers has given me inspiration and experiences I would not have had in any other mathematical field in Aarhus.

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Introduction

In the 1920's a new and revolutionary wave of ideas was gaining strong foothold in physics. Led by Niels Bohr, Paul Dirac and John von Neumann among others, quantum mechanics was given a solid foundation in these years. At that time it was known that interactions between particles creates phenomena not predicted by Newton's laws. Quantum mechanics gave way of describing physics that could explain the phenomena occurring at these small scales. But since Newtonian mechanics was superior in describing macroscopic interactions between objects, quantum mechanics should be consistent with this as well. This is the correspondence principle, a principle at the very heart of quantization.

Quantum mechanics is the quantization of classical mechanics, or rather is it a quantization. Whereas there is a natural way of raising the energy (letting Planck's constant go to 0), there is no canonical way of lowering the energy. Actually the ideas put forth in e.g. [13] contain algebraic contradictions, and there cannot exist a full quantization as the physicists in the 1920's want. But even with these contradictions quantum mechanics made predictions that were all verified in nature. Therefore even though there were problems in the mathematical interpretation, something was right. These problems and the ambiguity in choosing the right quantization led mathematicians to study quantization.

We will study different approaches to quantization, each of which has its share of good properties and flaws. We will study geometric quantization and deformation quantization where different parts of quantization is highlighted. We also give a way of bridging the gap between geometric quantization and deformation quantization on a compact Kähler manifold, namely Berezin–Toeplitz deformation quantization.

Classical mechanics in a mathematical language, is roughly speaking the study of functions on a symplectic manifold. Functions are the observables and the manifold is the phase space. The symplectic structure governs the dynamics of the system. When we do geometric quantization and Berezin–Toeplitz deformation quantization we have to choose a compatible complex structure and make the symplectic manifold Kähler. However, in the end the quantization should not depend on this complex structure. This leads to the introduction of the Hitchin connection. The Hitchin connection give a way of relating the quantum spaces produced from geometric quantization. Therefore we describe in rather great detail Andersen's differential geometric construction of the Hitchin connection.

In the final chapters we try generalize the Berezin–Toeplitz star product to be able to quantize another type of observables. Functions on the manifold can be seen as sections of a trivial line bundle on the manifold, or as sections of the zeroth tensor power of a line bundle. If we look at sections of a positive power of a line bundle, can we then make a deformation quantization of these? We try to answer this question in the setting of a principal polarized abelian variety. Studying these manifolds have the great advantages that everything can be explicitly calculated, and therefore serves as a great lab for testing theories. We do not give the final answer to the above question, but see indications of why it might be true.

In the very end we present calculations of several different objects and their expressions in the concrete case of a principal polarized abelian variety at hand.

Summary Here follows a summary of the content of this progress report.

Chapter 1 This chapter contains a collection of all the complex differential terms needed in this report. The purpose of this chapter is mainly to set the notation of the basic ingredients used.

Chapter 2 In this chapter we discuss various aspects of quantization. We begin by defining what a quantization should be, but cannot by the famous no-go theorem of Groenewold and van Hove. We describe two different approaches to avoid the no-go theorem, geometric and deformation quantization. In the end we describe Berezin–Toeplitz deformation quantization that in some sense unites the two approaches. We conclude the chapter with a brief discussion of the Hitchin connection.

Chapter 3 In Chapter 3 we repeat Andersen's construction of a Hitchin connection in the setup of Kähler quantization of a general symplectic manifold. Before describing the construction we introduce the needed notions of families of Kähler structures, and spell out what it means for them to be holomorphic and rigid. This chapter serves as a nice extension of the quantization discussion in Chapter 2. Furthermore as an introduction of the ingredients of Andersen's construction of the Hitchin connection, a Hitchin connection, which is explicitly calculated in Chapter 7 in the setting of a principal polarized abelian variety.

Chapter 4 Here we briefly discuss a very important application of Kähler quantization and the Hitchin connection, namely the quantization of moduli spaces of flat G-connections on a surface. We will only be interested in the case where G is $SU(n)$. We discuss two different views upon these moduli spaces, either as representations of the fundamental group of the surface, or as flat $SU(n)$-connections on the same surface. We also discuss compactness and smoothness properties of these moduli spaces and finally explain why they fit into the framework set up in Chapter 3.

Chapter 5 We discuss the basic properties of abelian varieties. As a start we use Kodaira's Embedding Theorem to give necessary and sufficient conditions on a torus $M = V/\Lambda$ where V is a complex vector space and Λ a lattice of full rank, to be an abelian variety. Secondly we investigate the construction of line bundles over M by functions on V satisfying som functional equations. Lastly we discuss the space of holomorphic section of a line bundle over M, and explain why the theta functions serve as a basis for this space.

Chapter 6 In this chapter we explicitly calculate the Berezin–Toeplitz star product on an abelian variety. We recall the needed theorems from [3] and reprove them. Following this we try to generalize the Berezin–Toeplitz star product to the space of smooth sections of a fixed power of the line bundle. We only obtain indications that it might be possible to do, and in the end put forth a conjecture.

Chapter 7 In this chapter we explicitly calculate the Hitchin connection from the recipe in Chapter 3. We furthermore calculate the curvature of this connection and see that it is indeed projectively flat. We compute the parallel transport operator induced by the flat connection defined by the heat equation, and see that it is a Toeplitz operator associated to a function on the square of Siegels Upper Half space. Lastly we investigate the Hilbert–Schmidt norm of Toeplitz operators, and prove an asymptotic result in the abelian case at hand. Chapter 6 and 7 is joint work with Andersen.

1

Complex Differential Geometry

In this chapter we will review the basic differential geometric facts we will need in the rest of this report. We begin by introducing the notion of an almost complex structure on an even dimensional smooth manifold.

1.1 Almost complex structures

Definition 1.1. A section, I, of the endomorphism bundle $\mathrm{End}(TM)$ on a manifold M, which is a complex structure in each fiber, that is $I_x : T_xM \to T_xM$ squares to $-Id$, is called *an almost complex structure* for M, and (M, I) is called an almost complex manifold.

Remark that $I^2 = -Id$ forces M to be even dimensional, and have a canonical orientation. The even dimensionality is because $\det(I - xId)$ is a polynomial of degree n. If n is odd this polynomial has a real root y, and with $\det(I - yId) = 0$ there is a vector $v \in TM$ such that $Iv = yv$. Again $I^2v = y^2v$, and since y is real $y^2 \neq -1$. The orientation is from the fact tha the structure group is reduced to $GL(n, \mathbb{C}) \subset GL^+(2n, \mathbb{R})$.

Let $TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the tangent bundle. I extends to a $\mathbb{C}$ linear bundle isomorphism on $TM_{\mathbb{C}}$ and has fiberwise eigenvalues $\pm i$ since $I^2 = -Id$. Let T'_I be the $+i$-eigenspace of $TM_{\mathbb{C}}$ and T''_I be the $-i$-eigenspace.

This extension of I induces a decomposition $TM_{\mathbb{C}} = T'_I \oplus T''_I$, and for each vector field X on M we denote the splitting into types as $X = X' + X''$. This is explicitly given by projecting X onto T'_I and T''_I with the projections

$$\pi_I^{1,0} = \frac{1}{2}(Id - iI) \quad \text{and} \quad \pi_I^{0,1} = \frac{1}{2}(Id + iI).$$

I also act on TM^* by $(I\alpha)(X) = \alpha(IX)$ and this also induce a splitting of $TM^*_{\mathbb{C}}$ into types, $TM^*_{\mathbb{C}} = T'^*_I \oplus T''^*_I$. The splitting of $TM^*_{\mathbb{C}}$ is compatible with the splitting of $TM_{\mathbb{C}}$ in the sence that T'^*_I consists of exactly those one forms that vanish on elements of T'' and likewise for T''^*_I.

Conjugation on $TM_{\mathbb{C}}$ given by fiberwise conjugation, give a conjugate linear isomorphism between the eigenspaces, $T'_I \to T''_I$, and fiberwise multiplication by I in T'_I regarded as multiplication by i makes T'_I into a complex vector bundle.

The decomposition of the cotangent bundle induce a decomposition in all tensor bundles, especially in the exterior algebra bundle

$$\wedge_I^k T^*M_{\mathbb{C}} = \bigoplus_{r+s=k} \wedge_I^{(r,s)} T^*M_{\mathbb{C}} \quad \text{with} \quad \wedge_I^{(r,s)} T^*M_{\mathbb{C}} = \wedge^r T'^*_I \otimes \wedge^s T''^*_I,$$

and thus also a splitting of the complex valued differential forms

$$\Omega_I^k(M) = \bigoplus_{r+s=k} \Omega_I^{(r,s)}(M) \quad \text{where} \quad \Omega_I^{(r,s)}(M) = C^\infty(M, \wedge_I^{(r,s)} T^*M_{\mathbb{C}}).$$

Elements of of $\Omega_I^{(r,s)}(M)$ are called differential forms of type (r,s). Define projections $\pi_I^{(r,s)}:\Omega_I^k(M)\to\Omega_I^{(r,s)}(M)$, $k=r+s$ picking out the (r,s)-part of a k form. Using these projections and the exterior differential $d:\Omega_I^{(r,s)}(M)\to\Omega_I^{r+s+1}(M)$ we define new operators

$$\partial_I:\Omega_I^{(r,s)}(M)\to\Omega_I^{(r+1,s)}(M)\quad\text{and}\quad\bar\partial_I:\Omega_I^{(r,s)}(M)\to\Omega_I^{(r,s+1)}(M),$$

by composing d with the projections, $\partial_I=\pi_I^{(r+1,s)}\circ d$, $\bar\partial_I=\pi_I^{(s,r+1)}\circ d$, and extending to all of $\Omega^*(M)=\oplus_{r=0}^{\dim M}\Omega^r(M)$ by complex linearity.

In general $d:\Omega_I^{(r,s)}(M)\to\Omega_I^{r+s+1}(M)$ can be decomposed as $d=\sum_{p+q=r+s+1}\pi_I^{(p,q)}\circ d=\partial_I+\bar\partial_I+\cdots$. If however $d=\partial_I+\bar\partial_I$ then $d^2=\partial_I^2+\partial_I\bar\partial_I+\bar\partial_I\partial_I+\bar\partial_I^2$ and since every operator projects to a different summand of $\Omega^{r+s+2}(M)$ we get $\partial_I^2=\partial_I\bar\partial_I+\bar\partial_I\partial_I=\bar\partial_I^2=0$. Note that the decomposition of d with projections depend on the almost complex structure, and if $d=\partial_I+\bar\partial_I$ we call the almost complex structure *integrable*.

Remark that everything above that depend on I has an I as subscript. Not to clutter the notation we will omit the dependence unless we need to be careful what complex structure we use.

1.1.1 Complex manifolds

Definition 1.2. A *complex manifold* is a smooth manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha:U_\alpha\to\mathbb{C}^n$ such that $\varphi_\alpha\circ\varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha\cap U_\beta)\subset\mathbb{C}^n$ for all α,β.

Every complex manifold induces an almost complex structure on its underlying smooth manifold. The complex structure is given by multiplication with i in the complex tangent bundle. In local coordinates $z_i=x_i+iy_i$ on the tangent bundle is $I:T_zM\to T_zM$ defined by

$$\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_n}\Big)\mapsto\Big(-\frac{\partial}{\partial y_1},\frac{\partial}{\partial x_1},\ldots,-\frac{\partial}{\partial y_n},\frac{\partial}{\partial x_n}\Big).$$

By the holomorphicity of the transistion functions for M, the definition of I is independent of the chosen coordinates, and hence TM is a complex vector bundle. The usual transition maps for the tangent bundle are furthermore holomorphic, and TM is a holomorphic vector bundle.

Theorem 1.3. *The induced almost complex structure on a complex manifold is integrable.*

The converse of this theorem is the classical theorem by Newlander and Nirenberg.

Theorem 1.4. *Let (M,I) be an integrable almost complex manifold. Then there exists a unique complex structure on M, which induces the almost complex structure I. Equivalently is I integrable if and only if the Nijenhuis tensor N_I vanishes*

$$N_I(X,Y)=[X,Y]+I([IX,Y]+[X,IY])-[IX,IY].$$

1.2 Connections in vector bundles

In this section we define the basic notions of a connection in a vector bundle, notions what we will use very frequently in the rest of this report.

We assume M is a complex manifold.

Definition 1.5. A *connection* ∇ on a complex vector bundle $E \to M$ is a map $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for all sections $s \in \Omega^0(E)(U)$ and all functions $f \in C^\infty(U)$, where $U \subset M$ is an open set and $\Omega^{p,q}(E)(U)$ are the smooth E-valued (p,q)-forms on U.

We can extend this operator to $\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$ by forcing the Leibniz rule to be true

$$\nabla(\alpha \wedge s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla s$$

for $\alpha \in \Omega^p(U)$ and a section $s \in \Omega^0(E)(U)$. Given a frame $\{e_i\}$ for E we can express ∇ as $\nabla e_i = \sum_j \theta_{ij} e_j$ and for a section $s = \sum f_j e_j$

$$\nabla s = \sum_j (df_j + \sum_i \theta_{ij} f_i) e_j \quad \text{or in matrix notation} \quad \nabla = d + \theta$$

where $\theta = (\theta_{ij})$ is the connection matrix.

The operator $\nabla^2 : \Omega^0(E) \to \Omega^2(E)$ is of particular importance. This operator is first of all linear in $\Omega^0(E)$, in the sense that for a section s of E and a smooth function f, then $\nabla^2(fs) = f\nabla^2(s)$, and consequently ∇^2 corresponds to a global section, Θ of $\wedge^2 TM^* \otimes \mathrm{Hom}(E,E) = \wedge^2 TM^* \otimes E^* \otimes E$. If $\{e_i\}$ is a frame for E we can represent Θ in terms of this frame as a matrix of two forms

$$\nabla^2 e_i = \sum_j \Theta_{ij} e_j,$$

where (Θ_{ij}) is the curvature matrix of ∇ in terms of the frame $\{e_i\}$. If we instead describe Θ in terms of the connection matrix we get

$$\nabla^2 e_i = \sum_j (d\theta_{ij} + \sum_k \theta_{ik} \wedge \theta_{kj}) \otimes e_j \quad \text{or} \quad \Theta = d\theta + \theta \wedge \theta,$$

these are the *Cartan structure equations.* In many cases it can however be of further simplification to look at the operator $\Theta(X,Y) : E \to E$, where X, Y are vector fields on M. By using a frame for E it can be shown by direct computations that this is

$$\Theta(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

If $E \to M$ is Hermitian and ∇ on E is compatible with the complex structure, i.e. $\nabla'' = \bar{\partial} : \Omega^0(E) \to \Omega^{0,1}(E)$, then $\nabla''^2 = 0$ and $\Theta^{0,2} = 0$. If furthermore ∇ is compatible with the metric h, i.e. $d\langle u, v\rangle = \langle \nabla u, v\rangle + \langle u, \nabla v\rangle$, the connection matrix θ is skew-symmetric in terms of a unitary frame for E, thus $\Theta^{2,0} = -(\bar{\Theta}^T)^{0,2} = 0$. Since types are invariant under change of frames the curvature matrix of this connection which is compatible with complex structure and metric is a Hermitian matrix of $(1,1)$-forms. It can furthermore be shown that $\theta = h^{-1}\partial h$ and $\Theta = \bar{\partial}\theta$, and therefore

$$\Theta = \bar{\partial}\partial \log h. \tag{1.1}$$

1.3 Cohomology and Chern Classes of line bundles

Let I be an integrable almost complex structure on a manifold M, the decomposition $d = \partial + \bar{\partial}$ and the fact that $\partial^2 = 0$ and $\bar{\partial}^2 = 0$, give for each positive integer p a cochain complex

$$\Omega^{(p,0)}(M) \xrightarrow{\bar{\partial}} \Omega^{(p,1)}(M) \xrightarrow{\bar{\partial}} \Omega^{(p,2)}(M) \xrightarrow{\bar{\partial}} \dots$$

The cohomology from this cochain complex is called Dolbeault cohomology of M, $H^{p,q}(M)$. Dolbeault cohomology is intimately connected to sheaf cohomology by the Dolbeault theorem giving an isomorphism

$$H^q(M, \Omega^p) \simeq H^{p,q}(M),$$

where Ω^p is the sheaf of holomorphic differential forms of type $(p, 0)$. Ω^0 is usually denoted $\mathcal{O}$.

If we have a short exact sequence of sheaves, as for instance

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0,$$

we get a long exact sequence on sheaf cohomology

$$\longrightarrow H^p(M, \mathbb{Z}) \longrightarrow H^p(M, \mathcal{O}) \longrightarrow H^p(M, \mathcal{O}^*) \xrightarrow{\delta} H^{p+1}(M, \mathbb{Z}) \longrightarrow \tag{1.2}$$

In this sequence we have the induced map

$$H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}).$$

This coboundary operator is of particular importance since we have a beautiful description of holomorphic line bundles as cohomology classes. There is a one to one correspondence between holomorphic line bundles on M and first cohomology classes in the sheaf cohomology of M with coefficients in the sheaf of non-vanishing holomorphic functions on M. If we compose δ with the induced map on cohomology given by the inclusion of constant sheafs $\mathbb{Z} \to \mathbb{R}$ we get a homomorphism

$$H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \mathbb{R}) \ .$$

By this homomorphism is each line bundle associated a two form on M, this two form is called the first Chern class of the line bundle.

It is rather peculiar that the first Chern class of a line bundle can be calculated as in the following

Proposition 1.6. *For every Hermitian holomorphic line bundle $\mathcal{L}$ on M, there exists a compatible connection with curvature form Θ, and the first Chern class is*

$$c_1(\mathcal{L}) = \left[\frac{i}{2\pi} \Theta \right],$$

and $c_1(\mathcal{L})$ is independent of the chosen compatible connection.

For a proof of this proposition see [30, Chapter 3.3].

1.4 Symplectic geometry

In this section the basic object is a symplectic manifold. This is a smooth manifold with some additional structure. In terms of physics a symplectic manifold is the phase space where classical mechanics take place.

Definition 1.7. A *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω is a closed non-degenerate 2-form defined on all of M, i.e. $d\omega = 0$ and $T_pM \to T_p^*M$ given by contraction $X \to X \cdot \omega$, is a linear isomorphism for every $p \in M$, or equivalent that $\omega^n \neq 0$, where $\dim_{\mathbb{C}} M = n$.

We define contraction $\cdot$, such that $X \cdot \omega$ is the 1-form $(X \cdot \omega)(Y) = \omega(X, Y)$ and $(\omega \cdot X)(Y) = \omega(Y, X)$.

Example 1.8. The standard example of a symplectic manifold is $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_{std} = \sum_{i=1}^{n} dx_i \wedge dy_i$.

To every function $f \in C^\infty(M)$ we associate a unique vector field X_f as the solution to $X_f \cdot \omega + df = 0$. The uniqueness is given by the non-degeneracy of ω. The vector field, X_f, is called the *Hamiltonian vector field* associated to f.

$C^\infty(M)$ is a Lie algebra under the Poisson bracket defined by the symplectic form and the Hamiltonian vector fields

$$\{f, g\} = -\omega(X_f, X_g).$$

It is easy to see that $\{\cdot,\cdot\}$ satisfies the Jacobi identity, since $\{f, g\} = \mathcal{L}_{X_f} g$ where $\mathcal{L}_X$ is the Lie derivative defined by $\mathcal{L}_X(f) = X(f)$ on functions and $\mathcal{L}_X(Y) = [X, Y]$ on vector fields, and by *Cartan's magical formula*

$$\mathcal{L}_X \omega = d(X \cdot \omega) + X \cdot d\omega,$$

on differential forms. The Jacobi identity follow from this. $C^\infty(M)$ with $\{\cdot,\cdot\}$ is a Poisson algebra.

1.4.1 Compatible almost complex structures

An almost complex structure I on (M, ω) is ω-compatible if $g(X, Y) = \omega(X, IY)$ defines a positive definite Riemannian metric.

Let $\mathrm{Riem}(M)$ be the space of Riemannian metrics. There exists a canonical surjective map $F : \mathrm{Riem}(M) \to \mathcal{I}(M, \omega)$ from the space of Riemannian metrics to the space of ω-compatible almost complex structures. This map is a left invers to $\mathcal{I} \to \mathrm{Riem}(M)$ given by associating $g = \omega \cdot I$ to an $I \in \mathcal{I}(M, \omega)$. This map defines a retraction of $\mathcal{I}(M, \omega)$ since $\mathrm{Riem}(M)$ is contractible – so any almost complex structure can be deformed to any other.

The contractibility of $\mathcal{I}(M, \omega)$ can be used to define topological invariants of (M, ω, I) defined in terms of I. We define the first Chern class of a symplectic manifold to be $c_1(M, \omega) = c_1(K_I^*) = -c_1(K_I)$ where K_I is the canonical line bundle on M, $K_I = \wedge^n T'^*$. This line bundle is Hermitian, by the Hermitian structure h on T' defined by $h(X, Y) = g(X, \bar{Y})$.

1.4.2 Kähler geometry

Definition 1.9. For each $I \in \mathcal{I}(M, \omega)$, the triple (M, ω, I) is called an *almost Kähler manifold.* If I is integrable (M, ω, I) is called a *Kähler manifold.* The metric $g = \omega \cdot I$ is called the Kähler metric, and ω the Kähler form.

In the following we need the notion of the Levi-Cevitta connection. It is the unique torsion free connection in the tangent bundle, which is compatible with the metric, i.e $\nabla g = 0$ or $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. Torsion-free meaning $\nabla_X Y - \nabla_Y X - [X, Y] = 0$. In the subsequent chapters we will often work with more than one connection at a time. In many cases we will not use indicies to destinguish between the different connections, since they are distinguished by the sections they act on.

In the case of a Kähler manifold the almost complex structure I is parallel with respect to ∇, that is $\nabla I = 0$ or $\nabla_X(IY) = I\nabla_X Y$ for all vector fields X, Y on M.

Since also g is parallel with respect to ∇, ω is it as well. The parallelism of I with respect to ∇ implies that ∇ respects the splitting of the tangent bundle.

A theorem which we will use several times in the following chapter is Kodaira's embedding theorem. Among other places it is used is in Chapter 5 to find necessary and sufficient conditions on the complex torus to be an abelian variety.

Theorem 1.10 (Kodaira's embedding theorem)**.** *A compact complex manifold has an embedding into projective space, if and only if it has a Hodge form, i.e. a closed positive* $(1,1)$*-form whose cohomology class is rational.*

A $(1,1)$-form $\omega = \frac{i}{2}\sum_{ij} h_{ij}(z) dz_i \wedge d\bar{z}_j$ is positive if $H(z) = (h_{ij}(z))$ is positive definite for every z.

Especially is every Kähler manifold with Kähler form a rational cohomology class embeddable in projective space.

1.4.3 Curvature of Levi-Cevitta connection

The Riemann curvature R on a Riemannian manifold M is a $(3,1)$-tensor defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In the case of a Kähler manifold Riemann curvature commutes with the integrable almost complex structure I, since I is parallel with respect to ∇, i.e.

$$R(X,Y)(IZ) = I(R(X,Y)Z) \quad \text{for all vector fields } X, Y, Z \text{ on } M.$$

This implies that R can be seen as 2-form of type $(1,1)$ with values in not only $\mathrm{End}(TM)$, but since R commutes with I it preserves the splitting og TM, so actually R takes its values in $\mathrm{End}(T') \oplus \mathrm{End}(T'')$.

By lowering an index by the metric we get a $(4,0)$-tensor known as the Riemann curvature tensor Rm

$$Rm(X,Y,Z,W) = g(R(X,Y)Z, W).$$

We could also raise an index and obtain a $(2,2)$-tensor, which we will call the curvature operator, R, which is a section of the endomorphism bundle of $(1,1)$-forms on M.

Definition 1.11. The *Ricci tensor* r is defined as the trace of the Riemann curvature,

$$r(X,Y) = \mathrm{Tr}(Z \mapsto R(X,Z)Y),$$

and the *Ricci form* as the skew-symmetric closed real $(1,1)$-form ρ, by

$$\rho(X,Y) = r(IX,Y).$$

In the Kähler case, a calculation shows that the Ricci form is minus the Riemann curvature operator on the Kähler form, $\rho = -R(\omega)$.

1.4.4 Divergence of a vector field

Given a vector field X on a Kähler manifold, we define a smooth function δX on M by $\mathcal{L}_X \omega^n = \delta X \omega^n$. By a calculation in local coordinates it can be shown that δX can be expressed only in terms of the Levi-Cevitta connection by

$$\delta X = \mathrm{Tr}(\nabla X).$$

We can extend the definition of divergence to tensors of arbitrary order. On vector fields we define it by

$$\delta(X_1 \otimes \cdots \otimes X_n) = \delta(X_1)X_2 \otimes \cdots \otimes X_n + \sum_j X_2 \otimes \cdots \otimes \nabla_{X_1} X_j \otimes \cdots \otimes X_n.$$

Later we shall especially need $\delta(X_1 \otimes X_2) = \delta(X_1)X_2 + \nabla_{X_1} X_2$.

A local formula for the divergence of a vector field can easily be derived from $\delta X = \mathrm{Tr}(\nabla X)$ to be

$$\delta(X) = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i}\left(X^i \sqrt{\det g}\right), \tag{1.3}$$

where $X = \sum_i X^i \frac{\partial}{\partial x_i}$, and $\{x_i\}$ are local coordinates on M.

1.5 Hodge theory

In this section we assume M is a *compact* Kähler manifold of complex dimension n.

Definition 1.12. For every $0 \le p \le n$ we define the *Hodge operator* $*$ as the unique vector bundle isomorphism $* : \wedge^p TM^* \to \wedge^{n-p} TM^*$ satisfying

$$\alpha \wedge (*\beta) = g(\alpha, \beta)\frac{\omega^n}{n!},$$

where $g(\alpha, \beta)$ is the pointwise inner product on forms induced by the Kähler metric.

Now define by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\bar{\beta},$$

for α, β p-forms on M, and let it be zero if they are forms of different degree. This is a positive definite symmetric sesquilinear form on $\Omega^*(M) = \bigoplus \Omega^p(M)$. With the inner product at hand we can make adjoints of operators acting on Ω^*, e.g. d^* which in the case of a compact oriented Riemannian manifold is given by $d^* = - * d *$ acting on $\Omega^p(M)$. With the adjoints we define three different Laplace operators

$$\Delta = dd^* + d^*d \quad \Box = \partial\partial^* + \partial^*\partial \quad \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

and a map $L : \Omega^{(p,q)}(M) \to \Omega^{(p+1,q+1)}(M)$ by $L(\alpha) = \omega \wedge \alpha$.

Theorem 1.13. *On a compact Kähler manifold, the Hodge Laplacian Δ commutes with $*$, d and L, and*

$$\Delta = 2\Box = 2\bar{\Box}.$$

We furthermore have the Kähler identities

$$[L, d] = 0 \quad [L^*, d^*] = 0.$$

Lemma 1.14. *The Hodge Laplacian of a function f, $\Delta f = - * d * df$ is the negative of metric Laplacian, so*

$$\Delta f = -\frac{1}{\sqrt{\det g}} \sum_{ij} \frac{\partial}{\partial x_i}\left(g^{ij} \sqrt{\det g}\frac{\partial f}{\partial x_j}\right),$$

where $\{x_i\}$ is local coordinates on M.

Proof. Let $\{x_i\}$ be local coordinates on M around a point p. Since f is a 0-form, $d^*f = 0$. A calculation in local coordinates using Cartan's magical formula yields $\delta X = -d^*(X \cdot g)$. Now

$$\Delta f = dd^*f + d^*df = d^*df = -d^*(X_f \cdot \omega) = -d^*(IX_f \cdot g) = \delta(IX_f). \tag{1.4}$$

Now $-IX_f$ satisfies $-IX_f \cdot g = df$ and by using this we can calculate $-IX_f$ by evaluating g on $-IX_f$ and $\frac{\partial}{\partial x_i}$ to get

$$-IX_f = \sum_{ij} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

If we now insert this in formula 1.3 we get the desired result. □

We get the as a corollary to the proof of Lemma 1.14 that

$$-IX_f = \operatorname{grad} f \qquad \text{and} \qquad \Delta f = -\delta(\operatorname{grad} f),$$

where $\operatorname{grad} f$ is the unique vector field associated to a function satisfying $(\operatorname{grad} f) \cdot g = df$.

1.5.1 Hodge decomposition

In this subsection we state the Hodge decomposition of de Rham cohomology on a compact Kähler manifold. But before doing that we need the notion of a harmonic form.

Definition 1.15. A form φ is *harmonic* if $\Delta\varphi = 0$. The vector space of harmonic r-forms on M is denoted by $\mathcal{H}^r(M)$.

By elliptic operator theory we have the following

Theorem 1.16. *Let M be a compact Riemannian manifold, then the de Rham cohomology group $H^r(M, \mathbb{C})$ is isomorphic to the vector space of harmonic r-forms on M, $H^r(M, \mathbb{C}) \simeq \mathcal{H}^r(M)$.*

This means that for every cohomology class in $H^r(M, \mathbb{C})$ exists a unique harmonic form representing the class, this form is even d-closed.

The Hodge decomposition is a decomposition of de Rham cohomology into types of Dolbeault cohomology groups.

Theorem 1.17. *Let M be a compact Kähler manifold. Then there is a direct sum decomposition of the de Rham cohomology group*

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M),$$

and each cohomology class has unique harmonic representative.

This theorem follows from the fact that there is a corresponding decomposition of harmonic forms.

Using Hodge theory it is possible to prove, that for any real $(1,1)$ d-exact form α exists a smooth real valued function f such that $\alpha = 2i\partial\bar{\partial}f$. Recall that the Ricci form ρ, is a closed real $(1,1)$-form, which by Hodge decomposition is equal to its harmonic part ρ^H plus a d-exact real $(1,1)$-form. Hence there exists a function $F \in C^\infty(M)$ such that

$$\rho = \rho^H + 2i\partial\bar{\partial}F.$$

F is called the *Ricci potential*, and is determined up to addition of a constant. If we demand $\int_M F\omega^n = 0$ we determine F uniquely.

2

Quantization

In this chapter we will discuss various aspects of quantization. The main part of this chapter is a discussion of two of the most mainstream quantizations, deformation quantization and geometric quantization. In the end we describe Berezin–Toeplitz deformation quantization. But before dwelving into the technical details of quantization we will make a general discussion about quantization. What have we learnt from physics, and what is our goal? In this context we will highlight features and flaws of deformation quantization and geometric quantization.

2.1 Quantization

Quantization is a process of turning a classical system into a quantum system, in such a way, that when we take an appropriate macroscopic limit of the quantum system ($\hbar \to 0$), we recover the original classical system. The problem however, is that not all quantum systems has classical counterparts. Furthermore could many different quantum systems reduce to the same classical theory, so which quantization should we choose? Thus there are a lot of problems contained in quantization. Why even bother to quantize when all these problems arise? The answer is provided by quantum mechanics, since it has proven to be a very accurate description of nature, so if we imitate the construction of Weyl, von Neumann and Dirac (e.g. [13]) we are bound to get something interesting.

We will describe axioms for quantizing $\mathbb{R}^{2n}$, and give a small view into the inconsistencies they contain. For a more general story see [17].

Let $A_{2n} = \mathbb{R}[q_1, \ldots, q_1, p_1, \ldots, p_n]$ be the Poisson algebra of polynomial functions on $\mathbb{R}^{2n}$, the Poisson bracket being the standard one

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i},$$

from $\omega_{std} = \sum_i dq_i \wedge dp_i$ and $\{f, g\} = -\omega(X_f, X_g)_{std}$.

This algebra contains the Heisenberg algebra $\mathcal{H}_{2n} = \mathbb{R}\{1, q_i, p_i \,|\, i = 1, \ldots, n\}$ of degree one polynomials on $\mathbb{R}^{2n}$. $\mathcal{H}_{2n}$ has the Heisenberg group H_{2n} of upper triangular matrices as its corresponding unique connected and simply connected Lie group.

Definition 2.1. Let $\mathfrak{g} \subset A_{2n}$ be a Lie algebra containing $\mathcal{H}_{2n}$. A *quantization* of $\mathfrak{g}$ is a linear map

$$Q : \mathfrak{g} \to Op(D),$$

where $Op(D)$ is a linear space of Hermitian operators acting on D. $\mathfrak{g}$ is called the set of *observables* and D the *quantum states*. D is a dense subset of a separable Hilbert space $\mathcal{H}$. Q should satisfy the following axioms (q1) $[Q(f), Q(g)] = -i\hbar Q(\{f, g\})$ for all $f, g \in \mathfrak{g}$.

(q2) $Q(1) = \mathbb{1}$, 1 is the constant function and $\mathbb{1}$ the identity operator.

(q3) If the Hamiltonian vector field $X_f = -\{f, \cdot\}$ is complete on $\mathbb{R}^{2n}$, $Q(f)$ has an extension to a self adjoint operator.

(q4) $Q|_{\mathcal{H}_{2n}}$ is irreducible.

(q5) $Q|_{\mathcal{H}_{2n}}$ is unitarily equivalent to the Schödinger representation on $L^2(\mathbb{R}^n)$.

Remark 2.2. One could also state (q5), as a requirement for $Q|_{\mathcal{H}_{2n}}$ to integrate to a unitary representation of H_{2n}. Since (q4) imposes an irreducibility condition on this representation, the Stone–von Neumann theorem forces $Q|_{\mathcal{H}_{2n}}$ to be unitary equivalent to the Schrödinger representation of $L^2(\mathbb{R}^n)$, $q_i \mapsto Q_i$, $p_i \mapsto P_i$, $1 \mapsto \mathbb{1}$. The operators $Q_i\psi = q_i\psi$, $P_i\psi = -i\hbar\frac{\partial\psi}{\partial q_i}$ are the usual position and momentum operators

In a quantization $\mathfrak{g}$, the domain of the space of quantizable observables, should be as large as possible. Ideally it should contain all the smooth functions on $\mathbb{R}^n$. $\hbar$ is the reduced Planck constant in physics, but we will treat it as a formal parameter.

2.1.1 Inconsistencies

Unfortunately the quantization axioms contain inconsistencies, which appear as algebraic contradictions, if you quantize certain simple polynomial functions. This is the idea of Groenewold and van Hove's celebrated no-go theorem.

Theorem 2.3. *There is no quantization for* $\mathbb{R}[q_1, \ldots, q_n, p_1, \ldots, p_n]$.

In $n = 1$ this theorem is proved by quantizing the classical equality

$$\frac{1}{9}\{q^3, p^3\} = \frac{1}{3}\{q^2p, p^2q\},$$

by finding that the left hand side give

$$Q(q)^2Q(p)^2 - 2i\hbar Q(q)Q(p) - \frac{2}{3}\hbar^2$$

and the right hand side is

$$Q(q)^2Q(p)^2 - 2i\hbar Q(q)Q(p) - \frac{1}{3}\hbar^2,$$

where Q is the supposed quantization. The theorem actually work without imposing axiom (q5). This theorem can to some extend also be generalized to the case of a symplectic manifold, we will not go into this, but for a reference see [17]. It should be noted that in terms of inconsistencies this is just the tip of the iceberg, for a more detailed account see [1].

An essential ingredient in making the above theorem work, is axiom (q1). This axiom is physically very important since it leads to the Heisenberg uncertainty principle. Physicists are therefore very keen on keeping these conditions. The other obvious ingredient to showing Theorem 2.3 is that the algebra we try to quantize contain polynomials of degree 3 or more together with monomials of mixed degree, e.g. p^2q and q^2p.

There are several different proposals to avoid the no-go theorem, two are rather obvious

(i) Keep all quantization axioms but quantize only few observables.

(ii) Replace axiom (q1) with something weaker.

Solution (i) will lead us towards one version of geometric quantization. In this solution we cannot simultaneously quantize p_j^2, q_j^2 and $p_j^2 q_j^2$ for any j. However taking the space of quantizable observables to be the set of all functions at most linear in p, i.e. $f(p,q) = f_0(q) + \sum_j f_j(q)p_j$, $f_0, f_j \in C^\infty(\mathbb{R}^n)$ and setting $Q(f) = f_0(Q(q)) + \frac{1}{2}\sum_j (f_j(Q(q))Q(p_j) + Q(p_j)f_j(Q(q)))$ we can show that all axioms are satisfied. We could have done a similar thing with functions linar in q.

In solution (ii) we keep (q2) – (q5) but require (q1) to hold only asymptotically as $\hbar$ goes to zero, i.e.

$$[Q(f), Q(g)] = -i\hbar Q(\{f,g\}) + O(\hbar^2) \quad \text{for } \hbar \to 0,$$

and we follow this line of thought in deformation quantization and a version of geometric quantization, which we call *Kähler quantization*.

Now we have two different approaches to handle the quantization problem for $\mathbb{R}^{2n}$. The next step is to generalize this to an arbitrary symplectic manifold M.

2.2 Geometric quantization

In this section we will investigate geometric quantization as Kostant [22] and Souriau [28] defined it. Let us review the axioms for quantization in the general setting of a symplectic manifold (M, ω).

Definition 2.4. A *quantization* of a symplectic manifold (M,ω) is an assignment of a separable Hilbert space $\mathcal{H}$, and a linear map $Q : f \mapsto Q(f)$ from a Lie subalgebra $\mathfrak{g} \subset C^\infty(M)$ under the Poisson bracket, to self-adjoint operators on a dense subset $D \subset \mathcal{H}$ satisfying the following axioms

(g1) $[Q(f), Q(g)] = -i\hbar Q(\{f,g\})$, for all $f, g \in \mathfrak{g}$.

(g2) $Q(1) = \mathbb{1}$ where 1 is the constant function and $\mathbb{1}$ is the identity operator on $\mathcal{H}$.

(g3) The assignment should be functorial in the sense that if $\varphi : (M,\omega) \to (\tilde{M}, \tilde{\omega})$ is a symplectomorphism, then for $\tilde{f} \in \tilde{\mathfrak{g}}$ we require $Q(\tilde{f} \circ \varphi)$ and $\tilde{Q}(\tilde{f})$ to be conjugate by a unitary operator from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ (or on a dense subset of these).

(g4) For $(\mathbb{R}^{2n}, \omega_{std})$ we recover the Schrödinger representation.

It should be noted that axiom (g3) is a substitute for the irreducibility criterion, which can be a bit hard to interpret on a general symplectic manifold. We will describe the construction by Kostant and Souriau, by first creating a prequantization by ignoring (g4), and afterwards including a polarization to get something that satisfies (g4).

2.2.1 Prequantization

Let $\mathcal{L} \to M$ be a complex Hermitian line bundle over M, with ∇ the canonical connection induced by the Hermitian metric. Locally over an open subset $U \subset M$ let θ be the connection matrix and s_U a non-vanishing section in $\mathcal{L}$. Then $\nabla_X(fs_U) = X(f)s_U + \theta(X)fs_U$. The prequantization operator $Q : C^\infty(M) \to Op(D)$, is given by

$$f \mapsto f - i\hbar\nabla_{X_f}. \tag{2.1}$$

Here $D = L_0^2(M, \mathcal{L}) \subset L^2(M, \mathcal{L})$ is the subset of smooth square integrable sections of $\mathcal{L}$ with compact support. The metric on $L^2(M, \mathcal{L})$ is given by

$$\langle s_1, s_2 \rangle = \int s_1 \bar{s}_2 \frac{\omega^n}{n!}.$$

A calculation, using that the integral of the Lie derivative of a top form over a manifold without boundary is 0, shows that for f real, $Q(f)$ is self-adjoint. The assignment is furthermore clearly linear.

With this assignment we need to see the axioms fulfilled. We will only focus on (g1) as this is the most important one.

$$\begin{aligned}[Q(f), Q(g)] &= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}] - 2i\hbar\{f, g\} \\ &= -2i\hbar\{f, g\} - \hbar^2 \Theta_{\mathcal{L}}(X_f, X_g) - \nabla_{X_{\{f,g\}}}\end{aligned}$$

If $i\hbar\Theta_{\mathcal{L}}(X_f, X_g) = \omega(X_f, X_g)$ (g1) is satisfied. In other words, (g1) is satisfied if

$$\left[\frac{i}{2\pi}\Theta_{\mathcal{L}}\right] = c_1(\mathcal{L}) = \left[\frac{1}{\hbar}\omega\right] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}).$$

This is called the *prequantum condition.* Remark that for $\frac{1}{\hbar}\omega$ to be an integer form $\frac{1}{\hbar}$ has to be an integer. This is really a quantization of the Planck constant as a physicist would understand, that is interpret quantization as a procedure of constraining something with continuous values to a set of discrete values. By looking at the prequantum line bundle cohomologically, it can be shown that the prequantum condition is not only sufficient for prequantization to work, it is also necessary.

Definition 2.5. A *prequantum line bundle* on a symplectic manifold (M, ω) is a Hermitian line bundle $(\mathcal{L}, \langle\cdot,\cdot\rangle)$ on M, with at compatible connection, ∇, satisfying the prequantum condition

$$\Theta_{\mathcal{L}} = \frac{-i}{\hbar}\omega$$

The cohomological investigation also reveals that, if a prequantum line bundle exists, the inequivalent choices of prequantum line bundles are parametrized by $H^1(M, U(1))$, see [31].

However, with prequantization (g4) is by no means fulfilled. We regard $\mathbb{R}^{2n}$ with $\omega_{std} = -d\theta$ where $\theta = \sum_j p_j dq_j$ and Hamiltonian vector field, X_f, for a function f

$$X_f = \sum_{j=1}^n \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j},$$

and since $\nabla_{X_f} = X_f - \frac{i}{\hbar}X_f \cdot \theta$

$$Q(f) = f + \sum_{j=1}^n p_j \frac{\partial f}{\partial p_j} - i\hbar\Big(\sum_{j=1}^n \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j}\Big),$$

which acts on $L^2(\mathbb{R}^{2n})$ and not on $L^2(\mathbb{R}^n)$. We therefore need to restrict Q to the space of functions only depending on the q-variables and is quadratic integrable over this variable as well – in all other ways q_i, p_j are not quantized as the Schrödinger representation, as we wanted.

2.2.2 Polarization

To interpret the restriction to half the variables on a general symplectic manifold, we need the notion of a polarization. There are both real and complex polarizations. We will only focus on complex polarizations.

Definition 2.6. Let (M,ω) be a symplectic manifold. A complex polarization $\mathcal{P}$ is a distribution of $TM_{\mathbb{C}}$ satisfying

- $\mathcal{P}$ is Lagrangian, i.e. $\mathcal{P} = \{X \in TM_{\mathbb{C}} \,|\, \omega(X,Y) = 0 \text{ for all } Y \in \mathcal{P}\}$.
- $\mathcal{P}$ is involutive, i.e. $[X,Y] \in \mathcal{P}$ for all $X,Y \in \mathcal{P}$.
- $\dim(\mathcal{P}_x \cap \bar{\mathcal{P}}_x \cap T_xM)$ is constant for all $x \in M$.

In the following we will focus on a particular type of polarization, the *Kähler polarization*.

Definition 2.7. Let $\mathcal{P}$ be a complex polarization on a symplectic manifold (M,ω). $\mathcal{P}$ is a *Kähler polarization* if the Hermitian form on $\mathcal{P}$ defined by $h(u,v) = i\omega(u,\bar{v})$ is positive definite.

With a Kähler polarization we can define a complex structure I on M by letting $\mathcal{P}$ be the $-i$-eigenspace of I, and $\bar{\mathcal{P}}$ the i-eigenspace of I. Involutivity of $\mathcal{P}$ give integrability of I, and by the Newlander–Nirenberg theorem, there exists a unique complex structure on M, which induce I. The metric g defined by $g(X,Y) = \omega(X,IY)$ for X,Y vector fields on M is a positive definite metric. It is furthermore Hermitian, and since ω is closed (M,ω,I) is a Kähler manifold. Conversely will every Kähler manifold (M,ω,I) have a Kähler polarization by choosing $\mathcal{P} = T''$.

With a Kähler polarization on M, the line bundle $\mathcal{L} \to M$ has a natural complex structure. A section s of $\mathcal{L}$ will be called holomorphic if $\nabla_X s = 0$ for all $X \in \mathcal{P}$. Two non-vanishing sections s, s' of $\mathcal{L}$ differ by a non-vanishing function φ, $s = \varphi s'$. Then if s, s' are holomorphic then

$$0 = \nabla_X s = \nabla_X(\varphi s') = (\nabla_X s')\varphi + X(\varphi) = X(\varphi)$$

φ is holomorphic. By choosing a trivialization of $\mathcal{L} \to M$ of holomorphic sections, the transition functions are also holomorphic, hence the name.

The space

$$D = \{s \in L^2_0(M,\mathcal{L}) \,|\, \nabla_X s = 0 \,\text{for all } X \in \mathcal{P}\},$$

is a closed subspace of $L^2_0(M,\mathcal{L})$, and hence is a Hilbert space, see [31]. Operators on D will be the target space of the quantization map.

The question is now which observables we can quantize. By calculating the covariant derivative of $Q(f)s$ with respect to $X \in \mathcal{P}$, we find that $Q(f)$ preserves the space of holomorphic sections if $[X, X_f] \in \mathcal{P}$

$$\begin{aligned}\nabla_X(Q(f)s) &= -i\hbar\nabla_X\nabla_{X_f}s + X(f)s + f\nabla_X s\\ &= Q(f)(\nabla_X s) + i\hbar\nabla_{[X_f,X]}s + (X_f\cdot\omega)(X)s + X(f)s\\ &= Q(f)(\nabla_X s) - i\hbar\nabla_{[X,X_f]}s\end{aligned}$$

So for $X \in \mathcal{P}$, $Q(f)$ preserves D if $[X, X_f] \in \mathcal{P}$.

Then the space of quantizable observables is

$$\mathfrak{g} = \{f \in C^\infty(M) \,|\, [X, X_f] \in \mathcal{P} \;\text{ for all } X \in \mathcal{P}\}.$$

This is an algebra with respect to the Poisson bracket. The problem is that this space is generally a very small space, and actually often 0-dimensional, see [31] for further details.

2.2.3 Weaken (g1)

In stead of limiting the space of observables, we can take another approach to make $Q(f)$ preserve D, by using that D is closed in $L_0^2(M,\mathcal{L})$. The cost is that (g1) can only hold asymptotically. For the rest of this section assume M is compact. When M is compact all smooth sections are square integrable and we will take $C^\infty(M,\mathcal{L})$ or sections of powers of $\mathcal{L}$, to mean the L^2-completion of this space. If M is a compact Kähler manifold and $\mathcal{L}\to M$ a prequantum line bundle, then Kodaira's embedding theorem make $\mathcal{L}$ ample. For now we will assume the prequantum line bundles to be very ample. We let $(\mathcal{L}^k, h^{(k)}, \nabla^{(k)})$ be the k'th tensor power of $\mathcal{L}$. If h is the metric for $\mathcal{L}$ with respect to the frame e, then $h^{(k)}$ is the metric for $\mathcal{L}^k$ with respect to the frame e^k, and $\nabla^{(k)}$ is the induced connection.

The space of global sections has a metric given by

$$\langle s_1, s_2\rangle = \frac{1}{n!}\int_M h^{(k)}(s_1,s_2)\omega^n.$$

Let $H^{(k)} = H^0(M,\mathcal{L}^k)$ be the, by compactness and elliptic operator theory, finite dimensional closed subspace of the L^2-completion of smooth global sections of $\mathcal{L}^k$, consisting of global holomorphic sections.

Since $H^{(k)}\subset D^{(k)} = C^\infty(M,\mathcal{L}^k)$ is closed there exists a projection $\pi^{(k)} : D^{(k)}\to H^{(k)}$. When we in the above construction needed $Q(f)$ to preserve the holomorphic sections, we had to limit the domain of Q to the space of observables, which preserved holomorphic sections. With the projection we will just compose the original quantum operator with the projection. To stick to the notation with $Q(f)$ being the quantum operator associated to $f\in\mathfrak{g}$ we define

$$P^{(k)}(f) = f - \frac{i}{k}\nabla^{(k)}_{X_f}.$$

We should also note, that $\frac{1}{k}$ from now plays the role of $\hbar$, and k is called the *level*. This amounts to require that $\hbar = 1$ and the prequantum condition for (M,ω) is $c_1(\mathcal{L}) = [\frac{\omega}{2\pi}]$, because then the prequantum condition for $(M,k\omega)$ is $c_1(\mathcal{L}) = [\frac{k\omega}{2\pi}]$. Now we explicitly replaced $\hbar$ with something discrete. For every $f\in C^\infty(M)$ define the prequantum operator

$$Q^{(k)}(f) = \pi^{(k)}\circ P^{(k)}(f).$$

These operators do not form an algebra but an asymptotic version of (g1) is satisfied

$$\|[Q^{(k)}(f), Q^{(k)}(g)] + \frac{i}{k}Q^{(k)}(\{f,g\})\| = O(\frac{1}{k^2}) \quad \text{for } k\to\infty.$$

The proof of the above equality and every other requirement on Q to be a quantization relies on the following theorem by Tuynman [29] since the contribution from $\frac{1}{2k}T^{(k)}_{\Delta f}$ vanish in the large k-limit by Theorem 2.11 of Bordemann, Meinrenken and Schlichenmaier.

Theorem 2.8 (Tuynman). *Let M be a prequantizable Kähler manifold. Then $Q^{(k)}(f)$ is a Toeplitz-operator*

$$Q^{(k)}(f) = T^{(k)}_{f+\frac{1}{2k}\Delta f},$$

where $\Delta = d^*d$ is the Hodge Laplacian associated to the Kähler form ω.

We shall have much more to say about Toeplitz operators and their properties, such as asymptotic expansion in the Section 2.4 about Berezin-Toeplitz deformation quantization. We call the above quantization procedure *Kähler Quantization.*

Remark that $H^{(k)}$ is a finite dimensional vector space, if not $\mathcal{L}$ was very ample we could have that for some of the lower values of k $H^{(k)}$ was zero dimensional. In that case we loose all information about the classical system we want to quantize. Since the prequantum bundle is ample the dimension of the space of holomorphic sections of $\mathcal{L}^k$ grows. The philosophy is that with the increase in power of $\mathcal{L}$ we recover more and more information about the classical system, and in the end we should be able to recover all information by considering all the tensor powers together. A result relating to this is Theorem 2.11 by Bordemann, Meinrenken and Schlichenmaier.

2.2.4 Shortcomings

Even though the above quantization procedures give operators depending on the right number of variables, there are however still problems. Quantizing a simple system as the 1-dimensional harmonic oscillator produce operators with a shifted spectrum compared with the right quantization provided by quantum mechanics. To shift the spectrum the prequantum line bundle $\mathcal{L}$ should be twisted with a square root of the canonical bundle of M, and then use the exact same quantization procedure as described above. This is called $\frac{1}{2}$-form quantization or metaplectic correction. We will however not go into further details with this.

A very important remark to geometric quantization is that it depend on the choice of complex structure on M, or equivalently on the choice of Kähler polarization. The classical system is just a set of observables on the phase space (M, ω), which does not have a complex structure, so the quantum system should not depend on a complex structure. This is why the quantum spaces in the following chapters will be denoted $H_I^{(k)}$. How to avoid the dependence of I will adressed in the Section 2.5 about the Hitchin connection.

2.3 Deformation quantization

Let M be a symplectic manifold. As noted above is deformation quantization a method of avoiding the contradictions of the no-go theorem for $\mathbb{R}^{2n}$. As in Kähler quantization we relax the canonical commutation relation (q1) to only be true asymptotically,

$$[Q(f), Q(g)] = -i\hbar Q(\{f, g\}) + O(\hbar^2). \tag{2.2}$$

One can try to obtain this by deforming the algebra structure on $\mathfrak{g} \subset C^\infty(M)$, $\mathfrak{g}$ is the algebra of observables, to get a formal associative non-commutative product $\star$ on formal power series $\mathfrak{g}[[\hbar]]$. This product should satisfy

$$f \star g = \sum_{j=0}^{\infty} \hbar^j C_j(f, g),$$

where $C_j : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ are bilinear operators satisfying

(dq 1) $C_0(f, g) = fg$

(dq 2) $C_1(f, g) - C_1(g, f) = -i\{f, g\}$ where $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(M)$

(dq 3) $C_j(1, f) = C_j(f, 1) = 0$ for $j \geq 1$.

Remark that (dq 2) is a consequence of the relaxed commutation relation (2.2).

If $\operatorname{supp} C_j(f, g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g$ for all $f, g \in C^\infty(M)$ and C_j are bidifferential operators the star product is called *local*. For every open set U in M, a local star product defines a star product on $C^\infty(U)$.

Example 2.9. An example of a local deformation quantization of $C^\infty(\mathbb{R}^{2n})$ with the Poisson bracket given by the symplectic structure ω_{std} is the *Moyal–Weyl product*

$$f * g = \mu \circ e^{\frac{i\hbar}{2}(P-P^*)}(f \otimes g)$$

where the operators

$$P = \sum_i \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial y_i} \quad \text{and} \quad P^* = \sum_i \frac{\partial}{\partial y_i} \otimes \frac{\partial}{\partial x_i}$$

and $\mu(f \otimes g) = fg$.

If (M, ω) is Kähler manifold then a local star product has *separation of variables* if

$$f \star h = fh \quad \text{and} \quad h \star g = hg,$$

for every locally defined holomorphic function g and anti-holomorphic function f and arbitrary function h.

Two star products $\star$ and $\star'$ on $C^\infty(M)$ are *equivalent*, if there exists a formal series $E = Id + \sum_{r=1}^\infty \lambda^r E_r$ with $\lambda \in \mathbb{C}$ and linear maps $E_r : C^\infty(M) \to C^\infty(M)$ called an equivalence transformation. These E_r's should satisfy

$$f \star' g = E^{-1}(Ef \star Eg) \quad \text{and} \quad E1 = 1,$$

for all $f, g \in C^\infty(M)$. For local star products the linear maps E_r should be differential operators. It should be remarked that separation of variables is not necessarily preserved under an equivalence transformation.

The power series in the definition of a star product is not required to exist for any value of $\hbar$, but the coefficients should be well-defined mappings. The study of such formal deformations is called *formal deformation quantization* and is a big industry in its own right. Instead of going into too much detail about this we turn our attention to making this deformed product fit together with quantization. The objective is for $\star$ to be an honest bilinear mapping $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and seek a linear assignment of each $f \in \mathfrak{g}$ an operator $Q(f)$ on a separable Hilbert space, such that the quantum observables satisfy

$$Q(f)Q(g) \sim Q(f \star g), \tag{2.3}$$

where $\sim$ is some sort of equality. In our case we will take it to be asymptotically a like, as defined below. The association of this Hilbert space is rather technical, and it requires $f \star g$ to actually exist for some small value of $\hbar$. This is not easy to verify for the formal star products, and usually the construction goes the other way around. You start with a geometric construction of the quantum operators, and checks that $\star$ defined by (2.3) is a (formal) star product.

To do this explicitly we look for an assignment $f \mapsto Q(f)$ depending on $\hbar$, of operators $Q(f)$ on a separable Hilbert space, to $f \in \mathfrak{g}$. The Q's should have an asymptotic expansion

$$Q(f)Q(g) \sim \sum_{j=0}^\infty \hbar^j Q(C_j(f,g))$$

for certain bilinear operators C_j, where $\sim$ is interpreted in sense of norms

$$\|Q(f)Q(g) - \sum_{j=0}^N \hbar^j Q(C_j(f,g))\| = O(\hbar^{N+1}) \quad \text{for } \hbar \to 0,$$

for all $N = 0, 1, 2 \cdots$. If M is a Kähler manifold this problem is solved by Berezin–Toeplitz quantization, which we will look at in the next section.

2.4 Berezin–Toeplitz Deformation Quantization

In this section we will discuss a way to bridge the gap between geometric quantization and deformation quantization, by using the space of quantum states $H^{(k)}$ from geometric quantization in Subsection 2.2.3, to create a deformation quantization. The setup is as in the Subsection 2.2.3, (M, ω, I) is a compact Kähler manifold with a prequantum line bundle $\mathcal{L} \to M$, and projection $\pi^{(k)} : D^{(k)} \to H^{(k)}$. $H^{(k)}$ and $\pi^{(k)}$ depend on the complex structure I it should be denoted $H_I^{(k)}$ and $\pi_I^{(k)}$.

Definition 2.10. For $f \in C^\infty(M)$ the Toeplitz operator at level k is defined by

$$T_{f,I}^{(k)} = \pi_I^{(k)} \circ M_f : H_I^{(k)} \to H_I^{(k)},$$

where M_f is multiplication by f.

As we shall discuss later the vector spaces $H_I^{(k)}$ depend on the complex structure and will form a vector bundle $H^{(k)}$ over the space of complex structures. We will in the rest of this section assume this.

Let $\{s_1, \cdots, s_N\}$ be a frame for $H^{(k)}$. Let $h_{ij}^I = (s_i, s_j)_I$, then the projection is defined as

$$\pi_I^{(k)}(s) = \sum_{ij} (s, s_i)_I (h_{ij}^I)^{-1} s_j.$$

In the case where the frame is orthogonal the sum reduce to

$$\pi_I^{(k)}(s) = \sum_j (s, s_j)_I s_j.$$

The Toeplitz operator $T_{f,I}^{(k)}$ associated to a function $f \in C^\infty(M)$, is the composition of multiplying a holomorphic section by f and projecting back onto $H_I^{(k)}$, that is

$$T_{f,I}^{(k)}(s) = \pi_I^{(k)}(fs) = \sum_j (fs, s_i)_I s_i|_I. \tag{2.4}$$

The linear map $T^{(k)} : C^\infty(M) \to \mathrm{End}(H^{(k)})$ given by $f \mapsto T_f^{(k)}$ that associates to a function a section of the endomorphism bundle of $H^{(k)}$ over the space of complex structures, is neither a Lie algebra homomorphism nor an associative algebra homomorphism since generally $T_f^{(k)} T_g^{(k)} = \pi^{(k)} M_f \pi^{(k)} M_g$ is not equal to $T_{fg}^{(k)} = \pi^{(k)} M_{fg}$. However we have asymptotic results relating $T_f^{(k)} T_g^{(k)}$ and $T_{fg}^{(k)}$.

Define for $f \in C^\infty(M)$ the supremums norm of f by

$$|f|_\infty = \sup_{x \in M} |f(x)|,$$

and the operator norm by

$$\|T_f^{(k)}\| = \sup_{\substack{s \in H^{(k)} \\ \|s\| = 1}} \|T_f^{(k)} s\|.$$

The following theorem was proved by Bordemann, Meinrenken and Schlichenmaier, [12].

Theorem 2.11 (Bordemann, Meinrenken, Schlichenmaier)**.**

(i) The collection of all Toeplitz operators associated to a smooth function f represents f faithfully, i.e. $\lim_{k \to \infty} \|T_f^{(k)}\| = |f|_\infty$.

(ii) For every $f, g \in C^\infty(M)$

$$\|[T_f^{(k)}, T_g^{(k)}] + \frac{i}{k} T_{\{f,g\}}^{(k)}\| = O(\frac{1}{k^2}) \quad \text{for } k \to \infty.$$

(iii) The map $C^\infty(M) \to \mathrm{End}(H^{(k)})$ *is surjective.*

The following simple calculation shows that $(T_f^{(k)})^* = T_{\bar{f}}^{(k)}$, and hence the Toeplitz operators are self-adjoint for f real. By Theorem 2.8 the geometric quantum operators $Q^{(k)}(f)$ are also self adjoint for f real. Let $s, s' \in H^{(k)}$, then

$$\left\langle s, T_f^{(k)} s' \right\rangle = \left\langle s, \pi^{(k)}(fs') \right\rangle = \langle s, fs' \rangle = \langle \bar{f} s, s' \rangle = \left\langle T_{\bar{f}}^{(k)} s, s' \right\rangle.$$

In [27] Schlichenmaier showed the following asymptotic expansion of the operators $T_f^{(k)} T_g^{(k)}$.

Theorem 2.12 (Schlichenmaier)**.** *For any f, g smooth functions on M, there exists a sequence of uniquely determined functions $C_j(f,g) \in C^\infty(M)$ such that for all $N \in \mathbb{N}$*

$$\|T_f^{(k)} T_g^{(k)} - \sum_{j=0}^{N} \left(\frac{1}{k}\right)^j T_{C_j(f,g)}^{(k)}\| = O\left(\frac{1}{k^{N+1}}\right) \quad \text{for } k \to \infty.$$

Furthermore the $C_j(f,g)$'s are the coefficients of a unique formal star product $\star_{BT}$ on $C^\infty(M)$ called the Berezin–Toeplitz *star product,*

$$f \star_{BT} g = \sum_{j=0}^{\infty} \hbar^j C_j(f,g). \tag{2.5}$$

It was later shown in [21] by Karabegov and Schlichenmaier that the star product had separation of variables.

Tuynman's Theorem 2.8 give a link between quantization and Toeplitz operators, and is a key ingredient in proving that (g1) is satisfied in Kähler quantization. The theorem follow since a differential operator followed by a projection is a Toeplitz operator.

Proposition 2.13. *If $X \in C^\infty(M, T')$ is a smooth section of the $(1,0)$-part of the tangent bundle, then*

$$\pi^{(k)} \nabla_X s = -T_{\delta X}^{(k)} s.$$

Tuynman's theorem follow by Equation (1.4). Assume s is a holomorphic section of $\mathcal{L}^k$. Since Hamiltonian vector fields are divergence-free we have $\delta X_f' = -\frac{i}{2} \delta I X_f$.

$$\begin{aligned} Q^{(k)}(f)s = \pi^{(k)}(f - \frac{i}{k}\nabla_{X_f})s &= \pi^{(k)}(f - \frac{i}{k}\nabla_{X_f'}) = \pi^{(k)}(f + \frac{i}{k}(\delta X_f')) \\ &= \pi^{(k)}(f + \frac{1}{2k}\delta I X_f) = \pi^{(k)}(f + \frac{1}{2k}\Delta f). \end{aligned}$$

2.5 Hitchin connections

As we have seen above, quantizing a classical system of observables on a phase space, yields a Hilbert space of quantum states, $H_I^{(k)}$ or a formal deformation of the observables by Berezin–Toeplitz Deformation Quantization. In each of these constructions we had to choose a compatible complex structure making the phase space a Kähler manifold, and not only a symplectic manifold as it was to begin with. There is however no physical reason why we should choose one complex structure over another, so we need a way of relating the different quantum spaces.

Suppose the spaces $H_I^{(k)}$ together form a vector bundle over the space of Kähler structures. Then one could try to relate quantum spaces for different I's by parallel transport given by a connection in this bundle. If the connection is flat, the quantum spaces are essentially identical.

This is exactly the idea behind the Hitchin Connection, which first appeared in [20] and independently in [10]. Here they constructed such a bundle over the Teichmüller space of a Riemann surface Σ by quantizing the moduli space of flat $SU(n)$-connections on Σ. The connection constructed was however only projectively flat, meaning that it induced a flat connection in the projectivization of the bundle or equivalent that the curvature is a 2-form on $\mathcal{T}$ times the identity. In that case the quantum space was taken to the covariant constant sections of the projectivized bundle, or equivalently one of the fibers of the projectivized bundle. In Chapter 3 we will take a more general approach to the Hitchin connection.

As with the quantum spaces identified by parallel transport, we should somehow have a way of identifying Berezin–Toeplitz star products associated to two different Kähler manifolds with the same underlying symplectic manifolds. With the notions of a formal Hitchin connection Andersen has proved how to do exactly this when the Hitchin connection is projectively flat. We will in the report not go into the details of a formal Hitchin connection. In Chapter 6 we will however create a formal trivialization corresponding to a formal Hitchin connection induced by the heat equation. We will use this to construct a star product independent of the Kähler structure, in the case of a principal polarized abelian variety. For more on the general theory see [2]. Explicit formuli for the formal Hitchin connection can be found in [7], see also [6] for an explicit calculation of a formal Hitchin connection on a principal polarized abelian variety.

3

The Hitchin Connection

To make the results consistent with [7] we change the normalization of the prequantum bundle, such that we now require

$$\Theta_{\mathcal{L}} = -i\omega \quad \text{or equivalently} \quad c_1(\mathcal{L}) = \left[\frac{\omega}{2\pi}\right].$$

At the end of this chapter we will state the results in the normalization where $c_1(\mathcal{L}) = [\omega]$.

In this chapter we will describe Andersen's construction of a Hitchin connection in the bundle of quantum states over the manifold parametrizing Kähler structures on the phase space. Before dwelving into the details we discuss general facts about the family of Kähler structures.

3.1 Families of Kähler structures

From now on we will assume $\mathcal{T}$ is a smooth manifold. Later we impose extra structure.

Definition 3.1. A *family of Kähler structures* on a symplectic manifold (M, ω) parametrized by $\mathcal{T}$ is a map

$$I : \mathcal{T} \to C^\infty(M, \operatorname{End} TM),$$

that to each element $\sigma \in \mathcal{T}$ associates an integrable and compatible almost complex structure. I is said to be smooth if I defines a smooth section of $\pi_M^* \operatorname{End}(TM)$,

$$\begin{array}{ccc} \pi_M^* \operatorname{End}(TM) & & \operatorname{End}(TM) \\ \downarrow & & \downarrow \\ \mathcal{T} \times M & \xrightarrow{\pi_M} & M \end{array}$$

where π_M is the projection.

For each point $\sigma \in \mathcal{T}$ we define M_σ to be M with the Kähler structure defined by ω and $I_\sigma := I(\sigma)$. The Kähler metric is denoted by g_σ.

Every I_σ is an almost complex structure and hence induce a splitting of the complexified tangent bundle $TM_{\mathbb{C}}$, denoted by $TM_{\mathbb{C}} = T'_\sigma \oplus T''_\sigma$, and the projection to each factor is given by

$$\pi_\sigma^{1,0} = \frac{1}{2}(Id - iI_\sigma) \quad \text{and} \quad \pi_\sigma^{0,1} = \frac{1}{2}(Id + iI_\sigma).$$

If $I_\sigma^2 = -Id$ is differentiated along a vector field V on $\mathcal{T}$, we get

$$V[I]_\sigma I_\sigma + I_\sigma V[I]_\sigma = 0,$$

and hence $V[I]_\sigma$ changes types on M_σ. Then for each σ, $V[I]_\sigma$ give an element of

$$C^\infty(M,((T''_\sigma)^*\otimes T'_\sigma)\oplus((T'_\sigma)^*\otimes T''_\sigma)),$$

and we have a splitting $V[I]_\sigma = V[I]'_\sigma + V[I]''_\sigma$ where

$$V[I]'_\sigma \in C^\infty(M,(T''_\sigma)^*\otimes T'_\sigma) \quad \text{and} \quad V[I]''_\sigma \in C^\infty(M,(T'_\sigma)^*\otimes T''_\sigma).$$

This splitting of $V[I]$ happens for every vector field on $\mathcal{T}$ and actually induce an almost complex structure on $\mathcal{T}$.

Since $V[I]_\sigma$ is a smooth section of $TM_\mathbb{C}\otimes T^*M_\mathbb{C}$ and the symplectic structure is a smooth section of $T^*M_\mathbb{C}\otimes T^*M_\mathbb{C}$ we can define a bivector field $\tilde{G}(V)$ by contraction with the symplectic form

$$\tilde{G}(V)\cdot\omega = V[I].$$

$\tilde{G}(V)$ is unique since ω is non-degenerate. By definition of the Kähler metric, g is the contraction of ω and I, $g=\omega\cdot I$, and since ω is independent of σ taking the derivative of this identity in the direction of a vector field V on $\mathcal{T}$ we obtain

$$V[g]=\omega\cdot V[I]=\omega\cdot\tilde{G}(V)\cdot\omega.$$

Since g is symmetric so is $V[g]$, and with ω being anti-symmetric $\tilde{G}(V)$ is symmetric. We furthermore have that ω is of type $(1,1)$ when regarded as a Kähler form on M_σ, using this and the fact that $V[I]_\sigma$ changes types on M_σ we get that $\tilde{G}(V)$ splits as $\tilde{G}(V)=G(V)+\bar{G}(V)$, where

$$G(V)\in C^\infty(M,S^2(T')) \quad \text{and} \quad \bar{G}(V)\in C^\infty(M,S^2(T'')).$$

In the above we have suppressed the dependence on σ, since it is valid for any σ.

3.1.1 Holomorphic families of Kähler structures

Let us now assume that $\mathcal{T}$ furthermore is a complex manifold. We can then ask $I:\mathcal{T}\to C^\infty(M,\mathrm{End}(TM))$ to be holomorphic. By using the splitting of $V[I]$ we make the following definition.

Definition 3.2. Let $\mathcal{T}$ be a complex manifold and I a smooth family of complex structures on M parametrized by $\mathcal{T}$. Then I is holomorphic if

$$V'[I]=V[I]' \quad \text{and} \quad V''[I]=V[I]''$$

for all vector fields V on $\mathcal{T}$.

Assume J is an integrable almost complex structure on $\mathcal{T}$ induced by the complex structure on $\mathcal{T}$. J induces an almost complex structure, $\hat{I}$ on $\mathcal{T}\times M$ by

$$\hat{I}(V\oplus X)=JV\oplus I_\sigma X,$$

where $V\oplus X\in T_{(\sigma,p)(\mathcal{T}\times M)}$. In [8] a simple calculation shows that the Nijenhuis tensor on $\mathcal{T}\times M$ vanish exactly when $\pi^{0,1}V'[I]X=0$ and $\pi^{1,0}V''[I]X=0$, which by the Newlander–Nirenberg theorem shows that $\hat{I}$ is integrable if and only if I is holomorphic, hence the name is well chosen.

Remark that for a holomorphic family of Kähler structures on (M,ω) we have

$$\tilde{G}(V')\cdot\omega=V'[I]=V[I]'=G(V)\cdot\omega,$$

which implies $\tilde{G}(V')=G(V)$. We can in the same way show that $\bar{G}(V)=\tilde{G}(V'')$.

3.1.2 Rigid families of Kähler structures

In constructing an explicit formula for the Hitchin connection we need the following rather restrictive assumption on our family of Kähler structures.

Definition 3.3. A family of Kähler structures I on M is called *rigid* if

$$\nabla_{X''} G(V) = 0$$

for all vector fields V on $\mathcal{T}$ and X on M_σ.

Equivalently we could give the above equation in terms of the induced $\bar{\partial}_\sigma$-operator on M_σ,

$$\bar{\partial}_\sigma(G(V)_\sigma) = 0,$$

for all $\sigma \in \mathcal{T}$ and all vector fields V on $\mathcal{T}$.

There are several examples of rigid families of Kähler structures. The family used in Chapter 6 is constant on M, and is therefore rigid. Several examples that are non-constant on M can also be given, see [8].

It should be remarked that the rigidity condition also is build into the arguments of [20].

3.2 Hitchin's connection

Now all tools are defined and we can construct the Hitchin connection. In Theorem 3.6 we need M to be compact, so let us assume this. Recall the quantum spaces

$$H_\sigma^{(k)} = \{s \in C^\infty(M, \mathcal{L}^k) \mid \nabla_\sigma^{0,1} s = 0\},$$

where $\nabla_\sigma^{0,1} = \frac{1}{2}(Id + iI_\sigma)\nabla$. Recall that in Subsection 2.2.2 $H^{(k)}$ was defined by the condition $\nabla_X s = 0$. This is just a reformulation of $\nabla^{0,1} s = 0$ since we can split $\nabla = \nabla^{1,0} + \nabla^{0,1}$ into types by the projections $\pi^{1,0}$ and $\pi^{0,1}$. Since ω is of type $(1,1)$ and it is the curvature of ∇, the $(0,2)$ part of the curvature vanishes, and $\nabla^{0,1}$ defines a $\bar{\partial}$-operator on each $C^\infty(M, \mathcal{L}^k)$, and in this language $\nabla_X s = 0$ for all $X \in \mathcal{P}$ is exactly $\nabla^{0,1} s = 0$.

It is not clear that these spaces form a vector bundle over $\mathcal{T}$. But by constructing a bundle, where these sit as subspaces of each of the fibers, and a connection in this bundle preserving $H_\sigma^{(k)}$, $H^{(k)}$ will be a subbundle over $\mathcal{T}$.

Define the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ of infinite rank. The finite dimensional subspaces $H_\sigma^{(k)}$ sits inside each of the fibers. This bundle has the trivial connection ∇^t. We seek a connection preserving $H_\sigma^{(k)}$.

Definition 3.4. A *Hitchin connection* is a connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$, which preserves the subspaces $H_\sigma^{(k)}$, and is of the form

$$\hat{\nabla} = \nabla^t + u,$$

where $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ is a 1-form on $\mathcal{T}$ with values in differential operators acting on sections of $\mathcal{L}^k$.

By analyzing the condition $\nabla_\sigma^{0,1} \hat{\nabla}_V s = 0$ for every vector field V on $\mathcal{T}$, we hope to find an explicit expression for u. If we express the above condition in terms of u, u should satisfy

$$0 = \nabla_\sigma^{0,1} V[s] + \nabla_\sigma^{0,1} u(V) s,$$

and if we differentiate $\nabla^{0,1}_\sigma s = 0$ along the a vector field V on $\mathcal{T}$ we get

$$0 = V[\nabla^{0,1}_\sigma s] = V[\frac{1}{2}(Id + iI_\sigma)\nabla s] = \frac{i}{2}V[I_\sigma]\nabla s + \nabla^{0,1}_\sigma V[s].$$

If we combine the previous two equations we get the following

Lemma 3.5. *The connection $\hat{\nabla} = \nabla^t + u$ preserves $H^{(k)}_\sigma$ for all $\sigma \in \mathcal{T}$ if and only if u satisfy the equation*

$$\nabla^{0,1}_\sigma u(V)s = \frac{i}{2}V[I]_\sigma \nabla^{1,0}_\sigma s \tag{3.1}$$

for all $\sigma \in \mathcal{T}$ and all vector fields V on $\mathcal{T}$.

If the conclusion is true the collection of subspaces $H^{(k)}_\sigma \subset C^\infty(M, \mathcal{L}^k)$ constitute a subbundle $H^{(k)}$ of $\mathcal{H}^{(k)}$.

Let us now assume that $\mathcal{T}$ is a complex manifold, and the family I is holomorphic. First of all, $\nabla^{1,0}_\sigma s$ is a section of $(T'_\sigma)^* \otimes \mathcal{L}^k$, so it is constant in the T''_σ-direction, which is why $V[I]''_\sigma \nabla^{1,0}_\sigma s = 0$, and by holomorphicity $V''[I] = V[I]''$, so $V''[I]_\sigma \nabla^{1,0}_\sigma s = 0$. Hence we can choose $u(V'') = 0$, and we therefore only need to focus on u in the V'-direction.

$u(V)$ should be a differential operator acting on sections of $\mathcal{L}^k$, and be related to I, so let us construct an operator from I.

Given a bivector field B on M we define an operator

$$\Delta_B = \nabla^2_B + \nabla_{\delta B}.$$

Locally $B = \sum_j X_j \otimes Y_j$ and ∇^2_B is defined by

$$\nabla^2_{X_1 \otimes Y_1} s = \nabla_{X_1}\nabla_{Y_1} s - \nabla_{\nabla_{X_1} Y_1} s,$$

and extending by linearity. A small calculation shows that ∇^2 is tensorial in $X_1 \otimes X_2$, i.e. $\nabla^2_{f(X_1 \otimes X_2)} s = \nabla^2_{(fX_1)\otimes X_2} s = \nabla^2_{X_1 \otimes (fX_2)}$. $\Delta_B : C^\infty(M, \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k)$ is a differential operator on smooth sections of $\mathcal{L}^k$.

Recall the bivector field $G(V)$ defined by $G(V)\cdot\omega = V'[I]$. The above construction give a differential operator $\Delta_{G(V)} : C^\infty(M, \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k)$. Locally $G(V) = \sum_j X_j \otimes Y_j$, and

$$\Delta_{G(V)} = \nabla^2_{G(V)} + \nabla_{\delta G(V)} = \sum_j \nabla_{X_j}\nabla_{Y_j} + \nabla_{\delta(X_j)Y_j}, \tag{3.2}$$

since $\delta(X_j \otimes Y_j) = \delta(X_j)Y_j + \nabla_{X_j} Y_j$. We state this local formula for later reference.

The second order differential operator $\Delta_{G(V)}$ is the cornerstone in the construction of $u(V)$. The idea in Andersen's construction is to calculate $\nabla^{0,1}\Delta_{G(V)} s$ and find remainder terms, which cancel other terms such that $\Delta_{G(V)}$ with these corrections satisfy equation (3.1), some central formulas are stated after the theorem. When calculating $\nabla^{0,1}\Delta_{G(V)} s$ the trace of the curvature of M_σ show up – that is the Ricci curvature ρ_σ. From Subsection 1.5.1 on Hodge decomposition $\rho_\sigma = \rho^H_\sigma + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma$ where F_σ is the Ricci potential. As with the family of Kähler structures the Ricci potentials F_σ is a family of Ricci potentials parametrized by $\mathcal{T}$ and can therefore be differentiated along a vector field on $\mathcal{T}$.

Define $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ by

$$u(V) = \frac{1}{4k + 2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} - 4kV'[F]). \tag{3.3}$$

Theorem 3.6 (Andersen [2]). *Let (M,ω) be a compact prequantizable symplectic manifold with $H^1(M,\mathbb{R})=0$ and first Chern class $c_1(M,\omega)=n[\frac{\omega}{2\pi}]$. Let I be a rigid holomorphic family of Kähler structures on M parametrized by a complex manifold $\mathcal{T}$. Then*

$$\hat{\nabla}_V = \nabla^t_V + \frac{1}{4k+2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F])$$

is a Hitchin connection in the bundle $H^{(k)}$ over $\mathcal{T}$.

The theorem is established through the following lemmas, which combine to show that $u(V)$ satisfy Equation (3.1). We do not prove the lemmas.

Lemma 3.7. *Assume that the first Chern class of (M,ω) is $n[\frac{\omega}{2\pi}]\in H^2(M,\mathbb{Z})$. For any $\sigma\in\mathcal{T}$ and any $G\in H^0(M_\sigma,S^2(T'_\sigma))$ we have the following identities*

$$\nabla^{0,1}\Delta_{G(V)}s = -2ik\omega\cdot G(V)\cdot\nabla s - i\rho\cdot G(V)\cdot\nabla s - ik\omega\cdot\delta(G(V))s$$

$$\nabla^{0,1}\nabla_{G(V)\cdot dF(s)} = -\partial\bar{\partial}F\cdot G(V)\cdot\nabla s - ik\omega\cdot G(V)\cdot dFs.$$

Lemma 3.8. *If $H^1(M,\mathbb{R})=0$ then for any vector field V on $\mathcal{T}$ we have*

$$\bar{\partial}_\sigma(V'[F]_\sigma) = \frac{i}{2}\omega\cdot(G(V)\cdot dF)_\sigma + \frac{i}{4}\omega\cdot\delta_\sigma(G(V))_\sigma.$$

Proof of Theorem 3.6. The idea is to piece the formulas in Lemma 3.7 and Lemma 3.8 together, and using that since the first Chern class is $n[\frac{\omega}{2\pi}]$ then $\rho = n\omega + 2i\partial\bar{\partial}F$

$$\nabla^{0,1}_\sigma u(V)s = -\frac{i}{2}\omega\cdot G(V)_\sigma\cdot\nabla_\sigma s = \frac{i}{2}V'[I]_\sigma s = \frac{i}{2}V[I]'_\sigma s = \frac{i}{2}V[I]_\sigma s$$

since s is holomorphic. This shows $u(V)$ satisfying Equation (3.1). □

Remark 3.9. The condition $c_1(M,\omega)=n[\frac{\omega}{2\pi}]=nc_1(\mathcal{L})$ can be removed by switching to the metaplectic correction. Here we make the same construction but now a square root of the canonical bundle of (M,ω) is tensored onto $\mathcal{L}^k$. Such a square root exists exactly if the second Stiefel–Whitney class is 0 – that is if M is spin. See [8] for further details.

Remark 3.10. The condition $H^1(M,\mathbb{R})=0$ is used to make the calculations in the proof easier, but there is no known examples of manifolds with $H^1(M,\mathbb{R})\neq 0$, which satisfy the remaining conditions where the Hitchin connection cannot be built in this way. An example is the torus T^{2n} which we will study in much greater detail in Chapter 6.

Remark 3.11. The rigidity condition on the family of Kähler structures is a serious condition. Current work is trying to remove this condition, or at least to rewrite it.

Remark 3.12. Originally the question about existence of Hitchin connection was asked in a TQFT setting. Hitchin ([20]) and Axelrod, Della Pietra and Witten ([10]) looked at quantization of the moduli space of $SU(n)$-connections on a Riemann surface. The goal was to quantize Chern–Simons theory to get a TQFT. If Andersen's construction is used in this setting it recovers the original results of [20] and [10].

Remark 3.13. To be able to really use the above Hitchin connection to geometrically quantize symplectic manifolds the Hitchin connection should be projectively flat. This is indeed the case in the original work of Hitchin [20] and Axelrod, Della Pietra and Witten [10]. By using Toeplitz operator techniques Gammelgaard has shown that this is indeed the case if there are no holomorphic vector fields on M_I for every I in the family, see [15, Theorem 6.22 and Theorem 6.14]. As we shall see in Section 7.1.1 this is also the case for the Hitchin connection build with Theorem 3.6 in the case of a principal polarized abelian variety.

Remark 3.14. In the other normalization $\Theta_{\mathcal{L}} = \frac{-i}{2\pi}\omega$ then $G(V)$ should be divided by 2π, so the Hitchin connection will be

$$\hat{\nabla}_V = \nabla^t_V + \frac{1}{\pi(8k+4n)}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 8\pi k V'[F])$$

In the presence of a symmetry group Γ, which acts by bundle automorphisms of $\mathcal{L}$ over M preserving both the Hermitian structure and the connection in $\mathcal{L}$. Then there is an induced action of Γ on (M, ω). If we further assume Γ acts on $\mathcal{T}$ and that I is Γ-equivariant, we have that the action of Γ on $\mathcal{H}^{(k)}$ preserves $H^{(k)}$ and the Hitchin connection. Such a symmetry group could be the mapping class group on a surface acting on the moduli space of flat connections. If the Hitchin connection is projectively flat, we get a sequence of projective representations

$$\rho_k : \Gamma \to \mathrm{Aut}\,(\mathbb{P}H^{(k)}).$$

These representations are the so called quantum representations, and is a very active field. In the case of the $SU(n)$ moduli space these representations are asymptotically faithful, meaning that an element of Γ lying in the kernel of ρ_k for all k is actually the identity (here we assume the genus of Σ to be greater that or equal to 3). This is a theorem shown by Andersen in [4] using Toeplitz operator techniques. The key ingredient in the proof is that Toeplitz operators asymptotically are projectively flat with respect to the flat Hitchin connection induced in the endomorphism bundle. For further details see [4].

4

Moduli space of flat G-connections

Let Σ be a compact Riemann surface of genus g. The moduli space of flat $G = SU(n)$ connections on Σ has two convenient descriptions. A topological, and a geometric.

Consider the set $\mathcal{A}_F$ of pairs (P, A) of a principal G bundle P over Σ and a flat connection A in P. Any such two pairs are isomorphic if there exists a bundle isomorphism between them pulling back the connection in the target bundle to the connection in the domain bundle. In particular are (P, A) and (P, A') equivalent if A and A' are gauge equivalent. The moduli space is then the set of equivalence classes $\mathcal{A}_F/\sim$ of flat G-bundles over Σ.

If G is simply connected all principal G-bundles over a 2-manifold are trivializable, and all principal G-bundles are isomorphic. Then looking at the set of pairs, we can restrict to connections in the trivial bundle $P = G \times \Sigma$. The space of connections in P, $\mathcal{A}$, can be identified with the space $\Omega^1 \otimes \mathfrak{g}$ of $\mathfrak{g}$-valued 1-forms on Σ, and $\mathcal{A}_F$ with the subset of $\mathcal{A}$ given by $\{A \in \mathcal{A} \,|\, F_A = dA + A \wedge A = 0\}$, where F_A is the curvature of A. The Gauge group $\mathcal{G} = \text{Maps}(\Sigma, G)$ acts on $\mathcal{A}_F$ by $g \in \mathcal{G}$ taking $A \in \mathcal{A}_F$ to $A^g = g^{-1}Ag + g^{-1}dg$. Then the moduli space is defined as $\mathcal{M}(\Sigma, G) = \mathcal{A}_F/\mathcal{G}$.

The topological description of the moduli space is as the set of conjugacy classes of homomorphisms of the fundamental group $\pi_1(\Sigma)$ into G under the quotient of the adjoint action of G on the set of homomorphisms

$$\text{Hom}(\pi_1(\Sigma), G)/G.$$

Since $G = SU(n)$ is a subset of the automorphisms of a vector space, $\text{Hom}(\pi_1(\Sigma), G)$ consists of representations and is called the representation variety. The quotient $\text{Hom}(\pi_1(\Sigma), G)/G$ is usually called the character variety since it is often given coordinates by taking the trace of the equivalence classes of representations.

From connections to representations

A connection in a principal G-bundle can be seen as parallel transport between the fibers of the bundle. Let $\gamma : [0, 1] \to \Sigma$ be a curve. If the connection A is flat parallel transport only depend on the connection and the homotopy class of γ. If $\gamma(0) = \gamma(1)$ there exists an element $g \in G$ relating the original element of $P_{\gamma(0)}$ with the transported element, by acting with g on the element. By fixing a flat connection A, the map assigning to each homotopy class of $\pi_1(\Sigma)$ an element of G is called the holonomy map, hol_A, and hol_A is a homomorphism. This give a map $hol : \mathcal{A}_F \to \text{Hom}(\pi_1(\Sigma), G)$. It can be shown that hol is also well-defined on the quotient $\mathcal{A}_F/\mathcal{G} \to \text{Hom}(\pi_1(\Sigma), G)/G$.

From representations to connections

It is also possible to construct a principal G-bundle from a representation $\rho : \pi_1(\Sigma) \to G$. Let $\pi : \tilde{\Sigma} \to \Sigma$ be the universal cover of Σ. Fix a base point $p_0 \in \tilde{\Sigma}$ and identify

$\tilde{\Sigma}$ with the space of homotopy classes of paths in Σ, starting at x_0, $x_0 = \pi(p_0)$. Then $\pi_1(\Sigma, x_0)$ naturally acts on $\tilde{\Sigma}$ from the right by concatenation of paths. For a representation $\rho : \pi_1(\Sigma, x_0) \to G$ we define the principal G-bundle to be the associated bundle to ρ, $P_\rho = \tilde{\Sigma} \times_\rho G = (\tilde{\Sigma} \times G)/\sim$, where $(\tilde{x}, g) \sim (\tilde{x} \cdot \alpha, \rho(\alpha)^{-1} g)$, $(\tilde{x}, g) \in \tilde{\Sigma} \times G$ and $\alpha \in \pi_1(\Sigma)$.

Pulling back the Maurer-Cartan form $\theta \in \Omega^1(G; \mathfrak{g})$ to $\tilde{\Sigma} \times G$ defines a natural flat connection $\tilde{A} = \pi_G^*(\theta)$ on $\tilde{\Sigma} \times G$, and since the Maurer-Cartan form is left-invariant this induces a flat connection on $\tilde{\Sigma} \times_\rho G$. It can be shown that this construction only depend on the conjugacy class of ρ, and is an inverse of the holonomy map, hol. Thus the holonomy map is a bijection $hol : \mathcal{M}(\Sigma, G) = \mathcal{A}_F/\mathcal{G} \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$. All details can be found in [19].

4.1 Properties of the moduli space

4.1.1 Smoothness and compactness

$\mathcal{M}(\Sigma, G)$ is in general not smooth, but a singular variety. It turns out that the irreducible representations $\mathcal{M}^{irr}(\Sigma, G) = \mathrm{Hom}^{irr}(\pi_1(\Sigma), G)/G \subset \mathcal{M}(\Sigma, G)$ constitute an open dense subset, and is a smooth manifold. $\mathcal{M}(\Sigma, G)$ is compact by the topological description of the moduli space. However, $\mathcal{M}^{irr}$ is not compact.

Another way to make the moduli space smooth, is to specify the values of the representations on curves around the boundary. For simplicity let us assume Σ has one boundary component. Let γ be a loop around the boundary component, and choose an element of a maximal torus of G (e.g. $G = SU(n)$ pick $D = e^{2\pi i \frac{d}{n}} Id$, where d, n are coprime) to be the corresponding element in G of γ. Then $\mathcal{M}_D(\Sigma, G) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma), G) \ : \ \rho(\gamma) = D\}/G$ is a smooth submanifold of $\mathcal{M}(\Sigma, G)$. As above $\mathcal{M}_D(\Sigma, G)$ compact.

4.1.2 Setup of geometric quantization

From now let us assume Σ has genus greater than 2. In [16] Goldman defines a symplectic structure on both of the above smooth manifolds, by identifying the tangent space $T_{[A]}\mathcal{M}(\Sigma, G)$ with first cohomology of Σ with coefficients in the adjoint-bundle $H^1(\Sigma, \mathrm{Ad}\, P)$. With the symplectic structure we could hope to quantize $\mathcal{M}(\Sigma, G)$. The tangent space of $\mathcal{M}_D(\Sigma, G)$ is a subspace of $H^1(\Sigma, \mathrm{Ad}\, P)$ consisting of cocycles which vanish on γ.

To proceed we need the moduli spaces to be prequantizable, i.e. the existence of a prequantum line bundle. Several people (e.g. [26], [14]) have produced these line bundles.

Atiyah and Bott [9] show that $\mathcal{M}_D(\Sigma, SU(n))$ is simply connected, and hence the requirement of $H^1(\mathcal{M}, \mathbb{R}) = 0$ is fulfilled. Furthermore they show that the second integer cohomology is $\mathbb{Z}$, $H^2(\mathcal{M}_D(\Sigma, SU(n)), \mathbb{Z}) = \mathbb{Z}$, and is generated by $n[\omega]$ which is half the Chern class of $\mathcal{M}_D(\Sigma, SU(n))$ and ω is the symplectic form.

The Teichmüller space $\mathcal{T}(\Sigma)$ of conformal equivalence classes of Riemannian metrics on Σ is a complex manifold. A Riemannian metric gives rise to a Hodge star operator $*$, and by extending this operator to forms with values in the adjoint bundle associated to P, $\mathrm{Ad}\, P$, we define the adjoint of the exterior covariant derivative d_A in $\mathrm{Ad}\, P$ by $d_A^* = - * d_A *$ and the Laplacian $\Delta_A = d_A d_A^* + d_A^* d_A$ as usual. The tangent space to $\mathcal{M}(\Sigma, SU(n))$ at a point $[A]$ is $H^1(\Sigma, \mathrm{Ad}\, P)$, and by using Hodge theory it can be identified with the space of harmonic forms. A harmonic form is preserved by $*$ and hence defining $I = -*$ we have an almost complex structure on $\mathcal{M}(\Sigma, SU(n))$. This almost complex structure is compatible with the symplectic structure, and by Narasimhan and Seshadri ([25]) it is integrable and hence Kähler.

It can be seen that it preserves the subspaces corresponding to the tangent space of $\mathcal{M}_D(\Sigma, SU(n))$, which are hence compact Kähler manifolds.

It was observed by Hitchin [20] that the family of Kähler structures on the moduli space $\mathcal{M}_D(\Sigma, SU(n))$ is rigid in the sense of Section 3.1.2.

All these ingredients combine to define a Hitchin connection in the bundle $H^{(k)}$ over Teichmüller space to Σ by Theorem 3.6. The bundle of quantum spaces $H^{(k)}$ was originally studied by Hitchin ([20]) and Axelrod, Della Pietra and Witten [10], where they also proved that the connection is projectively flat.

5

Abelian varieties, line bundles and theta functions

In this chapter we will discuss various properties of abelian varieties. We will start by investigating when a complex torus is an abelian variety. Afterwards we see how we can define line bundles over an abelian variety by functions on the universal cover, satisfying some functional equations. Lastly we will discuss the space of holomorphic sections of a line bundle over an abelian variety, and see that a basis for this space is given by theta functions. The chapter is mainly based on [18, Chapter 2.6].

5.1 Riemann conditions

Definition 5.1. Let V be a complex vector space of dimension n, $\Lambda \subset V$ a discrete lattice of maximal rank $2n$. The complex torus $M = V/\Lambda$ is called an *abelian variety* if it is a projective algebraic variety, i.e. can be embedded into projective space.

In this section we will specify necessary and sufficient conditions for embedding M into projective space. Kodaira's embedding theorem gives such a condition, and we will rewrite it to fit our purpose. The result is the *Riemann conditions.*

We are searching for existence of a cohomology class, and hence we better study the cohomology of M.

The first thing we need is to relate the cohomology of M with V. It can be shown by a small argument using harmonic forms that

$$H^*(M, \mathbb{C}) = \wedge^* V \otimes \wedge^* \bar{V}^*.$$

The way we proceed is to give a basis for $H^*(M, \mathbb{C})$ expressing the complex structure and one for $\wedge^* V \otimes \wedge^* \bar{V}^*$ expressing Λ and the rational structure of $H^1(M, \mathbb{Z})$.

V has Euclidean coordinates $z = (z_1, \dots, z_n)$ given by a complex basis $e_1, \dots, e_n$ and $dz_1, \dots, dz_n, d\bar{z}_1, \dots d\bar{z}_n$ are global 1-forms on M. $H^*(M, \mathbb{C}) = \text{span}_{\mathbb{C}}\{dz_A \wedge d\bar{z}_B\}_{A,B}$ A, B multi-indices. Let $\gamma \in H_1(M, \mathbb{Z})$ be a loop with basepoint $[0] \in M$, it lifts to a path $\tilde{\gamma}$ in V starting at 0 and ending at a $\lambda \in \Lambda \subset V$. V is the universal cover of M and hence Λ are the deck transformations, so $\Lambda = H_1(M, \mathbb{Z})$. Let $\lambda_1, \dots, \lambda_{2n}$ be an integral basis of Λ. Since Λ has full rank $\{\lambda_i\}$ is a real basis of V. Let $\{x_1, \dots, x_{2n}\}$ be the dual coordinates on V, and $\{dx_1, \dots, dx_{2n}\}$ the one-forms on M. By definition of the coordinates integrating dx_j around the loop λ_i give δ_{ij}, and hence $H^1(M, \mathbb{Z}) = \text{span}_{\mathbb{Z}}\{dx_1, \dots, dx_{2n}\}$, and generally $H^k(M, \mathbb{Z}) = \text{span}_{\mathbb{Z}}\{dx_I\}_{|I|=k}$.

This give us two different bases for the cohomology on M. $\{dz_\alpha, d\bar{z}_\alpha\}$ reflecting the complex structure on $H^*(M, \mathbb{C})$ and $\{dx_i\}$ reflecting the rational structure. It is the relation between the two, which will give us necessary and sufficient conditions for the existence of a Hodge form.

For the remaining part of this chapter greek indicies run from 1 to n, and latin indicies run from 1 to $2n$.

Let $\Pi = (\pi_{i\alpha})$ be the $2n \times n$-matrix such that $\tilde{\Pi} = (\Pi, \bar{\Pi})$ changes basis from $\{dz_\alpha, d\bar{z}_\alpha\}$ to $\{dx_i\}$. Let $\Omega = (w_{\alpha i})$ be the period matrix of $\Lambda \subset V$, i.e. $\lambda_i =$

$\sum_\alpha w_{\alpha i} e_\alpha$, and finally $\omega = \frac{1}{2}\sum_{ij} q_{ij} dx_i \wedge dx_j$ a two-form with $Q = (q_{ij})$ an integral skew-symmetric $2n \times 2n$-matrix.

Proposition 5.2 (Riemann Conditions)**.** *M is an abelian variety if and only if one of the following equivalent conditions are satisfied.*

(I) There exists an integral skew-symmetric matrix Q such that

$$\Pi Q \Pi = 0 \quad \text{and} \quad -i\Pi^T Q \bar{\Pi} > 0.$$

(II) There exists an integral skew-symmetric matrix Q such that

$$\Omega Q^{-1} \Omega^T = 0 \quad \text{and} \quad -i\Omega Q^{-1} \bar{\Omega}^T > 0.$$

(III) There exists an integral basis $\lambda_1, \ldots, \lambda_{2n}$ for Λ and a complex basis $e_1, \ldots, e_n$ for V such that $\Omega = (\Delta_\delta, Z)$ with Δ_δ diagonal with integer entries and Z symmetric and $\mathrm{Im}(Z) > 0$.

The two first conditions are natural consequences of playing around with the change of basis matrices, while the last is a consequence of the following

Lemma 5.3. *If Q is an integral skew-hermitian quadratic form on $\Lambda \simeq \mathbb{Z}^{2n}$, then there exists a basis $\lambda_1, \ldots, \lambda_{2n}$ for Λ in terms of which Q is given by the matrix* $Q = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}$, *where* $\Delta_\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}$.

With this lemma, the symmetry of Z follows from $\Omega Q^{-1} \Omega^T = 0$, which is a necessary condition for ω to be of type $(1,1)$. $\mathrm{Im}\, Z > 0$ follow from $-i\Omega Q^{-1} \bar{\Omega}^T > 0$, which is necessary for ω to be positive.

The cohomology class $[\omega]$ is called a polarization of M, and if all $\delta_i = 1$, M is called principal polarized.

If $(M_Z = V_Z/\Lambda, \omega)$ is a principal polarized abelian variety, Z reflects the complex structure on V. Since ω is not only a non-degenerate symplectic form but also positive, the metric defined by $g = \omega \cdot I_Z$ is positive definite, where I_Z is the complex structure defined by Z and hence (M_Z, ω) is Kähler. This is all incorporated in the proof the Riemann Conditions. Since for each choice of Z in Siegels Upper Half Space $\mathbb{H} = \{Z \in \mathrm{Mat}_n(\mathbb{C}) \,|\, Z^T = Z,\ \mathrm{Im}\, Z > 0\}$ the corresponding (M_Z, ω) is Kähler we will use $\mathbb{H}$ in Chapter 6 to parametrize the space of complex structures on the torus $M = V/\Lambda$.

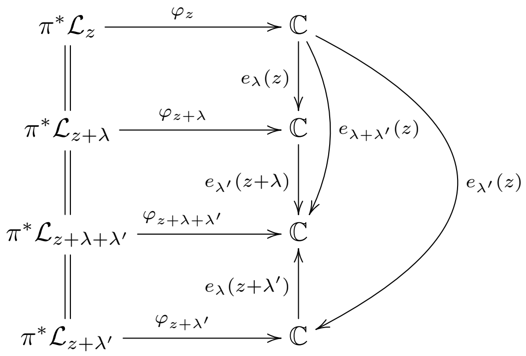
5.2 Line bundles

In the following we will construct line bundles on $M = V/\Lambda$ by complex functions satisfying certain functional equations, the so called multipliers.

Let $\mathcal{L} \to M$ be a line bundle. If we pull back $\mathcal{L}$ to V by the projection map $\pi : V \to M$, the line bundle $\pi^*\mathcal{L}$ is trivial since V is contractible. Pick a global trivialization of $\pi^*\mathcal{L}$, $\varphi : \pi^*\mathcal{L} \to V \times \mathbb{C}$. In each fiber, $(\pi^*\mathcal{L})_z$, we have an isomorphism $\varphi_z : (\pi^*\mathcal{L})_z \to \mathbb{C}$. Since $(\pi^*\mathcal{L})_z = \mathcal{L}_{\pi(z)} = \mathcal{L}_{\pi(z+\lambda)} = (\pi^*\mathcal{L})_{z+\lambda}$, for $\lambda \in \Lambda$, composing trivializations $\varphi_{z+\lambda} \circ \varphi_z^{-1}$ is an automorphism of $\mathbb{C}$, i.e. multiplication by a complex number. This complex number depend on λ and z, and is denoted $e_\lambda(z)$. If we vary z we get a family of functions $\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda \in \Lambda}$ called multipliers. These multipliers must satisfy the following relation

$$e_\lambda(z) e_{\lambda'}(z + \lambda) = e_{\lambda'}(z) e_\lambda(\lambda' + z) = e_{\lambda' + \lambda}(z). \tag{5.1}$$

These functional equations follow from the commutativity of the following diagram,

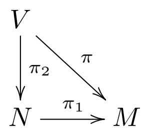


Assume given such a family of non-vanishing holomorphic functions $\{e_\lambda\}_{\lambda\in\Lambda}$ satisfying the above equations. Let $\mathcal{L} \to M$ be the quotient of $V \times \mathbb{C}$ by identifying $(z,\xi) \sim (z+\lambda, e_\lambda(z)\xi)$. Then $\mathcal{L}$ is a line bundle over M with the given functions as multipliers.

Our goal in this section is to construct a line bundle over M with first Chern class a specific positive 2-form ω. This construction rely on specifying the right multipliers.

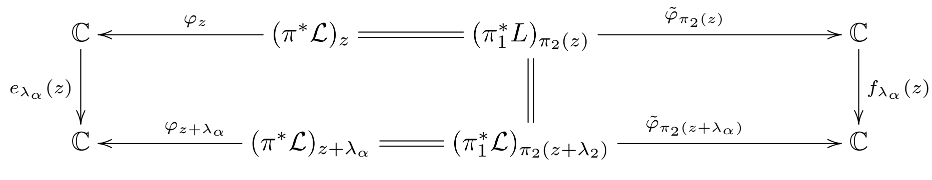
We begin our quest by showing that we only need to look at a certain type of multipliers, namely those with the first n constantly 1.

Let $\{\lambda_1,\dots,\lambda_{2n}\}$ be a basis for Λ over $\mathbb{Z}$ with $\lambda_1,\dots,\lambda_n$ linearly independent over $\mathbb{C}$. Then $N = V/\mathbb{Z}\{\lambda_1,\dots,\lambda_n\} \simeq (\mathbb{C}^*)^n$, and the projection $\pi : V \to M$ factorizes through N,



By Poincaré's $\bar\partial$-lemma $H^1((\mathbb{C}^*)^n,\mathcal{O}) = H^2((\mathbb{C}^*)^n,\mathcal{O}) = 0$ and combining this with the long exact sequence (1.2) $c_1 : H^1((\mathbb{C}^*)^n,\mathcal{O}^*) \to H^1((\mathbb{C}^*)^n,\mathbb{Z})$ is an isomorphism, and every line bundle on $(\mathbb{C}^*)^n$ is determined by its Chern class.

For every line bundle $\mathcal{L} \to M$ Lemma 5.3 provide a basis for Λ, $\{\lambda_1,\dots,\lambda_{2n}\}$ such that in terms of the dual coordinates on V, $x_1,\dots,x_{2n}$ the first Chern class is $c_1(\mathcal{L}) = \sum_{\alpha=1}^n \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}$. The functions $x_{n+\alpha}$ are well-defined global functions on N so $[dx_{n+\alpha}] = 0 \in H^1_{dR}(N)$. Then $c_1(\pi_1^*\mathcal{L}) = \pi_1^*(c_1(\mathcal{L})) = 0$ and $\pi_1^*\mathcal{L}$ is trivial. Let $\tilde\varphi : \pi_1^*\mathcal{L} \to (\mathbb{C}^*)^n \times \mathbb{C}$ be a trivialization and choose the trivialization of $\pi^*\mathcal{L}$ to extend $\tilde\varphi$, that is $\varphi_z = \tilde\varphi_{\pi_2(z)}$ and $\varphi_{z+\lambda_\alpha} = \tilde\varphi_{\pi_2(z+\lambda_\alpha)}$ for all $\alpha = 1,\dots,n$.



Since $\tilde\varphi_{\pi_2(z)} = \tilde\varphi_{\pi_2(z+\lambda_\alpha)}$, $f_{\lambda_\alpha}(z)$ is forced to be constantly 1. Commutativity and the fact that φ extends $\tilde\varphi$ implies that $e_{\lambda_\alpha}(z) = 1$. This is true for all $\alpha = 1,\dots,n$. This is not true for $\alpha > n$ since we do not have the equality $\pi_2(z) = \pi_2(z+\lambda_\alpha)$ which was essential to the argument. This shows that we only need to consider multipliers with the first n being constantly equal to 1.

Now assume ω to be an invariant integral form of type $(1,1)$ on V. Choose a basis $\lambda_1,\dots,\lambda_{2n}$ for Λ over $\mathbb{Z}$, such that in terms of the dual coordinates on V,

$x_1, \dots, x_{2n}$ then $\omega = \sum_{\alpha=1}^{n} \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}$, $\delta_\alpha \in \mathbb{Z}$. Furthermore we require the first n λ_α's to be linearly independent over $\mathbb{C}$. Since ω is non-degenerate each $\delta_\alpha \neq 0$, and we can define $e_\alpha = \delta_\alpha^{-1}\lambda_\alpha$, $\alpha = 1, \dots, n$. Let $z_1, \dots, z_n$ be the corresponding coordinates on V. As in Section 5.1 we can write $(\lambda_1, \dots, \lambda_{2n}) = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \Omega$ that is $\begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix} = \Omega^T(dx_1, \dots, dx_{2n})$, where $\Omega = (\Delta_\delta, Z)$ and third Riemann condition implies $Z = Z^T$ and $\operatorname{Im} Z > 0$.

Lemma 5.4. *The line bundle $\mathcal{L} \to M$ defined by multipliers $e_{\lambda_\alpha} = 1$ and $e_{\lambda_{n+\alpha}}(z) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}$, $\alpha = 1, \dots, n$ has Chern class $c_1(\mathcal{L}) = [\omega]$.*

Proof. First of all we need to see that these multipliers satisfy the line bundle condition (5.1), that is $e_{\lambda_\alpha}(z+\lambda_\beta)e_{\lambda_\beta}(z) = e_{\lambda_\beta}(z+\lambda_\alpha)e_{\lambda_\alpha}(z) = e_{\lambda_\alpha+\lambda_\beta}(z)$,

$$\begin{aligned} e_{\lambda_{n+\beta}}(z+\lambda_{n+\alpha})e_{\lambda_{n+\alpha}}(z) &= e^{-2\pi i(z_\beta + Z_{\beta\alpha})}e^{-\pi i Z_{\beta\beta}}e^{-2\pi i z_\alpha}e^{-\pi i Z_{\alpha\alpha}} \\ &= e^{-2\pi i(z_\alpha + Z_{\alpha\beta})}e^{-\pi i Z_{\beta\beta}}e^{-2\pi i z_\beta}e^{-\pi i Z_{\alpha\alpha}} \\ &= e_{\lambda_{n+\alpha}}(z+\lambda_{n+\beta})e_{\lambda_{n+\beta}}(z) \end{aligned}$$

and

$$\begin{aligned} e_{\lambda_{n+\alpha}}(z+\lambda_\beta)e_{\lambda_\beta}(z) = e_{\lambda_{n+\alpha}}(z+\lambda_\beta) &= e^{-2\pi i(z_\alpha + \delta_{\alpha,\beta}\delta_\alpha)}e^{-\pi i Z_{\alpha\alpha}} \\ = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}} &= e_{\lambda_{\alpha+n}}(z)e_{\lambda_\beta}(z+\lambda_{n+\alpha}). \end{aligned}$$

Let $\varphi : \pi^*\mathcal{L} \to V \times \mathbb{C}$ be a trivialization of $\pi^*\mathcal{L}$ which induces the given multipliers. For every section $\tilde{\theta}$ of $\pi : \mathcal{L} \to M$ over $U \subset M$, $\theta = \varphi^*(\pi^*(\tilde{\theta}))$ is an analytic function on $\pi^{-1}(U)$ satisfying $\theta(z+\lambda_\alpha) = \theta(z)$ and $\theta(z+\lambda_{n+\alpha}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}\theta(z)$ for every $\alpha = 1, \dots, n$. Conversely will any such function be a section of $\pi : \mathcal{L} \to M$. If $\|\cdot\|$ is a metric on $\mathcal{L}$ then $\|\tilde{\theta}(z)\|^2 = h(z)\,|\theta(z)|^2$ for every section $\tilde{\theta}$ of $\mathcal{L}$, where $|\cdot|$ is the standard inner product on $\mathbb{C}$ and θ is the function on V corresponding to $\tilde{\theta}$. h will be a positive smooth function of z and satisfy equations much like the equation for the line bundle

$$h(z)\,|\theta(z)|^2 = \|\tilde{\theta}(z)\|^2 = \|\tilde{\theta}(z+\lambda)\|^2 = h(z+\lambda)\,|\theta(z+\lambda)|^2 \tag{5.2}$$

for all $\lambda \in \Lambda$. That is

$$h(z+\lambda_\alpha) = h(z) \qquad h(z+\lambda_{n+\alpha}) = \left|e^{2\pi i z_\alpha + \pi i Z_{\alpha\alpha}}\right|^2 h(z).$$

Conversely, any such h satisfying the above equations will be a metric on $\mathcal{L}$.

Let $Z \in \mathbb{H} = \{Z \in \operatorname{Mat}_n(\mathbb{C}) \,|\, Z^T = Z, \operatorname{Im} Z > 0\}$ split as $Z = X + iY$, where X and Y are real matrices. Since $Y > 0$ it is invertible and define $W = (W_{\alpha\beta}) = Y^{-1}$. The function

$$h(z) = e^{\frac{\pi}{2}\sum W_{\alpha\beta}(z_\alpha - \bar{z}_\alpha)(z_\beta - \bar{z}_\beta)} = e^{-2\pi y \cdot Y y}$$

satisfy the above equations. It is clear that $h(z+\lambda_\alpha)=h(z)$ since $(z+\lambda_\alpha)_\beta = z_\alpha + \delta_{\alpha,\beta}\delta_\alpha$. The other equality is a direct computation

$$\begin{aligned}
\log h(z+\lambda_{n+\gamma}) =& \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha + Z_{\alpha\gamma} - \bar z_\alpha - \bar Z_{\alpha\gamma})(z_\beta + Z_{\beta\gamma} - \bar z_\beta - \bar Z_{\beta\gamma})\\
=& \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha - \bar z_\alpha + 2iY_{\alpha\gamma})(z_\beta - \bar z_\beta + 2iY_{\beta\gamma})\\
=& \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha - \bar z_\alpha)(z_\beta - \bar z_\beta) + \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha - \bar z_\alpha)2iY_{\beta\gamma}\\
&+ \frac{\pi}{2}\sum_{\alpha\beta} W_{\alpha\beta}2iY_{\alpha\gamma}(z_\beta - \bar z_\beta + 2iY_{\beta\gamma})\\
=& \log h(z) + \sum_\alpha \delta_{\alpha,\gamma} i\pi(z_\alpha - \bar z_\alpha) + \sum_\beta \delta_{\beta,\gamma} i\pi(z_\beta - \bar z_\beta + 2iY_{\beta\gamma})\\
=& \log h(z) + 2\pi i(z_\gamma - \bar z_\gamma) - 2\pi Y_{\gamma\gamma}
\end{aligned}$$

so $h(z+\lambda_{n+\gamma}) = h(z)e^{-4\pi\,\mathrm{Im}(z_\gamma) - 2\pi Y_{\gamma\gamma}} = h(z)\left|e^{2\pi i z_\gamma + \pi i Z_{\gamma\gamma}}\right|^2$.

h gives a metric on $\mathcal{L}\to M$. We then use the formula (1.1), to calculate the curvature $\Theta_{\mathcal{L}}$ of the canonical connection defined by h associated to this metric.

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \partial\bar\partial \log\frac{1}{h(z)}\\
&= -\frac{\pi}{2}\partial\bar\partial\Big(\sum_{\alpha\beta} W_{\alpha\beta}(z_\alpha - \bar z_\alpha)(z_\beta - \bar z_\beta)\Big)\\
&= -\frac{\pi}{2}\partial\Big(\sum_{\alpha\beta} W_{\alpha\beta}(-(z_\beta - \bar z_\beta)d\bar z_\alpha - (z_\alpha - \bar z_\alpha)d\bar z_\beta)\Big)\\
&= \frac{\pi}{2}\sum_{\alpha\beta} W_{\alpha\beta}(dz_\beta\wedge d\bar z_\alpha + dz_\alpha \wedge d\bar z_\beta)\\
&= \pi\sum_{\alpha\beta} W_{\alpha\beta}dz_\alpha\wedge d\bar z_\beta
\end{aligned}$$

The curvature should be related to ω. Changing basis to $\{dx_i\}$ you obtain

$$\Theta_{\mathcal{L}} = -2\pi i\omega.$$

This shows $[\omega] = [\frac{i}{2\pi}\Theta_{\mathcal{L}}] = C_1(\mathcal{L})$. ☐

5.3 Theta functions

Global holomorphic sections of $\mathcal{L}\to M = V/\Lambda$ are holomorphic functions on $V\simeq\mathbb{C}^n$ satisfying certain functional equations. A basis of the space of holomorphic sections are the theta functions.

Theorem 5.5. *Let $\mathcal{L}\to M$ be a line bundle whose first Chern class has a positive representative and let $\delta_1,\dots,\delta_n\in\mathbb{Z}$ be the integers defining the polarization $c_1(\mathcal{L})$ of M. Then*

(i) $\dim H^0(M,\mathcal{L}) = \Pi_\alpha\delta_\alpha$

(ii) $H^0(M,\mathcal{L}^k)$ *gives an embedding of M in a projective space for $k\geq 3$, that is the line bundle is ample.*

In the following we will describe some of the properties of the theta functions while proving (i). (ii) will not be proved, but is included here since it will later become crucial. A proof can be found in [18] and [23].

Proof. Choose an integral basis for Λ, such that in terms of the dual coordinates on V, $x_1, \dots, x_{2n}$ the first Chern class is expressed as $c_1(\mathcal{L}) = \sum_\alpha \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}$. We can also choose the basis such that the first n are linearly independent over $\mathbb{C}$. As before let $e_\alpha = \delta_\alpha^{-1}\lambda_\alpha$ and let $z_1, \dots, z_n$ be the complex coordinates on V. Then the period matrix Ω of $\Lambda \subset V$ is of the form (Δ_δ, Z) with $Z = X + iY$ symmetric and $Y > 0$.

It can be shown that the line bundle $\mathcal{L}$ is a translation of the line bundle $\mathcal{L}_0$ given by multipliers $e_{\lambda_\alpha} = 1$, $e_{\lambda_{n+\alpha}} = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}$, since any line bundle is defined by first Chern class up to translation (translation τ_μ with $\mu \in \Lambda$ is homotopic to the identity so $c_1(\tau_\mu^* \mathcal{L}) = c_1(\mathcal{L})$). With the Chern class fixed we only need to consider the previous set of multipliers. Global sections $\tilde{\theta}$ of $\mathcal{L}$ are given by global holomorphic functions θ on V satisfying $\theta(z + \lambda_\alpha) = \theta(z)$ and $\theta(z + \lambda_{\alpha+n}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}\theta(z)$. To understand θ better we could look at its power series expansion. If we define $z_\alpha^* = e^{2\pi i \delta_\alpha^{-1} z_\alpha}$, we see that

$$(z+\lambda)_\alpha^* = e^{2\pi i \delta_\alpha^{-1}(z_\alpha + \delta_\alpha)} = e^{2\pi i \delta_\alpha^{-1} z_\alpha} = z_\alpha^*,$$

and since θ has the same invariance it has a power series expansion in this variable,

$$\theta(z) = \sum_{l \in \mathbb{Z}^n} a_l (z_1^*)^{l_1} \cdots (z_n^*)^{l_n} = \sum_{l \in \mathbb{Z}^n} a_l e^{2\pi i \sum_\alpha l_\alpha \delta_\alpha^{-1} z_\alpha} = \sum_{l \in \mathbb{Z}^n} a_l e^{2\pi i l \cdot \Delta_\delta^{-1} z}.$$

Second condition give a recursion formula between the coefficients of θ since

$$\theta(z + \lambda_{n+\alpha}) = \sum_{l \in \mathbb{Z}^n} a_l e^{2\pi i\, l \cdot \Delta_\delta^{-1}(z + \lambda_{n+\alpha})} = \sum_{l \in \mathbb{Z}^n} a_l e^{2\pi i\, l \cdot \Delta_\delta^{-1} \lambda_{n+\alpha}} e^{2\pi i\, l \cdot \Delta_\delta^{-1} z}.$$

The condition give

$$\begin{aligned}
\theta(z + \lambda_{n+\alpha}) &= e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}} \theta(z) \\
&= e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}} \sum_{l \in \mathbb{Z}^n} a_l e^{2\pi i\, l \cdot \Delta_\delta^{-1} z} \\
&= \sum_{l \in \mathbb{Z}^n} a_l e^{-2\pi i z_\alpha} e^{2\pi i l_\alpha \delta_\alpha^{-1} z_\alpha} e^{2\pi i \sum_{\beta \neq \alpha} l_\beta \delta_\beta^{-1} z_\beta} e^{-i\pi Z_{\alpha\alpha}} \\
&= \sum_{l \in \mathbb{Z}^n} a_l e^{2\pi i z_\alpha \delta_\alpha^{-1}(l_\alpha - \delta_\alpha)} e^{2\pi i \sum_{\beta \neq \alpha} l_\beta \delta_\beta^{-1} z_\beta} e^{-\pi i Z_{\alpha\alpha}} \\
&= \sum_{l \in \mathbb{Z}^n} a_{l + \delta_\alpha e_\alpha} e^{2\pi i z_\alpha \delta_\alpha^{-1} l_\alpha} e^{2\pi i \sum_{\beta \neq \alpha} l_\beta \delta_\beta^{-1} z_\beta} e^{-\pi i Z_{\alpha\alpha}} \\
&= \sum_{l \in \mathbb{Z}^n} a_{l + \delta_\alpha e_\alpha} e^{-i\pi Z_{\alpha\alpha}} e^{2\pi i\, l \cdot \Delta_\delta^{-1} z}
\end{aligned}$$

The two power series expansions are equal if and only if the coefficients are equal, so

$$a_{l+\delta_\alpha e_\alpha} = e^{i\pi Z_{\alpha\alpha}} e^{2\pi i\, l \cdot \Delta_\delta^{-1} \lambda_{n+\alpha}} a_l$$

This recursion relation determine the theta function θ completely by specifying coefficients $\{a_l\}_{l \in \mathbb{Z}^n : 0 \leq l_\alpha < \delta_\alpha}$, which is why $\dim H^0(M, \mathcal{O}(\mathcal{L})) \leq \Pi_\alpha \delta_\alpha$. To show equality we need to show that θ will converge for any choice of coefficients.

Given an integral vector l_0 with entries $0 \leq l_{0_\alpha} < \delta_\alpha$ we define the function

$$\theta_{l_0}(z) = e^{2\pi i\, l_0\cdot\Delta_\delta^{-1} z} \sum_{l\in\mathbb{Z}^n} \tilde{a}_{l_0+\Delta_\delta l} e^{2\pi i\, l\cdot z} = \sum_{l\in\mathbb{Z}^n} \tilde{a}_{l_0+\Delta_\delta l} e^{2\pi i\,(l_0+\Delta_\delta l)\cdot\Delta_\delta^{-1} z},$$

by $\tilde{a}_{l_0} = 1$ and the recursion relation. Since $\theta(z) = \sum_{0\leq l_{0_\alpha}<\delta_\alpha} a_{l_0}\theta_{l_0}(z)$, we only need to show convergence of $\theta_{l_0}(z)$.

Define $b_l = \tilde{a}_{l_0+\Delta_\delta l}$. Then the recursion relation is

$$b_{l+e_\alpha} = \tilde{a}_{l_0+\Delta_\delta l+\Delta_\delta e_\alpha} = e^{2\pi i\,(l_0+\Delta_\delta l)\cdot\Delta_\delta^{-1}\lambda_{n+\alpha}} e^{\pi i Z_{\alpha\alpha}} b_l$$

These relations can be solved by setting $b_l = e^{\pi i\, l\cdot Zl + 2\pi i\,\Delta_\delta^{-1} l_0\cdot Zl}$. This is a solution since

$$\begin{aligned} b_{l+e_\alpha} &= e^{\pi i\,(l+e_\alpha)\cdot Z(l+e_\alpha) + 2\pi i\,\Delta_\delta^{-1} l_0\cdot Z(l+e_\alpha)} \\ &= e^{\pi i\, l\cdot Zl} e^{2\pi i\, l\cdot Ze_\alpha} e^{\pi i\, e_\alpha\cdot Ze_\alpha} e^{2\pi i\,\Delta_\delta^{-1} l_0\cdot Zl} e^{2\pi i\,\Delta_\delta^{-1} l_0\cdot Ze_\alpha} \\ &= b_l e^{\pi i Z_{\alpha\alpha}} e^{2\pi i\, l\cdot\lambda_{n+\alpha}} e^{2\pi i\,\Delta_\delta^{-1} l_0\cdot\lambda_{\alpha+n}} \end{aligned}$$

since Z is symmetric and $Ze_\alpha = \lambda_{n+\alpha}$.

Uniform convergence on compact sets of $\mathbb{C}^n$ follow from Weierstrass' M-test, if we give an upper bound on the absolute values on the coefficients b_l. First of all $|b_l| = e^{-\pi\, l\cdot Yl - 2\pi\,\Delta_\delta^{-1} l_0\cdot Yl}$. Secondly Y is positive definite so there exists a $c' > 0$ such that $\left\langle \frac{l}{\|l\|}, Y\frac{l}{\|l\|}\right\rangle > c'$ for all $l \neq 0$, so $\langle l, Yl\rangle > c'\|l\|^2$. Thirdly since $l \mapsto \left|\Delta_\delta^{-1} l_0\cdot Yl\right|$ is continuous as a function from the unit sphere in $\mathbb{R}^n$ the image is bounded, which give the existence of a c'' such that $|\Delta_\delta l_0\cdot Yl| < c''\|l\|$. Since for $l \in \mathbb{Z}^n$ $\|l\|^2 \geq \|l\|$ we have

$$-c''\|l\|^2 < \Delta_\delta^{-1} l_0\cdot Yl < c''\|l\|^2.$$

Putting this together we get

$$|b_l| < e^{-c'\|l\|^2 - 2\pi c''\|l\|^2} = e^{-(c'+2\pi c'')\|l\|^2},$$

which by the M-test forces θ_{l_0} to converge on compact subsets of $\mathbb{C}^n$. □

Remark that if $c_1(\mathcal{L})$ is a principal polarization of M, $H^0(M,\mathcal{L})$ is 1-dimensional and generated by the section $\tilde{\theta}$ with corresponding

$$\theta(z) = \sum_{l\in\mathbb{Z}^n} e^{i\pi l\cdot Zl} e^{2\pi i\, l\cdot z}$$

satisfying the functional equations $\theta(z+\lambda_\alpha) = \theta(z)$ and $\theta(z+\lambda_{\alpha+n}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}\theta(z)$. These functions are called the Riemann theta functions of the principal polarized abelian variety (M,ω). The theta functions are smooth functions in Z, and will be denoted by $\theta(z,Z)$.

5.4 Multipliers and sections for powers of line bundles

In Chapter 6 we will concentrate on a principal polarized abelian variety, (M_Z,ω), where we emphasize the dependence of Z. We will study the vector space of holomorphic sections of fixed power, k, of a line bundle $\mathcal{L}_Z \to M_Z$ given by the multipliers in Lemma 5.4. Taking tensor powers of the line bundle amount to taking powers of the multipliers, i.e. multipliers for $\mathcal{L}_Z^k \to M_Z$ are

$$e_{\lambda_\alpha}(z) = 1 \quad e_{\lambda_{n+\alpha}}(z) = e^{-2\pi i k z_\alpha - \pi i k Z_{\alpha\alpha}}.$$

The space of holomorphic section of $\mathcal{L}_Z^k \to M_Z$ is then

$$H^0(M_Z, \mathcal{L}_Z^k) = \{\theta \in \mathcal{O}(V_Z) \,|\, \theta(z+\lambda_\alpha) = \theta(z),\, \theta(z+\lambda_{n+\alpha}) = e^{-2\pi i k z_\alpha - \pi i k Z_{\alpha\alpha}}\theta(z)\}.$$

A basis for this space is provided by the theta functions with rational characteristics. These are just translates of Riemann's theta functions and then multiplied by an elementary exponential factor.

$$\theta_\alpha^{(k)}(z, Z) = e^{\pi i k \alpha \cdot Z\alpha + 2\pi i k \alpha \cdot z}\theta(kz + kZ\alpha, kZ) = \sum_{l \in \mathbb{Z}^n} e^{\pi i k (l+\alpha)\cdot Z(l+\alpha)} e^{2\pi i k (l+\alpha)\cdot z},$$

where $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$.

These functions satisfy the relations to be in $H^0(M_Z, \mathcal{L}_Z^k)$. This can be seen by writing out the identities. Theta functions for different α and β are clearly linearly independent, and they constitute a basis for $H^0(M_Z, \mathcal{L}_Z^k)$. This can be proven by a variation of the argument used to prove that θ is characterized up to a scalar by its functional equation: one makes a Fourier expansion of f for the lattice Λ in V, and expresses the remaining functional equations as recursion relations on the Fourier coefficients. These leave only k^n coefficients to be determined.

A last but important property about $\theta_\alpha^{(k)}$ is that it satisfies a heat equation

$$\frac{\partial \theta_\alpha^{(k)}}{\partial Z_{ij}} = \frac{1}{4\pi i k}\frac{\partial^2 \theta_\alpha^{(k)}}{\partial z_i \partial z_j},$$

which can be shown be direct computation of both sides of the equation.

To summarize we have the following

Theorem 5.6. *Let (M_Z, ω) be a principal polarized abelian variety, and $\mathcal{L}_Z \to M_Z$ a line bundle given by the multipliers in Lemma 5.4. Then*

(i) The basis of $H_Z^{(k)} = H^0(M_Z, \mathcal{L}_Z^k)$ are the theta functions with rational characteristics

$$\theta_\alpha^{(k)}(z, Z) = \sum_{l \in \mathbb{Z}^n} e^{\pi i k (l+\alpha)\cdot Z(l+\alpha)} e^{2\pi i k (l+\alpha)\cdot z},$$

with $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$.

(ii) $\dim H_Z^{(k)} = k^n$.

(iii) $\mathcal{L}_Z \to M_Z$ is ample, $\mathcal{L}_Z^3 \to M_Z$ is very ample.

(iv) $\theta_\alpha^{(k)}(z, Z)$ converges absolutely and uniformly on compact sets.

(v) $\theta_\alpha^{(k)}$ satisfies a heat equation

$$\frac{\partial \theta_\alpha^{(k)}}{\partial Z_{ij}} = \frac{1}{4\pi i k}\frac{\partial^2 \theta_\alpha^{(k)}}{\partial z_i \partial z_j}$$

6

Berezin–Toeplitz Deformation Quantization of Abelian Varieties

In this chapter we present partial results on obtaining a generalized version of Berezin-Toeplitz Deformation Quantization of an abelian variety. In the first part of this chapter we follow [3] to obtain explicit formulas for the Berezin–Toeplitz star product on $C^\infty(M)$. Our idea is to extend this star product to a star product on $A = \bigoplus_{k=0}^{\infty} C^\infty(M, \mathcal{L}^k)$. We begin with a recap of [3] and reproving some crucial theorems. Afterwards we use these results to do calculations to obtain indications of a star product lurking behind the scenes and finally putting forth a conjecture.

6.1 Berezin–Toeplitz Deformation Quantization

In this chapter M is a torus as in Chapter 5, i.e. $M = V/\Lambda$ where V is a real vector space with a symplectic structure ω, and Λ is a discrete lattice in V of maximal rank. We use the results of Chapter 5 to give complex coordinates $z = x + Zy$ on M, where $Z \in \mathbb{H}$ is a symmetric $n \times n$-matrix with positive definite imaginary part. $\mathbb{H}$ parametrizes the space of compatible complex structures on M, such that for each $Z \in \mathbb{H}$, $(M, \omega, I(Z))$ is a principal polarized abelian variety, where the complex structure associated to Z is denoted by $I(Z)$.

For each $Z \in \mathbb{H}$ let us explicitly construct a prequantum line bundle on $M_{I(Z)}$. As we saw in Chapter 5 we only need to specify a set of multipliers and a Hermitian structure. The multipliers depend on Z and are

$$\begin{aligned} e_{\lambda_i}(z) &= 1, & i &= 1, \dots, n, \\ e_{\lambda_{n+i}}(z) &= e^{-2\pi i z_i - \pi i Z_{ii}}, & i &= 1, \dots, n, \end{aligned}$$

where $\{\lambda_i\} \subset \Lambda$ is an integral basis for Λ. The constructed line bundle is denoted $\mathcal{L}_Z$. If we define $h(z) = e^{-2\pi y \cdot Y y}$, where $Z = X + iY$, it will define a Hermitian structure on $V \times \mathbb{C}$ by $h(z) \cdot \langle \cdot, \cdot \rangle_{\mathbb{C}}$ where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is the standard inner product on $\mathbb{C}^n$. As we saw in Chapter 5 this function satisfies the functional equation

$$h(z + \lambda) = \frac{1}{|e_\lambda(z)|^2} h(z),$$

and the inner product on $V \times \mathbb{C}$ is invariant under the action of Λ and hence induces a Hermitian structure $\langle \cdot, \cdot \rangle$ on $\mathcal{L}$. We have furthermore seen that this line bundle $(\mathcal{L}_Z, \langle \cdot, \cdot \rangle)$ has curvature $-2\pi i \omega$, and hence is a prequantum line bundle in the normalization $\hbar = 1$

As we have seen the space of holomorphic sections of $\mathcal{L}_Z^k$ has dimension k^n, and fit together as fibers of a vector bundle over $\mathbb{H}$ by letting $H^{(k)}_{I(Z)} = H^0(M_Z, \mathcal{L}_Z^k)$. The L^2-inner product on $H^0(M_Z, \mathcal{L}_Z^k)$ is given by

$$(s_1, s_2) = \int_{M_Z} s_1(z)\overline{s_2(z)} h(z) dx dy,$$

for $s_1, s_2 \in H^0(M_Z, \mathcal{L}_Z^k)$. We have also seen that a basis for the space of sections are the *theta functions*, $\theta_\alpha^{(k)}(z, Z)$ where $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$. We also saw that the theta functions satisfies the heat equation,

$$\frac{\partial \theta_\alpha^{(k)}}{\partial Z_{ij}} = \frac{1}{4\pi i k} \frac{\partial^2 \theta_\alpha^{(k)}}{\partial z_i \partial z_j}.$$

We can geometrically interpret this differential equation as defining a connection $\tilde{\nabla}$ in the trivial $C^\infty(\mathbb{C}^n)$-bundle over $\mathbb{H}$, by

$$\tilde{\nabla}_{\frac{\partial}{\partial Z_{ij}}} = \frac{\partial}{\partial Z_{ij}} - \frac{1}{4\pi i k}\frac{\partial^2}{\partial z_i \partial z_j} \quad \text{and} \quad \tilde{\nabla}_{\frac{\partial}{\partial \bar{Z}_{ij}}} = \frac{\partial}{\partial \bar{Z}_{ij}}.$$

The coordinates $z = x + Zy$ identify $H^0(M_Z, \mathcal{L}_Z^k)$ as a subspace of $C^\infty(\mathbb{C}^n)$ and $H^{(k)}$ as a subbundle of the trivial $C^\infty(\mathbb{C}^n)$-bundle on $\mathbb{H}$. This bundle is preserved by $\tilde{\nabla}$ and hence induces a connection ∇ in $H^{(k)}$. The covariant constant sections of $H^{(k)}$ with respect to ∇ will, under the embedding induced by the coordinates, be identified with the theta functions. Since now ∇ has a global frame of covariantly constant sections it is flat. Remember that $\mathbb{H}$ is contractible, so since parallel transport with a flat connection only depend on the homotopy class of the curve transported along, we get a canonical way to identify all $H^0(M_Z, \mathcal{L}_Z^k)$, and hence there is no ambiguity in defining the quantum space of geometric quantization to be $H^0(M_Z, \mathcal{L}_Z^k)$.

Our goal is to express Toeplitz operators associated to a section of a fixed power of the line bundle $\mathcal{L}$, in terms of a orthonormal basis of $H^0(M_Z, \mathcal{L}_Z^k)$. The theta functions provide us with such a basis.

Lemma 6.1. *The theta functions $\theta_\alpha^{(k)}(z, Z)$, $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$ define an orthonormal basis with respect to the inner product on $H^0(M_Z, \mathcal{L}_Z^k)$ defined by*

$$(s_1, s_2)_Y = (s_1, s_2)\sqrt{2^n k^n \det Y},$$

where $Y = \operatorname{Im} Z$.

Proof. This is a straight forward computation of the inner product $(\theta_\alpha^{(k)}, \theta_\beta^{(k)})$. First we use the absolute convergence of the theta functions to switch the integration and summations

$$(\theta_\alpha^{(k)}, \theta_\beta^{(k)}) = \sum_{l,m \in \mathbb{Z}^n} \int_{M_Z} e^{\pi i k(l+\alpha)\cdot Z(l+\alpha)} e^{-\pi i k(m+\beta)\cdot \bar{Z}(m+\beta)} e^{2\pi i k(l+\alpha)\cdot z} e^{-2\pi i k(m+\beta)\cdot \bar{z}} \\ e^{-2\pi k y \cdot Y y} dx dy$$

We consider the x and y-integration separately. The integrand in the x-integration is

$$e^{2\pi i k(l+\alpha)\cdot x} e^{-2\pi i k(m+\beta)\cdot x} = \Pi_{i=1}^n e^{2\pi i k(l_i + \alpha_i - m_i - \beta_i)x_i},$$

and since each factor is integrated over $[0,1]$ we get 1 if $l_i + \alpha_i = m_i + \beta_i$ for all i, and 0 otherwise. Since we for two different l and m's have the same α_i's and β_i's the integral is 1 if and only if $m = l$ and $\alpha = \beta$. Consequently we can calculate the remaining part of the integral as a Gaussian over $\mathbb{R}^n$

$$
\begin{aligned}
(\theta_\alpha^{(k)}, \theta_\beta^{(k)}) &= \delta_{\alpha,\beta} \sum_l \int_{[0,1]^n} e^{\pi i k(l+\alpha)\cdot(Z-\bar{Z})(l+\alpha)} e^{2\pi i k(l+\alpha)\cdot(Z-\bar{Z})y} e^{-2\pi k y\cdot Yy} dy \\
&= \delta_{\alpha,\beta} \sum_l \int_{[0,1]^n} e^{-2\pi k((l+\alpha)\cdot Y(l+\alpha)+2(l+\alpha)\cdot Yy+y\cdot Yy)} dy \\
&= \delta_{\alpha,\beta} \sum_l \int_{[0,1]^n} e^{-2\pi k(l+\alpha+y)\cdot Y(l+\alpha+y)} dy \\
&= \delta_{\alpha,\beta} \int_{\mathbb{R}^n} e^{-2\pi k(\alpha+y)\cdot Y(\alpha+y)} dy \\
&= \delta_{\alpha,\beta} \int_{\mathbb{R}^n} e^{-2\pi k y\cdot Yy} dy \\
&= \delta_{\alpha,\beta} \frac{1}{\sqrt{2^n k^n \det(Y)}}
\end{aligned}
$$

where the second equality is symmetry of Y, fifth equality is translation invariance of the Lebesgue measure and last equality is a standard formula for Gaussian integrals since Y is positive definite. □

The metric $(\cdot,\cdot)_Y$ on $H^{(k)}$ is compatible with ∇ since the theta functions are an orthonormal basis with respect to $(\cdot,\cdot)_Y$ and are covariantly constant with respect to ∇.

Let $(r,s) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and consider the function $F_{r,s} \in C^\infty(M)$ given in (x,y)-coordinates by

$$F_{r,s}(x,y) = e^{2\pi i(x\cdot r + s\cdot y)}.$$

We have previously defined Toeplitz operators associated to a function $f \in C^\infty(M)$, $T_f^{(k)} : H^0(M_Z, \mathcal{L}_Z^k) \to H^0(M_Z, \mathcal{L}_Z^k)$. We shall now explicitly compute the matrix coefficients of these operators in terms of the basis consisting of theta functions.

To get our hands on the matrix coefficients we only need to calculate $(F_{r,s}\theta_\alpha^{(k)}, \theta_\beta^{(k)})$, since this indeed is $(T_{F_{r,s}}^{(k)})_{\beta\alpha}$. This is essentially done in the exact same way as in Lemma 6.1, where we used absolute convergence of the theta functions to interchange summation and integration. Instead of the x-integral being 1 when $\alpha = \beta$ we have a contribution from $F_{r,s}$, so the integral is 1 if $\alpha - \beta = -[\frac{r}{k}]$, where $[\frac{r}{k}]$ is the residue class of $\frac{r}{k}$ mod $\mathbb{Z}^n$ – otherwise the integral is 0. The remaining single sum of exponentials are treated in the exact same way as above, and rewrites to a single sum over $\mathbb{R}^n$ – only slightly more involved. Evaluation of this integral give

$$(F_{r,s}\theta_\alpha^{(k)}, \theta_\beta^{(k)})_Y = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot\bar{Z}r} e^{-2\pi i s\cdot\alpha} e^{-\pi^2(s-\bar{Z}r)\cdot(2\pi kY)^{-1}(s-\bar{Z}r)}. \tag{6.1}$$

This is a rather lengthy expression and it turns out that if we rescale the phase functions by an appropriate constant depending on r, s, Z, k we get a truly nice expression. The scaling factor is

$$f(r,s,Z)(k) = e^{\frac{\pi}{2k}(s-Xr)\cdot Y^{-1}(s-Xr)} e^{\frac{\pi}{2k} r\cdot Yr},$$

and defining $\hat{F}_{r,s} = f(r,s,Z)(k)F_{r,s}$. Then

$$(T_{\hat{F}_{r,s}}^{(k)})_{\beta\alpha} = (T_{f(r,s,Z)(k)F_{r,s}}^{(k)})_{\beta\alpha} = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot s} e^{-2\pi i s\cdot\alpha}. \tag{6.2}$$

Remark 6.2. Since $T_{F_{r,s}}^{(k)}$ does depend on Z they are not covariant constant with respect to the connection ∇^e induced in the endomorphism bundle. However we note that the twisted Toeplitz operators $\hat{T}_{F_{r,s}}^{(k)} = T_{f(r,s,Z)(k)F_{r,s}}^{(k)}$ does not depend on Z and are therefore covariant constant with respect to ∇^e.

With these twisted Toeplitz-operators $\hat{T}^{(k)}_{F_{r,s}}$ we prove equivalence of the Berezin–Toeplitz star product $\star_Z$ and the Moyal–Weyl product $*$ on $C^\infty(M)$.

Theorem 6.3 ([3](Theorem 5))**.** *Let $\star_Z$ be the star product obtained by applying Berezin–Toeplitz deformation quantization to M_Z. Then for $f,g \in C^\infty(M)$ we have that*

$$E_Z^{-1}(E_Z(f) \star_Z E_Z(g)) = f * g,$$

where $*$ is the Moyal-Weyl product and $E_Z = e^{\frac{1}{2k}\Delta_Z}$ an exponential of the Hodge Laplacian Δ_Z associated to the metric $g_Z(\cdot,\cdot) = 4\pi\omega(\cdot, I(Z)\cdot)$.

To prove this theorem we need the following proposition, which give an explicit calculation of the Hodge Laplacian Δ_Z.

Proposition 6.4 ([3](Proposition 1))**.** *Let Δ_Z be the Laplace operator given by the metric $g_Z(\cdot,\cdot) = 4\pi\omega(\cdot, I(Z)\cdot)$ on M_Z. Then*

$$e^{\frac{1}{2k}\Delta_Z} F_{r,s} = f(r,s,Z)(k) F_{r,s}.$$

Proof. We begin by recalling that the Hodge Laplacian on functions are given globally by $\Delta_Z f = -*d*df$, where $*$ is the Hodge star, and by Lemma 1.14 it can be calculated locally as

$$\Delta_Z f = -\frac{1}{\sqrt{\det g}}\Bigg(\sum_{ij} \frac{\partial}{\partial x_i}\big(g^{ij}\sqrt{\det g}\frac{\partial f}{\partial x_j}\big)\Bigg).$$

In this case the metric is constant on M_Z and we have

$$\Delta_Z = -\sum_{ij} g^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f.$$

In the case where we use Euclidean coordinates on M it is an easy calculation to determine g and from this determine Δ_Z to be

$$\Delta_Z = -\frac{1}{4\pi}\Bigg((\frac{\partial}{\partial y} - X\frac{\partial}{\partial x}) \cdot Y^{-1}(\frac{\partial}{\partial y} - X\frac{\partial}{\partial x}) + \frac{\partial}{\partial x} \cdot Y\frac{\partial}{\partial x}\Bigg).$$

We compute that

$$(\frac{\partial}{\partial y} - X\frac{\partial}{\partial y}) \cdot Y^{-1}(\frac{\partial}{\partial y} - X\frac{\partial}{\partial x})F_{r,s}(x,y) = -4\pi^2\Bigg((s - Xr) \cdot Y^{-1}(s - Xr)\Bigg)F_{r,s}(x,y),$$

and that

$$\frac{\partial}{\partial x} \cdot Y\frac{\partial}{\partial x}F_{r,s}(x,y) = -4\pi^2(r \cdot Yr)F_{r,s}(x,y).$$

Hence we obtain the desired formula. □

Proof of Theorem 6.3. Define $\hat{T}^{(k)}_{F_{r,s}} = T^{(k)}_{e^{\frac{1}{2k}\Delta_Z}F_{r,s}} : H^0(M_Z, \mathcal{L}_Z^k) \to H^0(M_Z, \mathcal{L}_Z^k)$. Then

$$(\hat{T}^{(k)}_{F_{r,s}})_{\beta\alpha} = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot s} e^{-2\pi i s\cdot\alpha}.$$

We then compute the matrix coefficients of the product of two of these Toeplitz operators

$$\begin{aligned}
(\hat{T}^{(k)}_{F_{r,s}}\hat{T}^{(k)}_{F_{t,u}})_{\beta\alpha} &= \sum_{\varphi} (\hat{T}^{(k)}_{F_{r,s}})_{\beta\varphi}(\hat{T}^{(k)}_{F_{t,u}})_{\varphi\alpha} \\
&= \sum_{\varphi} \delta_{\varphi-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot s} e^{-2\pi i s\cdot\varphi} \delta_{\alpha-\varphi,-[\frac{t}{k}]} e^{-\frac{\pi i}{k} t\cdot u} e^{-2\pi i u\cdot\alpha} \\
&= \delta_{\alpha-\beta,-[\frac{r+t}{k}]} e^{-\frac{\pi i}{k}(r\cdot s + t\cdot u)} e^{-2\pi i (s+u)\cdot\alpha} e^{-\frac{2\pi i}{k} s\cdot t}
\end{aligned}$$

We also know that

$$(\hat{T}^{(k)}_{F_{r+t,s+u}})_{\beta\alpha} = \delta_{\alpha-\beta,-[\frac{r+t}{k}]} e^{-\frac{\pi i}{k}(r+t)\cdot(s+u)} e^{-2\pi i(s+u)\cdot\alpha}.$$

Combining the above we get that

$$(\hat{T}^{(k)}_{F_{r,s}} \hat{T}^{(k)}_{F_{t,u}})_{\beta\alpha} = e^{\frac{\pi i}{k}(r\cdot u - s\cdot t)} (\hat{T}^{(k)}_{F_{r+t,s+u}})_{\beta\alpha}.$$

So substituting h for $\frac{1}{k}$ we get

$$E_Z(F_{r,s}) \star_Z E_Z(F_{t,u}) = e^{\pi i h(r\cdot u - s\cdot t)} E_Z(F_{r+t,s+u}) = E_Z(F_{r,s} * F_{t,u}),$$

since an easy calculation shows that $F_{r,s} * F_{t,u} = e^{\pi i h(r\cdot u - s\cdot t)} F_{r+t,s+u}$.

The phase functions provide a Fourier basis for the smooth functions on M, thus we have proved the equivalence of $\star_Z$ and $*$. $\square$

The Berezin–Toeplitz star product is by Karabegov–Schlichenmaier [21] a differential star product. This is verified in this case by using Theorem 2.11 and the uniqueness of Theorem 2.12, and the differential operators turn out to be

$$D = -\frac{1}{2\pi i}\frac{\partial}{\partial z} \otimes (Z - \bar{Z})\frac{\partial}{\partial \bar{z}},$$

that is

$$f \star_Z g = \mu \circ e^{\frac{1}{k}D}(f \otimes g).$$

This is easily verified by direct calculations.

6.1.1 Generalizing Berezin–Toeplitz

In the same fashion as we define Toeplitz operators for a function $f \in C^\infty(M)$ we define Toeplitz operators associated to a section s of a fixed power, l, of the prequantum line bundle $s \in C^\infty(M_Z, \mathcal{L}_Z^l)$, $T_s^{(k+l)} : H^0(M_Z, \mathcal{L}_Z^k) \to H^0(M_Z, \mathcal{L}_Z^{k+l})$. We do this by composing the multiplication operator M_s with the projection $\pi^{(k+l)}$, $T_s^{k+l} = \pi^{(k+l)} \circ M_s$.

We first investigate wether it is possible to make a star product on $A = \bigoplus_{k=0}^\infty C^\infty(M_Z, \mathcal{L}_Z^l)$ by analyzing the product of two such Toeplitz operators.

The matrix coefficients of $T^{(k+l)}_{F_{r,s}\theta_\gamma^{(l)}} : H^0(M_Z, \mathcal{L}_Z^k) \to H^0(M_Z, \mathcal{L}_Z^{l+k})$ in the basis of theta functions are as above $(F_{r,s}\theta_\gamma^{(l)}\theta_\alpha^{(k)}, \theta_\beta^{(k+l)})$. The product $\theta_\gamma^{(l)}\theta_\alpha^{(k)}$ is a holomorphic section of $\mathcal{L}^{k+l} \to M$, and it would be helpful to know the ring structure of $R(M_Z, \mathcal{L}_Z) = \bigoplus_{k=0}^\infty H^0(M_Z, \mathcal{L}_Z^k)$, since we could then expand $\theta_\gamma^{(k)}\theta_\alpha^{(l)}$ as a sum of theta functions at level $k + l$.

The ring $R(M_Z, \mathcal{L}_Z)$ is a also known as the homogeneous coordinate ring of the abelian variety M_Z. The structure coefficients are calculated in [24, Chap. 10].

Proposition 6.5 ([24]). *Let M_Z be an abelian variety, and $\mathcal{L}_Z$ an ample line bundle of degree 1. Then*

$$\theta_\gamma^{(l)}(z, Z)\theta_\alpha^{(k)}(z, Z) = \sum_{\eta \in \frac{1}{k+l}\mathbb{Z}^n/\mathbb{Z}^n} \theta^{(k+l)}_{\frac{l\gamma - k\alpha}{k+l}+\eta}(0, Z)\, \theta^{(k+l)}_{\frac{l\gamma+k\alpha}{k+l}+l\eta}(z, Z), \tag{6.3}$$

and the structure coefficients are theta functions evaluated at 0.

With the above proposition at hand we can easily calculate the matrix coefficients of the Toeplitz operator $T^{(k+l)}_{F_{r,s}\theta^{(l)}_\gamma}$

$$
\begin{aligned}
(T^{(k+l)}_{F_{r,s}\theta^{(l)}_\gamma})_{\beta\alpha} &= (F_{r,s}\theta^{(l)}_\gamma \theta^{(k)}_\alpha, \theta^{(k+l)}_\beta)_Y \\
&= \sum_{\eta\in\frac{1}{k+l}\mathbb{Z}^n/\mathbb{Z}^n} \theta^{(k+l)}_{\frac{l\gamma-k\alpha}{k+l}+\eta}(0,Z)\delta_{\frac{l\gamma+k\alpha}{k+l}+l\eta-\beta,-[\frac{r}{k+l}]} e^{-\frac{\pi i}{k+l}r\cdot\bar{Z}r} \qquad (6.4)\\
&\qquad e^{-\pi^2(s-\bar{Z}r)\cdot(2\pi(k+l)Y)^{-1}(s-\bar{Z}r)} e^{-2\pi i s\cdot(\frac{l\gamma+k\alpha}{k+l}+l\eta)}.
\end{aligned}
$$

To construct the Berezin–Toeplitz star product we should make an asymptotic expansion of the product of two such operators. The coefficients are then the bidifferential operators needed in the definition of the star product. Unfortunately multiplication of $T^{(k+l_1+l_2)}_{F_{t,u}\theta^{(l_2)}_\eta}$ and $T^{(k+l_1)}_{F_{r,s}\theta^{(l_1)}_\gamma}$ does not give anything from which it is obvious what to do. A better approach would be to set $l_2 = 0$, and see if we can obtain reasonable results, and then afterwards include the section.

By associativity of Toeplitz operators, and by the fact that $T^{(k+l)}_{F_{r,s}\theta^{(l)}_\gamma} = T^{(k+l)}_{F_{r,s}} T^{(k+l)}_{\theta^{(l)}_\gamma}$ since $\theta^{(l)}_\gamma$ is holomorphic, we have

$$
T^{(k+l)}_{\hat{F}_{t,u}} T^{(k+l)}_{\hat{F}_{r,s}\theta^{(l)}_\gamma} = (T^{(k+l)}_{\hat{F}_{t,u}} T^{(k+l)}_{\hat{F}_{r,s}}) T^{(k+l)}_{\theta^{(l)}_\gamma} = e^{\frac{\pi i}{k+l}(s\cdot t - r\cdot u)} T^{(k+l)}_{\hat{F}_{r+t,s+u}\theta^{(l)}_\gamma},
$$

where the last equality follows from the proof of Theorem 6.3. From the above we see that there is not a big difference to include a section on the first factor. We might have the desired expansion of Toeplitz-operators, which should give us the product. This leads us to conjecture that the new product should be

$$
f \star_Z g = \mu \circ e^{\frac{1}{k+l}D}(f\otimes s) \quad \text{where} \quad D = -\frac{1}{2\pi i}\frac{\partial}{\partial z}\otimes (Z-\bar{Z})\nabla_{\frac{\partial}{\partial \bar{z}}}.
$$

We would furthermore like to find a transformation of this product, making the new product equivalent to a Z-independent product. This leads us to the following

Conjecture 6.6. For a function $f \in C^\infty(M)$ and a section $s \in C^\infty(M_Z, \mathcal{L}^l_Z)$ we have the following equivalence of products

$$
E_Z(f) \star_Z A_Z(s) = A_Z(f * s),
$$

where $*$ is a generalized Moyal–Weyl product defined by the operators $P = \frac{\partial}{\partial x}\otimes\nabla_{\frac{\partial}{\partial y}}$ and $P^* = \frac{\partial}{\partial y}\otimes\nabla_{\frac{\partial}{\partial x}}$. When we write $\frac{\partial}{\partial x}\otimes\nabla_{\frac{\partial}{\partial y}}$ we mean the inner product between the vectors $\frac{\partial}{\partial x}$ with $\frac{\partial}{\partial x_i}$ in the i'th entry and the vector $\nabla_{\frac{\partial}{\partial y}}$ with $\nabla_{\frac{\partial}{\partial y_i}}$ in the i'th entry,

$$
\frac{\partial}{\partial x}\otimes\nabla_{\frac{\partial}{\partial y}} := \sum_i \frac{\partial}{\partial x_i}\otimes\nabla_{\frac{\partial}{\partial y_i}}.
$$

Finally A_Z is a formal differential operator like the exponential of the Laplacian for functions E_Z, now we just use the Laplacian defined from the Hermitian line bundle $\mathcal{L}^l \to M$, which on a Kähler manifold is

$$
\Delta^l_Z = -\sum_{ij} g^{ij}(x)(\nabla^l_{\frac{\partial}{\partial x_i}}\nabla^l_{\frac{\partial}{\partial x_j}} - \sum_k \Gamma^k_{ij}\nabla^l_{\frac{\partial}{\partial x_k}}),
$$

where $\{x_i\}$ are arbitrary coordinates on M, ∇^l the connection in $\mathcal{L}^l \to M$ whose curvature is $\frac{i}{2\pi}l\omega$, and Christoffel symbols are structure constants from the Levi-Cevitta connection in the tangent bundle. In particular we define $A_Z = e^{\frac{1}{2(k+l)}\Delta^l_Z}$.

Remark 6.7. By simply checking terms of each of the expressions, it can be shown that the conjecture is true up to first order in $\frac{1}{k}$. This is an easy but tedious exercise and will not be done here.

6.1.2 Future work

One part of the future work is to verify the above conjecture. One way of going about this is by finding eigensections of Δ^l_Z, and by using these we could hope to copy the proof of Theorem 6.3 directly. We are currently on the hunt for these sections.

Another part is to not only find the above product, but try to create a deformation quantization of $\bigoplus_{k=0}^{\infty} C^\infty(M, \mathcal{L}^k)$. The idea is to use the coherent states and the symbol calculus for Toeplitz operators by Guillemin and Boutet de Monvel, to obtain a star product in the same way as Schlichenmaier does in [27]. This is still work in progress.

The product $\star_Z$ in Conjecture 6.6 is what we expect should partly the deformation quantization on $\bigoplus_{k=0}^{\infty} C^\infty(M, \mathcal{L}^k)$. If the conjecture is true, we have a chance of creating a Z-independent star product on $\bigoplus_{k=0}^{\infty} C^\infty(M, \mathcal{L}^k)$, by using A_Z as the formal trivialization in the language of formal Hitchin connections from [2].

7

Geometric quantization of abelian varieties

In the previous Chapter we only stated the most necessary facts about geometric quantization of abelian varieties, which was needed for the Berezin–Toeplitz deformation quantization. In this chapter we will go deeper into geometric quantization and develop an explicit expression for the Hitchin connection in the bundle $H^{(k)} \to \mathbb{H}$. We use Andersen's construction in Theorem 3.6. This is opposed to the connection, ∇, given by the heat equation. Ee explicitly calculate the parallel transport given by the connection from the heat equation. In the last section we investigate the asymptotics of the Hilbert–Schmidt norm of Toeplitz-operators.

7.1 Hitchin Connection

Following [7] in Chapter 3 we gave Andersens geometric construction of the Hitchin connection. We will use the construction to get explicit formulas for the Hitchin connection, and calculate its curvature to see if it is actually flat as is the case of the connection from the heat equation, ∇. Surprisingly it is not. At this point we should remark that even though $H^1(M, \mathbb{R}) \neq 0$ for M a torus, the construction give us a Hitchin connection. The reason is that the requirement $H^1(M, \mathbb{R}) = 0$ is only used to prove Lemma 3.8, and this lemma is irrelevant in our torus case. We explain this below.

As we saw in Section 3.2 the Hitchin connection in $H^{(k)}$ is (if it exists) given by

$$\hat{\nabla}_V = \nabla^t_V + u(V),$$

where V is any vector field on $\mathcal{T}$, the space which parametrizes the Kähler structures. ∇^t is the trivial connection and u a smooth one-form with values in the vector space of differential operators on $C^\infty(M, \mathcal{L}^k)$. We also saw above that u has a very specific form, namely

$$u(V) = \frac{1}{\pi(8k+4n)}\big(\Delta_{G(V)} + 2\nabla_{G(V)dF} + 8\pi k V'[F]\big), \tag{7.1}$$

where the first Chern class of (M, ω) is $n[\omega]$, F is the Ricci-potential and G a symmetric bivector field defined by contraction with the symplectic form.

The torus case is very favorable since the metric is flat. This implies that the first Chern class i 0, and hence $n = 0$ and likewise the Ricci potential. The one form u is particularly simple,

$$u(V) = \frac{1}{8\pi k}\Delta_{G(V)}.$$

By Lemma 3.7 we immediately get that $u(V)$ satisfy Equation (3.1) – provided the family of Kähler structures are rigid. In Equation (7.2) we see that the family is constant on M and therefore is rigid.

Since $G(V)$ is defined by

$$G(V)\cdot\omega = V'[I],$$

where I is the complex structure, we need to calculate I and especially the derivative of I with respect to a vector field on $\mathbb{H}$. We have globally defined frame of vector fields $\frac{\partial}{\partial Z_{ij}}$ for the holomorphic tangent bundle to $\mathbb{H}$, so by linearity we only need to solve

$$G(\frac{\partial}{\partial Z_{ij}})\cdot\omega = \frac{\partial I(Z)}{\partial Z_{ij}}.$$

The complex structure $I(Z): TM \to TM$ is defined by $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{z}_i}$ being frames of the i and $-i$ eigenspaces of $I(Z)$ respectively, i.e. $I(Z)(\frac{\partial}{\partial z_i}) = i\frac{\partial}{\partial z_i}$ and $I(Z)(\frac{\partial}{\partial \bar{z}_i}) = -i\frac{\partial}{\partial \bar{z}_i}$. We cannot differentiate $I(Z)$ in this basis, since the basis depend on Z. We therefore need to rewrite everything in terms of Z-independent quantities, in other words we will express $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{z}_i}$ in terms of $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$.

Rewriting the dz_i's is easy since $z_i = x_i + \sum_{j=1}^n Z_{ij}y_j$, so $dz_i = dx_i + \sum_{j=1}^n Z_{ij}dy_j$. By assuming

$$\frac{\partial}{\partial z_i} = \sum_{j=1}^n a_{ij}\frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial y_j}$$

and solving the equations $\delta_{i,j} = dz_j(\frac{\partial}{\partial z_i})$ and $0 = d\bar{z}_j(\frac{\partial}{\partial z_i})$ we get that $B = (b_{ij}) = \frac{1}{2i}Y^{-1}$ and $A = (a_{ij}) = \frac{i}{2}Y^{-1}\bar{Z}$, in other words

$$\frac{\partial}{\partial z} = \frac{i}{2}Y^{-1}\bar{Z}\frac{\partial}{\partial x} + \frac{1}{2i}Y^{-1}\frac{\partial}{\partial y} = \frac{i}{2}(Y^{-1}X - i)\frac{\partial}{\partial x} + \frac{1}{2i}Y^{-1}\frac{\partial}{\partial y}.$$

In the exact same way you obtain

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2i}Y^{-1}Z\frac{\partial}{\partial x} + \frac{i}{2}Y^{-1}\frac{\partial}{\partial y} = \frac{1}{2i}(Y^{-1}X + i)\frac{\partial}{\partial x} + \frac{i}{2}Y^{-1}\frac{\partial}{\partial y}.$$

Using linearity of $I(Z)$ and $I(Z)(\frac{\partial}{\partial z}) = i\frac{\partial}{\partial z}$ we get by inserting the above

$$\frac{i}{2}(Y^{-1}X - i)I(Z)(\frac{\partial}{\partial x}) + \frac{1}{2i}Y^{-1}I(Z)(\frac{\partial}{\partial y}) = \frac{-1}{2}(Y^{-1}X - i)\frac{\partial}{\partial x} + \frac{1}{2}Y^{-1}\frac{\partial}{\partial y}$$
$$\frac{1}{2i}(Y^{-1}X + i)I(Z)(\frac{\partial}{\partial x}) + \frac{i}{2}Y^{-1}I(Z)(\frac{\partial}{\partial y}) = \frac{-1}{2}(Y^{-1}X + i)\frac{\partial}{\partial x} + \frac{1}{2}Y^{-1}\frac{\partial}{\partial y}.$$

Solving these equations for $I(Z)(\frac{\partial}{\partial x})$ and $I(Z)(\frac{\partial}{\partial y})$ and writing $I(Z)$ as a tensor we get

$$I(Z) = \begin{pmatrix} -Y^{-1}X & -(Y + XY^{-1}X) \\ Y^{-1} & XY^{-1} \end{pmatrix} \tag{7.2}$$

which is true for all $Z = X + iY \in \mathbb{H}$. The bivector field $G(\frac{\partial}{\partial Z_{ij}})$ is calculated by taking the Z_{ij} derivative of $I(Z)$. We can now do this because $I(Z)$ is written in an Z-independent basis.

To this end we need to recall the following derivation property for matrices. If $A = (a_{ij})$ is a symmetric invertible $n \times n$-matrix then

$$\frac{\partial A^{-1}}{\partial a_{ij}} = -A^{-1}\frac{\partial A}{\partial a_{ij}}A^{-1} = -A^{-1}\Delta_{ij}A^{-1},$$

where Δ_{ij} is an $n \times n$-matrix with all entries 0 except the ij'th and ji'th which is 1, if $i \neq j$ and Δ_{ii} is an $n \times n$-matrix with all entries 0 except the ii'th diagonal entry which is 1. This follows easily from $A^{-1}A = Id$. Using this rule and that $Y^{-1} = 2i(Z - \bar{Z})^{-1}$ we get

$$\frac{\partial Y^{-1}}{\partial Z_{ij}} = -\frac{1}{2i} Y^{-1} \Delta_{ij} Y^{-1}.$$

Derivation of the above equations with respect to Z_{ij} becomes rather messy if we do not also require Z to be normal, that is since Z is symmetric $[Z, \bar{Z}] = 0$, which is equivalent to $[X, Y] = 0$, or $[X, Y^{-1}] = 0$. A consequence of this is that everything will commute even $[Y^{-1}, \Delta_{ij}] = 0$ since the imaginary part of derivation of $Z\bar{Z} = \bar{Z}Z$ with respect to Z_{ij} give $Y\Delta_{ij} = \Delta_{ij}Y$, and hence $[Y^{-1}, \Delta_{ij}] = 0$. Written as a tensor

$$\frac{\partial I(Z)}{\partial Z_{ij}} = \frac{1}{2i} Y^{-1} \Delta_{ij} Y^{-1} \begin{pmatrix} \bar{Z} & \bar{Z}^2 \\ -1 & -\bar{Z} \end{pmatrix}.$$

As we should contract this derivative with the symplectic form $\omega = -\frac{1}{2i}\sum_{ij=1}^{n} w_{ij} dz_i \wedge d\bar{z}_j$ where $Y^{-1} = W = (w_{ij})$, we want to know its appearance in the Z-dependent $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ frame. It is clear from above that

$$\frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial z}) = \frac{i}{2} Y^{-1} \bar{Z} \frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial x}) + \frac{1}{2i} Y^{-1} \frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial y}) = 0,$$

and an easy calculation shows that

$$\frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial \bar{z}}) = -Y^{-1} \Delta_{ij} \frac{\partial}{\partial z}.$$

In other words

$$\frac{\partial I(Z)}{\partial Z_{ij}} = \begin{cases} -\sum_{k=1}^{n} (w_{ki} \frac{\partial}{\partial z_j} \otimes d\bar{z}_k + w_{kj} \frac{\partial}{\partial z_i} \otimes d\bar{z}_k) & \text{for } i \neq j \\ -\sum_{k=1}^{n} w_{ki} \frac{\partial}{\partial z_i} \otimes d\bar{z}_k & \text{for } i = j \end{cases}$$

Remark that since $I(Z)^2 = -Id$, $\frac{\partial I(Z)}{\partial Z_{ij}}$ and $I(Z)$ anti-commute. This is clearly reflected in the above expressions for $\frac{\partial I(Z)}{\partial Z_{ij}}$. Now since $G(\frac{\partial}{\partial Z_{ij}})$ is defined by

$$-G(\frac{\partial}{\partial Z_{ij}}) \cdot \frac{1}{2i} \sum_{kl=1}^{n} w_{kl} dz_k \wedge d\bar{z}_l = \frac{\partial I(Z)}{\partial Z_{ij}}$$

it is

$$G(\frac{\partial}{\partial Z_{ij}}) = \begin{cases} 2i \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j} + 2i \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z_i} & \text{for } i \neq j \\ 2i \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_i} & \text{for } i = j. \end{cases}$$

With $G(\frac{\partial}{\partial Z_{ij}})$ being expressed in complex coordinates, we should mentioned that the family of Kähler structures parametrized by $\mathbb{H}$ in the way described above, actually is rigid, i.e. $\bar{\partial}_Z(G(V)_Z) = 0$ for all vector fields V on $\mathbb{H}$. This is clear since $G(\frac{\partial}{\partial Z_{ij}})$ is zero in $\bar{z}_i$ directions and $G(\frac{\partial}{\partial Z_{ij}}) = 0$.

Now by Equation (3.2)

$$\Delta_{G(\frac{\partial}{\partial Z_{ij}})} = \begin{cases} 2i \nabla_{\frac{\partial}{\partial z_i}} \nabla_{\frac{\partial}{\partial z_j}} + 2i \nabla_{\frac{\partial}{\partial z_j}} \nabla_{\frac{\partial}{\partial z_i}} & \text{for } i \neq j \\ 2i \nabla_{\frac{\partial}{\partial z_i}} \nabla_{\frac{\partial}{\partial z_i}} & \text{for } i = j, \end{cases}$$

and since the Hitchin connection is flat in the $\frac{\partial}{\partial \bar{Z}_{ij}}$ direction we have explicitly derived the Hitchin connection

$$\hat{\nabla}_{\frac{\partial}{\partial Z_{ij}}} = \begin{cases} \frac{\partial}{\partial Z_{ij}} - \frac{1}{4\pi ik}\nabla_{\frac{\partial}{\partial z_i}}\nabla_{\frac{\partial}{\partial z_j}} - \frac{1}{4\pi ik}\nabla_{\frac{\partial}{\partial z_j}}\nabla_{\frac{\partial}{\partial z_i}} & \text{for } i \neq j \\ \frac{\partial}{\partial Z_{ij}} - \frac{1}{4\pi ik}\nabla_{\frac{\partial}{\partial z_i}}\nabla_{\frac{\partial}{\partial z_i}} & \text{for } i = j \end{cases}$$

$$\hat{\nabla}_{\frac{\partial}{\partial \bar{Z}_{ij}}} = \frac{\partial}{\partial \bar{Z}_{ij}} \tag{7.3}$$

7.1.1 Curvature of the Hitchin Connection

With the explicit formula for the Hitchin Connection it is possible to explicitly calculate its curvature. We see that it is not 0 as the curvature of the connection induced by the heat equation, but it is rather projectively flat, meaning that the curvature is a two form on $\mathbb{H}$.

Proposition 7.1. *The Hitchin connection $\hat{\nabla}$ is projectively flat.*

Proof. The curvature of $\hat{\nabla}$ is

$$R_{\hat{\nabla}}(U,V)s = \hat{\nabla}_U\hat{\nabla}_V s - \hat{\nabla}_V\hat{\nabla}_U s - \hat{\nabla}_{[U,V]}s,$$

where U, V are vector fields on $\mathbb{H}$ and s a section of $H^{(k)}$. Since $R_{\hat{\nabla}}$ is a $(1,1)$-form we only need to calculate

$$R_{\hat{\nabla}}\Big(\frac{\partial}{\partial \bar{Z}_{kl}}, \frac{\partial}{\partial Z_{ij}}\Big)s = \frac{\partial}{\partial \bar{Z}_{kl}}\hat{\nabla}_{\frac{\partial}{\partial Z_{ij}}}s - \hat{\nabla}_{\frac{\partial}{\partial Z_{ij}}}\frac{\partial}{\partial \bar{Z}_{kl}}s = \left[\frac{\partial}{\partial \bar{Z}_{kl}}, \hat{\nabla}_{\frac{\partial}{\partial Z_{ij}}}\right]s. \tag{7.4}$$

If we assume $i \neq j$ then

$$\left[\frac{\partial}{\partial \bar{Z}_{kl}}, \hat{\nabla}_{\frac{\partial}{\partial Z_{ij}}}\right] = -\frac{1}{4\pi ik}\left(\left[\frac{\partial}{\partial \bar{Z}_{kl}}, \nabla_{\frac{\partial}{\partial z_i}}\nabla_{\frac{\partial}{\partial z_j}}\right] + \left[\frac{\partial}{\partial \bar{Z}_{kl}}, \nabla_{\frac{\partial}{\partial z_j}}\nabla_{\frac{\partial}{\partial z_i}}\right]\right).$$

Since

$$\left[\frac{\partial}{\partial \bar{Z}_{kl}}, \nabla_{\frac{\partial}{\partial z_i}}\nabla_{\frac{\partial}{\partial z_j}}\right] = \left[\frac{\partial}{\partial \bar{Z}_{kl}}, \nabla_{\frac{\partial}{\partial z_i}}\right]\nabla_{\frac{\partial}{\partial z_j}} + \nabla_{\frac{\partial}{\partial z_i}}\left[\frac{\partial}{\partial \bar{Z}_{kl}}, \nabla_{\frac{\partial}{\partial z_j}}\right]$$

we really only need to calculate $[\frac{\partial}{\partial \bar{Z}_{kl}}, \nabla_{\frac{\partial}{\partial z_i}}]$. To do this we find an expression for $\nabla_{\frac{\partial}{\partial z_i}}$.

If we define $\alpha = -\pi i \sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$ it is a potential for the symplectic form $d\alpha = -2\pi i\omega$. Then the connection ∇ in $\mathcal{L}^k \to M$ is defined as $\nabla = d + k\alpha$. That is $\nabla_{\frac{\partial}{\partial z_i}} = \frac{\partial}{\partial z_i} + k\alpha(\frac{\partial}{\partial z_i})$.

Lengthy but easy calculations using expressions for x and y in terms of $z, \bar{z}, Z, \bar{Z}$, show that

$$\alpha = \frac{\pi}{2}\sum_{ij=1}^{n} w_{ij}(z_j d\bar{z}_i - \bar{z}_j dz_i),$$

and hence

$$\nabla_{\frac{\partial}{\partial z_i}} = \frac{\partial}{\partial z_i} - \frac{k\pi}{2}\sum_{j=1}^{n} w_{ij}\bar{z}_j \quad \text{and} \quad \nabla_{\frac{\partial}{\partial \bar{z}_i}} = \frac{\partial}{\partial \bar{z}_i} + \frac{k\pi}{2}\sum_{j=1}^{n} w_{ij}z_j, \tag{7.5}$$

or in vector notation

$$\nabla_{\frac{\partial}{\partial z}} = \frac{\partial}{\partial z} - k\pi i(Z - \bar{Z})^{-1}\bar{z} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial \bar{z}}} = \frac{\partial}{\partial \bar{z}} + k\pi i(Z - \bar{Z})^{-1}z.$$

Using this vector notation a long calculation based on swithching to x, y-coordinates, differentiating and switching back, shows

$$\frac{\partial}{\partial \bar{Z}_{kl}} \nabla_{\frac{\partial}{\partial z}} = \frac{i}{2} Y^{-1} \Delta_{kl} \nabla_{\frac{\partial}{\partial \bar{z}}},$$

and

$$\frac{\partial}{\partial \bar{Z}_{kl}} \nabla_{\frac{\partial}{\partial z_i}} = \begin{cases} \frac{i}{2} w_{ki} \nabla_{\frac{\partial}{\partial \bar{z}_l}} + \frac{i}{2} w_{li} \nabla_{\frac{\partial}{\partial \bar{z}_k}} & \text{for } k \neq l \\ \frac{i}{2} w_{ki} \nabla_{\frac{\partial}{\partial \bar{z}_k}} & \text{for } k = l. \end{cases}$$

Since the commutator $[\frac{\partial}{\partial \bar{Z}_{kl}}, \nabla_{\frac{\partial}{\partial z_i}}]$ acts on holomorphic sections of $H^{(k)}$ this commutator is $\frac{\partial}{\partial \bar{Z}_{kl}} \nabla_{\frac{\partial}{\partial z_i}}$. Each fiber of $H^{(k)}$ consists of holomorphic sections of $\mathcal{L}^k \to M$, and thus every term with $\nabla_{\frac{\partial}{\partial \bar{z}_i}}$ to the very right vanish, i.e. $[\nabla_{\frac{\partial}{\partial \bar{z}_i}}, \nabla_{\frac{\partial}{\partial z_j}}]s = \nabla_{\frac{\partial}{\partial \bar{z}_i}} \nabla_{\frac{\partial}{\partial z_j}} s$. By the explicit formulas in equation (7.5) the commutator is

$$\left[\nabla_{\frac{\partial}{\partial \bar{z}_k}}, \nabla_{\frac{\partial}{\partial z_i}}\right] = -k\pi w_{ki}.$$

Combining these ingredients we get the following expressions for the coefficients of $R_{\hat{\nabla}}$.

$$R_{\hat{\nabla}}(\frac{\partial}{\partial \bar{Z}_{kl}}, \frac{\partial}{\partial Z_{ij}}) = \begin{cases} \frac{1}{4}(w_{ki} w_{lj} + w_{li} w_{kj}) & \text{for } i \neq j, k \neq l \\ \frac{1}{4} w_{ki} w_{kj} & \text{for } i \neq j, k = l \\ \frac{1}{4} w_{ki} w_{li} & \text{for } i = j, k \neq l \\ \frac{1}{8} w_{ki}^2 & \text{for } i = j, k = l. \end{cases}$$

Since $R_{\hat{\nabla}} = \sum_{ijkl} R_{\hat{\nabla}}(\frac{\partial}{\partial \bar{Z}_{kl}}, \frac{\partial}{\partial Z_{ij}}) d\bar{Z}_{kl} \wedge dZ_{ij}$ and all of the coefficients are independent of the coordinates on M, $R_{\hat{\nabla}}$ is a two form on $\mathbb{H}$, in other words, the Hitchin connection $\hat{\nabla}$ is projectively flat. □

Remark 7.2. By the time of differentiation of the complex structure we assumed Z to be normal. In the case of $n = 1$ this is no restriction, but in higher dimensions not every element in $\mathbb{H}$ is normal. The above calculations can be carried out when Z is not normal, but the formulas become extremely long. Dealing with a general Z it is not clear that $\frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial z}) = 0$ – most likely it is not. It eased our calculations a lot that the bivector field $G(\frac{\partial}{\partial Z_{ij}})$ was a really nice expression, we would not get that if Z was not normal.

7.2 Parallel transport

In the description of geometric quantization, the projectively flat Hitchin connection gave a way of identifying quantum spaces associated to two different Kähler structures Z_1, Z_2 by parallel transport along a path from Z_1 to Z_2. Since the space parametrizing the Kähler structures is contractible and the connection is projectively flat we get an identification of all quantum spaces. If we however use the flat connection defined by the heat equation we get an isomorphism between quantum spaces associated to different complex structures, which is independent of the chosen path.

Proposition 7.3. *Parallel transport PT_{∇} induced by the connection from the heat equation ∇ in the bundle $H^{(k)}$, is the Toeplitz operator associated to $f \in C^\infty(\mathbb{H} \times \mathbb{H})$ given by $f(Z_1, Z_2) = \sqrt{\frac{\det(Z_2 - \bar{Z}_2)}{\det(Z_1 - \bar{Z}_2)}}$.*

Proof. The theta functions $\theta_\alpha^{(k)}$ provide a frame of covariant constant sections for ∇, and hence the parallel transport is trivial in this frame. That is $PT_\nabla(Z_1, Z_2) : H_{Z_1}^{(k)} \to H_{Z_2}^{(k)}$ maps $\theta_\alpha^{(k)}(x + Z_1 y, Z_1)$ to $\theta_\alpha^{(k)}(x + Z_1 y, Z_1) \in H_{Z_2}^{(k)}$. We find the map in terms of the basis of theta functions for $H_{Z_1}^{(k)}$ and $H_{Z_2}^{(k)}$, that is we need to calculate the matrix coefficients

$$(\theta_\alpha^{(k)}(x + Z_1 y, Z_1), \theta_\beta^{(k)}(x + Z_2 y, Z_2))_{Y_2}.$$

To ease the calculations we pick another trivialization of $\mathcal{L}_Z^k \to M_Z$, by multiplying sections with the function $g(z, Z) = e^{i\pi k z\cdot(Z-\bar{Z})^{-1}(z-\bar{z})} = e^{i\pi k(x+Zy)\cdot y}$. A basis for $H_Z^{(k)}$ is now $S_\alpha^{(k)}(z, Z) = g(z, Z)\theta_\alpha^{(k)}(z, Z)$. By changing the trivialization we change the multipliers for $\mathcal{L}_Z^k$. A small calculation shows

$$g(z + \lambda_i) = g(z)e^{i\pi k y_i} \quad \text{and} \quad g(z + \lambda_{n+i}) = g(z)e^{2\pi i k z_i + \pi i k Z_{ii}}e^{-i\pi k x_i},$$

for $i = 1, \ldots, n$, and the sections $S_\alpha^{(k)}(z, Z) = g(z, Z)\theta_\alpha^{(k)}(z, Z)$ now satisfy the equations

$$S_\alpha^{(k)}(z + \lambda_i) = e^{\pi i k y_i}S_\alpha^{(k)}(z, Z) \quad \text{and} \quad S_\alpha^{(k)}(z + \lambda_{n+i}) = e^{-\pi i k x_i}S_\alpha^{(k)}(z, Z).$$

Multipliers for $\mathcal{L}_Z^k$ in this trivialization is then $e_{\lambda_i} = e^{\pi i k y_i}$ and $e_{\lambda_{n+i}} = e^{-\pi i k x_i}$. A metric on $\mathcal{L}_Z^k$ should satisfy Equation (5.2). Since $\left|S_\alpha^{(k)}(z + \lambda, Z)\right|^2 = \left|S_\alpha^{(k)}(z, Z)\right|^2$ for all $\lambda \in \Lambda$ and all $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$ we can choose $h(z) = 1$ as the function inducing a Hermitian structure in each fiber of $\mathcal{L}_Z^k$.

$$(S_\alpha^{(k)}(z, Z), S_\beta^{(k)}(z, Z)) = \delta_{\alpha,\beta}\frac{1}{\sqrt{(2k)^n \det(Y)}},$$

since $g(z, Z)\overline{g(z, Z)} = e^{-2\pi k y\cdot Y y}$, and in the same way as above we normalize the inner product to get $(\cdot, \cdot)_Y$ with respect to which $S_\alpha^{(k)}$ is an orthonormal frame.

The objective is to calculate $(S_\alpha^{(k)}(x + Z_1 y, Z_1), S_\beta^{(k)}(x + Z_2 y, Z_2))_{Y_2}$. As usual we use the absolute convergence of theta functions to interchange integration and summation, and since the matrices Z_1, Z_2 do not appear in the x-integrals, these again contribute $\delta_{\alpha,\beta}$. Therefore

$$\begin{aligned}
&(S_\alpha^{(k)}(x + Z_1 y, Z_1), S_\alpha^{(k)}(x + Z_2 y, Z_2))_{Y_2} \\
&= \sqrt{(2k)^n \det Y_2}\sum_{l\in\mathbb{Z}}\int_{[0,1]^n} e^{i\pi k y\cdot(Z_1-\bar{Z}_2)y}e^{\pi i k(l+\alpha)\cdot(Z_1-\bar{Z}_2)(l+\alpha)}e^{2\pi i k(l+\alpha)\cdot(Z_1-\bar{Z}_2)y}dy \\
&= \sqrt{(2k)^n \det Y_2}\sum_{l\in\mathbb{Z}}\int_{[0,1]^n} e^{\pi i k(y+l+\alpha)\cdot(Z_1-\bar{Z}_2)(y+l+\alpha)}dy \\
&= \sqrt{(2k)^n \det Y_2}\int_{\mathbb{R}^n} e^{i\pi k y\cdot(Z_1-\bar{Z}_2)y}dy,
\end{aligned}$$

where second equality is $Z_1 - \bar{Z}_2$ being symmetric, and last equality is translation invariance of the Lebesgue measure.

To evaluate the integral we use the following lemma about complex Gaussian integrals.

Lemma 7.4. *Let H be an $n \times n$ complex matrix with positive definite Hermitian part $\frac{1}{2}(H + H^*)$. Then*

$$\int_{\mathbb{C}^n} e^{-\bar{z}\cdot Hz+\bar{J}\cdot z+J\cdot\bar{z}}(2\pi i)^{-n}dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n = \frac{1}{\det H}e^{\bar{J}\cdot H^{-1}J},$$

and if H furthermore is symmetric

$$\int_{\mathbb{R}^n} e^{-x\cdot Hx}dx = \sqrt{\frac{\pi^n}{\det H}}.$$

It is crucial to convergence of the integral in Lemma 7.4 that H has positive definite Hermitian part. If H is Hermitian this lemma follows immediately by transforming H to diagonal form and evaluating a diagonal integral of the form

$$\int_{\mathbb{C}} e^{-\bar{z}az}\frac{1}{2\pi i}dzd\bar{z} = \int_{\mathbb{R}^2} e^{-a(u^2+v^2)}\frac{1}{\pi}dudv = \frac{1}{a}.$$

The second equality in Lemma 7.4 follows from the first by noting that when $J=0$ and H symmetric, $\bar{z}\cdot Hz = x\cdot Hx + y\cdot Hy$,

$$\frac{1}{\det H} = \int_{\mathbb{C}^n} e^{-\bar{z}\cdot Hz}(2\pi i)^{-n}dzd\bar{z} = \pi^{-n}\int_{\mathbb{R}^n} e^{-x\cdot Hx}dx\int_{\mathbb{R}^n} e^{-y\cdot Hy}dy.$$

In our case $H = -i\pi k(Z_1-\bar{Z}_2)$, and since $\frac{1}{2}(H+\bar{H}) = \pi k(Y_1+Y_2)$ is positive definite we can use Lemma 7.4 to calculate the integral.

$$\begin{aligned}(S_\alpha^{(k)}(x+Z_1y,Z_1), S_\alpha^{(k)}(x+Z_2y,Z_2))_{Z_2} &= \sqrt{\frac{\pi^n(2k)^n\det Y_2}{(-i\pi k)^n\det(Z_1-\bar{Z}_2)}}\\ &= \sqrt{\frac{\det(Z_2-\bar{Z}_2)}{\det(Z_1-\bar{Z}_2)}}.\end{aligned}$$

In other words parallel transport, $PT_\nabla(Z_1,Z_2) : H_{Z_1}^{(k)} \to H_{Z_2}^{(k)}$ is nothing but multiplication by a function on $\mathbb{H}\times\mathbb{H}$.

Equation 2.4 show that parallel transport $PT_\nabla(Z_1,Z_2)$ is the Toeplitz operator associated to the function $f:\mathbb{H}\times\mathbb{H}\to\mathbb{C}$ given by $f(Z_1,Z_2) = \sqrt{\frac{\det(Z_2-\bar{Z}_2)}{\det(Z_1-\bar{Z}_2)}}$, $PT_\nabla(Z_1,Z_2) = T_{f,Z_2}^{(k)}$. □

7.3 Asymptotics of Hilbert–Schmidt norms

In this section we study the asymptotics of Toeplitz operators and the Hilbert–Schmidt norm, and in particular to prove the following theorem in the setting of an abelian variety at hand.

Theorem 7.5. *For any two smooth functions $f,g\in C^\infty(M)$ and any $Z\in\mathbb{H}$ one has that*

$$\langle f,g\rangle = \lim_{k\to\infty} k^{-n}\left\langle T_{f,I(Z)}^{(k)}, T_{g,I(Z)}^{(k)}\right\rangle,$$

where the real dimension of M is $2n$ and $\langle\cdot,\cdot\rangle$ is the Hilbert–Schmidt inner product.

Definition 7.6. The Hilbert–Schmidt inner product of two operator A,B is

$$\langle A,B\rangle = \mathrm{Tr}(AB^*).$$

If we introduce the notation

$$\begin{aligned}\eta_k(r,s) &= \mathrm{Re}(e^{-\frac{\pi i}{k}r\cdot\bar{Z}r}e^{-2\pi i s\cdot\alpha}e^{-\pi^2(s-\bar{Z}r)\cdot(2\pi kY)^{-1}(s-\bar{Z}r)})\\ &= e^{-\frac{\pi}{2k}((s-Xr)\cdot Y^{-1}(s-Xr)+r\cdot Yr)}\end{aligned}$$

and recall the matrix coefficients of the Toeplitz operators $T^{(k)}_{F_{r,s}}$ in terms of the basis of theta functions then

$$(F_{r,s}\theta^{(k)}_\alpha, \theta^{(k)}_\beta)_Y = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-2\pi i s\cdot\alpha} e^{-\frac{\pi i}{k} r\cdot s}\eta_k(r,s).$$

Note that $f(r,s,Z)(k) = \eta_k(r,s)^{-1}$. Here we suppress the Z dependence in $\eta_k(r,s)$ since we from now only consider fixed Kähler structure.

Lemma 7.7.

$$\mathrm{Tr}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{t,u}})^*) = \begin{cases} k^n\eta_k(r,s)\eta_k(t,u)\varepsilon(r,s,t,u) & (r,s)\equiv(t,u) \ \mathit{mod}\ k \\ 0 & \mathit{else} \end{cases}$$

where $\varepsilon(r,s,t,u)\in\{\pm1\}$ *and is* 1 *for* $(r,s)=(t,u)$.

Proof. We start by calculating the matrix coefficients of the product of the Toeplitz operators

$$\begin{aligned}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{t,u}})^*)_{\beta\alpha} &= \sum_\varphi (T^{(k)}_{F_{r,s}})_{\beta\varphi}\overline{(T^{(k)}_{F_{t,u}})_{\alpha\varphi}} \\ &= \delta_{\alpha-\beta,-[\frac{r-t}{k}]} e^{-2\pi i\alpha\cdot(s-u)} e^{-\frac{\pi i}{k}(r\cdot s-2s\cdot t+t\cdot u)}\eta_k(r,s)\eta_k(t,u).\end{aligned}$$

Now when taking the trace $\alpha=\beta$ and to get something non-zero we must have $r\equiv t$ mod k. In that case

$$\mathrm{Tr}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{t,u}})^*) = \varepsilon(r,s,t,u)\eta_k(r,s)\eta_k(t,u)e^{\frac{\pi i}{k}r\cdot(s-u)}\sum_\alpha e^{-2\pi i\alpha(s-u)},$$

the ε is obtained since $t=r+kv$ only determines the equality

$$e^{-\frac{\pi i}{k}(r\cdot s-2s\cdot t+t\cdot u)} = \pm e^{\frac{\pi i}{k}(r\cdot(s-u))}$$

up to a sign. Now if $s\not\equiv u$ the last term is zero since it is n sums of all k'th roots of unity, and hence 0. If $s\equiv u$ each term in the sum is 1, and we get the desired result. □

Using the above lemma and the following limits

$$\lim_{k\to\infty}\eta_k(r,s) = 1 \qquad\text{and}\qquad \lim_{k\to\infty}\eta_k(r+kt,s+ku)=0, \tag{7.6}$$

for all $r,s\in\mathbb{Z}^n$, we can prove Theorem 7.5

Proof. From Lemma 7.7 we get in particular

$$\|T^{(k)}_{F_{r,s}}\|_k = k^{-n/2}\sqrt{\mathrm{Tr}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{r,s}})^*)} = \eta_k(r,s),$$

and

$$\|T^{(k)}_{E(F_{r,s})}\|_k = 1,$$

where $\|\cdot\|_k = k^{-n/2}\sqrt{\langle\cdot,\cdot\rangle}$ is the k-scaled Hilbert–Schmidt norm.

Let $f,g\in C^\infty(M)$ be an arbitrary elements and expand them in Fourier series

$$f = \sum_{(r,s)\in\mathbb{Z}^{2n}}\lambda_{r,s}F_{r,s} \qquad\text{and}\qquad g=\sum_{(t,u)\in\mathbb{Z}^{2n}}\mu_{t,u}F_{t,u}.$$

$\eta_k(r,s)$ and $\eta_k(t,u)$ decays very fast for increasing $r,s \in \mathbb{Z}^n$ and we have

$$\begin{aligned} k^{-n}\operatorname{Tr}(T_f^{(k)}(T_g^{(k)})^*) &= k^{-n} \sum_{(r,s),(t,u)\in\mathbb{Z}^{2n}} \lambda_{r,s}\bar{\mu}_{t,u} \operatorname{Tr}(T_{F_{r,s}}^{(k)}(T_{F_{t,u}}^{(k)})^*) \\ &= \sum_{(r,s)\in\mathbb{Z}^{2n}} \lambda_{r,s}\bar{\mu}_{t,u}\eta_k(r,s)^2 \\ &\quad + \sum_{\substack{(r,s),(t,u)\in\mathbb{Z}^{2n} \\ (t,u)\neq(0,0)}} \lambda_{r,s}\bar{\mu}_{r+kt,s+ku}\eta_k(r,s)\eta_k(r+kt,s+ku)\varepsilon(r,s,t,u). \end{aligned}$$

This sum converges uniformly so if we take the large k limit we can interchange limit and summation. Now by Equation 7.6 and since

$$\lim_{k\to\infty} \mu_{r+kt,s+ku} = 0$$

by pointwise convergence of the Fourier series we finally get

$$\lim_{k\to\infty} k^{-n}\operatorname{Tr}(T_f^{(k)}(T_g^{(k)})^*) = \sum_{(r,s)\in\mathbb{Z}^{2n}} \lambda_{r,s}\bar{\mu}_{r,s}.$$

Now since the pure phase functions are orthogonal we get to desired result. □

It should be remarked that Theorem 7.5 just is a particular case of a theorem of the same wording, with M being a compact Kähler manifold, see e.g. [5]. Theorem 7.5 was also proved in [11] but only for a small class of principal polarized abelian varieties.

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