Homogeneous Spaces

Unitary group representations – F09

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1 Introduction

In this note we investigate the notion of a group action on a topological space and espcially transitive actions. First of all what it is, but also see that almost all such spaces have the structure of a quotient space. The spaces which have a quotient space structure are called homogeneous spaces. Last but not least we will be dealing with some examples of homogeneous spaces.

The following note is mainly based on [Helgason(1962)] and examples are inspired by [Warner(1971)] and [Bröcker and Dieck(1985)].

2 Group actions on topological spaces

Definition 2.1. Let G be a locally compact group and X a locally compact Hausdorff space. A *left action* of G on X is a continuous map $G \times X \to X$ denoted by $(g, x) \to g \cdot x$ satisfying

(i) $x \mapsto g \cdot x$ is a homeomorphism of X for any $g \in G$.

(ii) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Note that (ii) state that the action shall have a homomorphism property and (ii) also implies that $e \cdot x = x$ for any $x \in X$ and $e \in G$ the identity element.

A locally compact Hausdorff space equipped with an action of G is called a *G-space*. A *transitive* G-space, is a G-space, where for every $x, y \in X$ there exists a $g \in G$ such that $g \cdot x = y$.

The subgroup $H_x = \{g \in G : g \cdot x = x\}$ is called the *isotropy group* of a point $x \in X$. In the following it will be shown that H_x is closed for any x.

Before giving the definition of a homogeneous space we state and prove a theorem on these G-spaces. But first an example of a G-space.

Example 2.2 (Transitive G-space)**.** The quotient space G/H where H is a closed subgroup of G, and G is a locally compact group acting on G/H by left multiplication. We shall in the following see that this is actually almost the only examples there is.

Let now X be a locally compact Hausdorff space and G a locally compact group acting transitively on X. Let $x_0 \in X$ be given, and let H be the isotropy group of x_0. Define $\varphi : G \to X$ by $\varphi(g) = g \cdot x_0$. φ is continuous because of the definition of G-action and $H = \varphi^{-1}(x_0)$, so H is closed. With the action beeing transitive φ is surjective and constant on left cosets of H. Hence φ induces a continuous bijection $\Phi : G/H \to X$. Φ is surjective because the action is transitive and φ is constant on left cosets of H. Φ is injective because if $\varphi(g_1) = g_1 \cdot x_0 = g_2 \cdot x_0 = \varphi(g_2)$ then $g_2^{-1} g_1 \in H$ and $g_1^{-1} g_2 \in H$. This implies that $g_1 \in g_2 H$ and $g_2 \in g_1 H$ so $g_1 H = g_2 H$ and thus Φ is injective. Φ is continuous from the general topological fact that given a quotient map $\pi : G \to G/H$, then a map $f : G/H \to X$ is continuous iff $\tilde{f} : G \to X$ is continuous.

Now we have a continuous bijection between G/H and X. To see that G/H and X are topologically the same we must show that Φ^{-1} is continuous. If this always was so, then every transitive G-space would be a quotient space. But unfortunately Φ^{-1} is not generally continuous. An example is the locally compact group $\mathbb{R}_d$ (the real line with the discrete topology) acting on $\mathbb{R}$ by translations. But the isotropy group of an element $x \in \mathbb{R}$ is just $H = \{0\}$, so $\Phi^{-1} : \mathbb{R} \to \mathbb{R}_d/H = \mathbb{R}_d$ should be continuous. But with $\mathbb{R}_d$ given the discrete topology and $\mathbb{R}$ its normal topology this cannot happen. Therefore Φ is not always a homeomorphism, further requirements are needed.

Theorem 2.3. *Let G be a locally compact group and let X be a transitive G-space. Let $x_0 \in X$ and let H be the isotropy group of x_0. If G is σ-compact then $\Phi : G/H \to X$ given by $gH \to g \cdot x_0$ is a homeomorphism.*

Definition 2.4. A *homogeneous space* is a transitive G-space X that is isomorphic to a quotient space G/H – that is, a space where the above map Φ is a homeomorphism.

Before we prove this theorem we need a lemma that will be crucial in the proof of the theorem.

Lemma 2.5 (The Baire Category Theorem)**.** *If a locally compact Hausdorff space M is a countable union $M = \cup_{n=1}^{\infty} M_n$ where each M_n is a closed subset of M, then at least one of the M_n's contain an open subset of M.*

Proof. Assume that no M_n contains an open subset of M. Let U_1 be an open subset of M with $\overline{U_1}$ compact. This can be found since M is locally compact. Choose now a_1 in the open set $U_1 \setminus M_1$ and a neighborhood U_2 of a_1 such that $\overline{U_2} \subseteq U_1 \setminus M_1$. Then $\overline{U_2} \cap M_1 = \emptyset$. If $U_1 \setminus M_1 = \emptyset$ then $U_1 \subset M_1$, and we have assumed that M_1 did not contain an open subset. The choice of U_2 can be done because the M is locally compact, by [Thomsen(2008), Shriking lemma]. Now choose again an a_2 in the open non-empty set $U_2 \setminus M_2$ and a neighborhood U_3 of a_2 such that $\overline{U_3} \subseteq U_2$ and $\overline{U_3} \cap M_2 = \emptyset$, and so forth. This gives us a sequence of decreasing compact subsets that are non-empty.

$$\overline{U_1} \supset \overline{U_2} \supset \cdots \supset \overline{U_n} \supset \cdots \neq \emptyset.$$

All the compact sets are non-empty, and $\overline{U_n}$ is contained in $\overline{U_m}$ if $m \leq n$, so there is a point $b \in M$ which is in every $\overline{U_n}$. But this means that $b \notin M_n$ for every n which is a contradiction because the union of the M_n's is M and b is in M. This completes the proof. □

Now to the proof of the main theorem.

Proof of Theorem 2.3. Note that π is a qoutient map, and if V is an open set in G then $\pi^{-1}(\pi(V)) = VH$ is open in G as well, which means that $\pi(V)$ is open in G/H. So π is an open map. To show that Φ is an open map it is then enough that just φ, which induces Φ, is open.

Let $U \subset G$ be an open set. Choose $u_0 \in U$ and with G being locally compact we can choose V a compact neighborhood of $1 \in G$, such that V is symmetric and $u_0 VV \subset U$ [Folland(1995), Proposition 2.1.b]. Now with G being σ-compact there is a countable set $\{y_k\} \subset G$ such that $y_k V$ covers G. This is so because we know that there is a countable set of compact sets that covers G, so $G = \cup_n W_n$. Now each W_n is contained in the union $\cup_{\beta_n \in W_n} \beta_n \mathring{V}$ of open sets, so there is a finite cover of W_n with sets of the form $\beta_{n_i} \mathring{V}$, so just choose $\{y_k\} = \{\beta_{n_i}\}_{i,n}$ which is a countable set, then G is also covered by the compact sets $y_k V$.

With the action beeing transitive $X = \cup_{n=1}^{\infty} \varphi(y_n V)$. From the definition of an action $x \mapsto g \cdot x$ is a homeomorphism of X, which means that $\varphi(y_n V)$ is homeomorphic to $\varphi(V)$, and with V compact and φ continuous $\varphi(V)$ is compact and hence $\varphi(y_n V)$ is compact and since X is Hausdorff $\varphi(y_n V)$ is closed.

From the lemma we get that there is an m such that $\varphi(y_m V)$ contains an open subset of X and hence $\varphi(V)$ contains an open subset of X (they are homeomorphic). This means that there is a $u_1 \in V$ such that $\varphi(u_1)$ is an inner point of $\varphi(V)$, which implies that $\varphi(1)$ is an inner point of $\varphi(u_1^{-1}V)$. But then $\varphi(u_0)$ is an inner point of $\varphi(u_0u_1^{-1}V)$ and $u_0u_1^{-1}V \subset u_0VV \subset U$, so $\varphi(u_0)$ is an inner point af $\varphi(U)$. All this follow from [Folland(1995), Proposition 2.1.a] Thus $\varphi(U)$ is open. $\square$

We can immediately extract the following corollary.

Corollary 2.6. *Let G and X be two locally compact groups. Assume that G is σ-compact. Then every continuous surjective homomorphism ψ from G onto X is open.*

Proof. This is a corollary to the proof of the previous theorem, because the action of G on a space X is a homomorphism. If we further require that the homomorphism is surjective, then the action is transitive and from the previous proof this was all that was used about the map to show that it was open. $\square$

The following example is an application of Theorem 2.3, where we by observing that the isotropy group of $SO(n)$ is $SO(n-1)$, get a homeomorphism between $SO(n)/SO(n-1)$ and S^{n-1}.

Example 2.7. Let $\{e_i : i = 1, \dots, n\}$ be the canonical basis for $\mathbb{R}^n$, that is $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the i'te place. Each $\sigma \in GL(n, \mathbb{R})$ uniquely determines a linear transformation of $\mathbb{R}^n$ by the requirement that $\sigma(e_j) = \sum_i \sigma_{ij} e_i$. This transformation is also denoted by σ. If we regard n-tuples of $\mathbb{R}^n$ as $n \times 1$-matrices then σ acts on $\mathbb{R}^n$ in a natural way – that is by left matrix multiplication

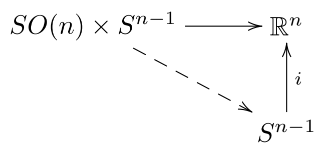
$$GL(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n.$$

Let $\langle\, , \rangle$ be the standard inner product on $\mathbb{R}^n$. With respect to this inner product the basis $\{e_i\}$ is orthonormal. If $\sigma \in GL(n, \mathbb{R})$ then $\langle \sigma(v), w \rangle = \langle v, \sigma^T(w) \rangle$ where σ^T is the transpose of σ.

If $\sigma \in SO(n)$ then $\sigma^T\sigma = I$ and

$$\langle \sigma(v), \sigma(v) \rangle = \langle v, \sigma^T\sigma(v) \rangle = \langle v, v \rangle ,$$

so σ preserves the length of vectors. Then if we just look at $S^{n-1} \subset \mathbb{R}^n$ then the action factor through S^{n-1}.



It can be shown, that the action actually is smooth, so that we in the end not only get at homemorphism but actually a diffeomorphism.

The action is transitive, because choose a vector $v_1 \in S^{n-1}$ and let $\{v_1, v_2, \dots, v_n\}$ be a orthonormal basis for $\mathbb{R}^n$ with v_1 as the first element. Let $v_i = \sum_j \sigma_{ji} e_j$. Then the matrix (σ_{ji}) is orthogonal. Because $\det(\sigma^T) = \det(\sigma)$ being orthogonal means that $\det(\sigma) = \pm 1$. If $\det(\sigma) = 1$ then $\sigma \in SO(n)$. If $\det(\sigma) = -1$ then change v_2 to $-v_2$. The basis $\{v_1, -v_2, v_3, \dots, v_n\}$ is still orthonormal but now $\det(\sigma) = 1$ – just calculate the determinant by the second column. This implies that we can choose the orthonormal basis in such a way that $\det(\sigma) = 1$ that is $\sigma \in SO(n)$.

We can further see that in any case $\sigma(e_1) = v_1$. So each point on S^{n-1} can be connected to one of the poles by a special orthogonal matrix. So if v, w are two points on S^{n-1} then there is a special orthogonal matrix σ such that $\sigma(v) = w$. Just find a special orthogonal matrix σ_1 such that $\sigma_1(e_1) = v$ and a special orthogonal matrix σ_2

such that $\sigma_2(e_1) = w$, then $\sigma = \sigma_2\sigma_1^{-1} \in SO(n)$ and $\sigma(v) = \sigma_2\sigma_1^{-1}(v) = \sigma_2(e_1) = w$. This means that the action is transitive.

Now we are left with the task of determining the isotropy group, H, of the action of the north pole, e_n. But because of the symmetry of the sphere we only refer to H as *the* isotropy group of the action.

If we regard the inclusion of $SO(n-1)$ in $SO(n)$ by sending $\tilde{\sigma} \mapsto \sigma$ where

$$\sigma = \begin{pmatrix} \begin{pmatrix} & & \\ & \tilde{\sigma} & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \end{matrix} & 1 \end{pmatrix}.$$

The claim is now that $SO(n-1)$ included in $SO(n)$ is exactly the isotropy group of the action of $SO(n)$ on S^{n-1}. Elements of $SO(n-1)$ definitely satisfy the isotropy condition that the north pole should be preserved under the action, that is $\sigma(e_n) = e_n$, by the construction of the inclusion. Assume conversly that $\sigma \in SO(n)$ and that $\sigma(e_n) = e_n = \sum_i \sigma_{in}e_i$ where $\sigma_{in} = 0$ for $i < n$ and $\sigma_{nn} = 1$. With σ being orthogonal $\sigma\sigma^T = I$ so $\sum_i \sigma_{ni}^2 = 1$, and with $\sigma_{nn} = 1$ then $\sigma_{ni} = 0$ for $i < n$. So $\sigma \in SO(n-1) \subset SO(n)$.

We have now seen that the isotropy group of the action of $SO(n)$ on the sphere S^{n-1} at the north pole is $SO(n-1)$ included in $SO(n)$. The thoerem then says that the map $SO(n)/SO(n-1) \to S^{n-1}$ given by $\sigma SO(n-1) \mapsto \sigma(e_n)$ is a homeomorphism. Actually we also need to show that $SO(n)$ is σ-compact, but $SO(n) \subset GL(n,\mathbb{R}) \subset \mathbb{R}^{n^2}$ and the euclidean space $\mathbb{R}^{n^2}$ is σ-compact and hence also the subspace $SO(n)$. As stated, the map can also be shown to be a diffeomorphism.

In the next example we see that $\mathbb{R}/\mathbb{Z} \cong S^1$.

Example 2.8. The real numbers act on S^1 by rotation. The action is the map $\mathbb{R} \times S^1 \to S^1$ given by

$$(t, s) \mapsto e^{is\pi t}s,$$

where we regard S^1 as a subset of $\mathbb{C}$. Its clear that the action is transitive, and it is also clear that the isotropy group of each point is the same, namely $\mathbb{Z} \subset \mathbb{R}$. $\mathbb{R}$ is σ-compact and then the theorem gives a homeomorphism between $\mathbb{R}/\mathbb{Z}$ and S^1.

Example 2.9. Moving to the complex case we could in a much similar way as the above, show that the special unitary matrices act transitively on S^{2n-1} (again the natural action by multiplication) and that the isotropy group of the action is $SU(n-1)$. Then the theorem says that we have a homeomorphism between $SU(n)/SU(n-1)$ and S^{n-1}. As before we can show that this actually is a diffeomorphism.

As a special case of this, since $SU(1)$ only consists of the 1×1 identity matrix, S^3 is diffeomorphic with $SU(2)$ and in this way we can give S^3 a Lie group structure.

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