Intersection Cuts for Mixed Integer Conic Quadratic Sets

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Abstract

Balas introduced intersection cuts for mixed integer linear sets. Intersection cuts are given by closed form formulas and form an important class of cuts for solving mixed integer linear programs. In this paper we introduce an extension of intersection cuts to mixed integer conic quadratic sets. We identify the formula for the conic quadratic intersection cut by formulating a system of polynomial equations with additional variables that are satisfied by points on a certain piece of the boundary defined by the intersection cut. Using a software package from algebraic geometry we then eliminate variables from the system and get a formula for the intersection cut in dimension three. This formula is finally generalized and proved for any dimension. The intersection cut we present generalizes a conic quadratic cut introduced by Modaresi, Kilinc and Vielma.

1 Introduction

In this paper we study a mixed integer set obtained from a single conic quadratic inequality defined from rational data $A \in \mathbb{Q}^{m\times n}$ and $d \in \mathbb{Q}^m$:

$$Q_I := \{x \in \mathbb{R}^n : Ax - d \in L^m \text{ and } x_j \in \mathbb{Z} \text{ for } j \in I\}, \tag{1}$$

where L^m is the m-dimensional Lorentz cone $L^m := \{y \in \mathbb{R}^m : y_m \geq \sqrt{\sum_{j=1}^{m-1} y_j^2}\}$ and I is an index set for the integer constrained variables. The continuous relaxation of Q_I is given by $Q := \{x \in \mathbb{R}^n : Ax - d \in L^m\}$. A mixed integer conic quadratic set of the form Q_I can be obtained from a single constraint of the continuous relaxation of a Mixed Integer Conic Quadratic Optimisation (MICQO) problem. Valid inequalities for Q_I (linear or non-linear) can therefore be used as cuts for solving MICQO problems.

The present paper gives an extension of the *intersection cut* of Balas [4] from Mixed Integer Linear Optimisation (MILO) problems to MICQO problems. Several previous papers have aimed at extending cuts from MILO to MICQO. An extension of the mixed integer rounding cuts of Nemhauser and Wolsey [13] to MICQO was given by Atamtürk and Narayanan [2, 3]. Çezik and Iyengar [7] studied the extension of the Chvátal-Gomory procedure from MILO to MICQO. The Lift-and-Project algorithm of Balas et al. [5] developed for MILO was generalized by Stubbs and Mehrotra [15] to MICQO.

Intersection cuts form a very important class of cutting planes for solving MILO problems [6]. Mixed Integer Rounding (MIR) cuts [13], Mixed Integer Gomory (MIG) cuts [9], Lift-and-Project cuts [5] and Split Cuts [8] are all intersection cuts. Since the intersection cuts we propose for MICQO are also given by a closed form formula and derived using similar principles as for MILO problems, we hope they can be equally useful for solving MICQO problems.

As our study is inspired by the intersection cut introduced by Balas [4] for a mixed integer linear set $P_I := \{x \in P : x_j \in \mathbb{Z} \text{ for } j \in I\}$ with a polyhedral relaxation $P := \{x \in \mathbb{R}^n : Ax - d \in \mathbb{R}^m_+\}$, we first review the derivation of intersection cuts in a linear setting.

Intersection cuts for P_I are derived from a maximal choice B of linearly independent rows $\{a_i\}_{i \in B}$ of the matrix A. A given choice B gives a relaxation:

$$P^B := \{x \in \mathbb{R}^n : a_i^T x - d_i \geq 0 \text{ for } i \in B\} \tag{2}$$

of P obtained by removing constraints *not* indexed by B. An intersection cut is then obtained from P^B and a choice of a *split disjunction*. A split disjunction is a disjunction of the form $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$, with $(\pi, \pi_0) \in \mathbb{R}^{n+1}$ chosen so that there are no mixed integer points strictly between the hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$. The simple geometry of P^B gives that the convex hull $\text{conv}(P_1^B \cup P_B^2)$ of the sets:

$$P_1^B := \{x \in P^B : \pi^T x \leq \pi_0\} \text{ and } P_2^B := \{x \in P^B : \pi^T x \geq \pi_0 + 1\} \tag{3}$$

can be described with at most one additional linear inequality, and such an inequality is called the intersection cut obtained from B and (π, π_0).

Our proposal for an intersection cut for the mixed integer conic quadratic set Q_I is now the following. Again we consider a maximal choice B of linearly independent rows $\{a_i\}_{i \in B}$ of the matrix A. We require $m \in B$ since it is necessary to include the m^{th} row of $Ax - d$ in B in a conic quadratic setting for natural reasons. The choice B leads to the relaxation Q^B of Q:

$$Q^B := \{x \in \mathbb{R}^n : A_B \cdot x - d_B \in L^{|B|}\}, \tag{4}$$

where (A_B, d_B) is obtained from (A, d) by deleting rows *not* indexed by B. Given a choice of split disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$, we will show that the convex hull $\text{conv}(Q_1^B \cup Q_B^2)$ of the sets:

$$Q_1^B := \{x \in Q^B : \pi^T x \leq \pi_0\} \text{ and } Q_2^B := \{x \in Q^B : \pi^T x \geq \pi_0 + 1\} \tag{5}$$

can be described with at most one additional inequality given by a closed form formula, and we call such an inequality a *conic quadratic intersection cut*.

We now present our main result: The inequality description of $\text{conv}(Q_1^B \cup Q_2^B)$. For simplicity assume in the following that there is only one choice of constraint set B, *i.e.*, assume the matrix A has full row rank. Let the sets $Q_1 := \{x \in Q : \pi^T x \leq \pi_0\}$ and $Q_2 := \{x \in Q : \pi^T x \geq \pi_0 + 1\}$ be the points in Q satisfying $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$. We now give a characterization of $\text{conv}(Q_1 \cup Q_2)$.

We first answer the question: When does $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ give an intersection cut for Q_I, *i.e.*, when do we have $\text{conv}(Q_1 \cup Q_2) \neq Q$? The answer depends on the geometry of the nullspace $\mathcal{L} := \{x \in \mathbb{R}^n : Ax = 0_n\}$ of A and on the geometry of the affine set $\mathcal{A} := \{x \in \mathbb{R}^n : Ax = d\}$, and is as follows.

- (No intersection cut). We have $\text{conv}(Q_1 \cup Q_2) = Q$ if and only if either $\pi \notin \mathcal{L}^\perp$ or $\mathcal{A}$ is *not* strictly between the hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$.

To obtain a description of $\text{conv}(Q_1 \cup Q_2)$ when $\text{conv}(Q_1 \cup Q_2) \neq Q$ we apply an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ to reduce the problem to the following question: Given a disjunction $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$ on $\mathbb{R}^m$ such that the apex 0_m of the Lorentz cone L^m lies strictly between the hyperplanes $\delta^T y = r_1$ and $\delta^T y = r_2$, what is the inequality description of the convex hull $\text{conv}(S_1 \cup S_2)$ of the sets

$$S_1 := \{y \in L^m : \delta^T y \leq r_1\} \text{ and } S_2 := \{y \in L^m : \delta^T y \geq r_2\} \tag{6}$$

of points in the Lorentz cone L^m satisfying the disjunction $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$? The precise formula for $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$ is given in Sect. 2.

There are two types of intersection cuts that may be needed to describe $\text{conv}(Q_1 \cup Q_2)$: A linear inequality or a conic quadratic inequality. A linear inequality appears when either δ or $-\delta$ belongs to the Lorentz cone as follows.

- (Linear intersection cut). Suppose $\text{conv}(Q_1 \cup Q_2) \neq Q$. If $\delta \in L^m$, then we have $\text{conv}(Q_1 \cup Q_2) = \{x \in Q : \pi^T x \geq \pi_0 + 1\}$, and if $-\delta \in L^m$, then $\text{conv}(Q_1 \cup Q_2) = \{x \in Q : \pi^T x \leq \pi_0\}$.

The most interesting case is the following situation where a conic quadratic inequality is needed to describe $\text{conv}(Q_1 \cup Q_2)$.

- (Conic quadratic intersection cut). If $\text{conv}(Q_1 \cup Q_2) \neq Q$ and $\pm\delta \notin L^m$, then $\text{conv}(Q_1 \cup Q_2)$ is the set of $x \in Q$ such that $y := Ax - d$ satisfies

$$4 \cdot r_1 \cdot r_2 \cdot (\delta^T y - r_1)(\delta^T y - r_2) + (r_1 - r_2)^2 (\sum_{i=1}^{m} y_i^2 - y_m^2) \cdot (\sum_{i=1}^{m} \delta_i^2 - \delta_m^2) \leq 0 \quad (7)$$

Observe that (7) is not in conic quadratic form. We present conic quadratic forms of (7) in Sect. 5. The validity of (7) for $S_1 \cup S_2$ is easy to see: The constant $4r_1 r_2$ is negative since $r_1 < 0 < r_2$, and for any $y \in \mathbb{R}^m$ satisfying $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$ we have $(\delta^T y - r_1)(\delta^T y - r_2) \geq 0$. Furthermore, the condition $\pm\delta \notin L^m$ gives $\sum_{i=1}^m \delta_i^2 - \delta_m^2 > 0$, and finally any $y \in L^m$ satisfies $\sum_{i=1}^m y_i^2 - y_m^2 \leq 0$.

Intersection cut (7) was also obtained independently by Modaresi et al. [12] for the special case when: (a) the matrix A is non-singular, (b) the n^{th} row and column of A are both the n^{th} unit vector, and (c) the split disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ has $\pi_n = 0$. In this case the (transformed) disjunction $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$ always has $\delta_m = 0$, and the hyperplanes $\delta^T y = r_1$ and $\delta^T y = r_2$ are therefore always parallel to the coordinate axis associated with the last variable y_m. Since the last coordinate is very different from the other coordinates for points in a Lorentz cone, the geometry becomes substantially more complex when one allows $\delta_m \neq 0$, and it is not clear which inequality is needed. This is why we decided to consider Gröbner bases from algebraic geometry to identify (7). Our approach is inspired by a paper of Ranestad and Sturmfels [14] on determining the boundary of the convex hull of a variety.

For mixed integer linear sets intersection cuts and split cuts are equivalent [1]. We give a counterexample in Sect. 6 which shows that this is no longer the case in a conic quadratic setting.

The remainder of the paper is organized as follows. In Sect. 2 we reduce the problem of characterizing the set $\text{conv}(Q_1 \cup Q_2)$ in $\mathbb{R}^n$ to the problem of characterizing the set $\text{conv}(S_1 \cup S_2)$ in $\mathbb{R}^m$. In Sect. 3 we characterize the boundary of $\text{conv}(S_1 \cup S_2)$ by using the algebraic geometry software called Singular. We present our main result in Sect. 4. In Sect. 5 we discuss conic quadratic forms of our conic quadratic inequality, and finally in Sect. 6 we give an example to show that conic quadratic split cuts and intersection cuts are not equivalent.

2 Reduction to the main case

We continue studying a mixed integer conic quadratic set $Q_I = \{x \in Q : x_j \in \mathbb{Z} \text{ for } i \in I\}$ with relaxation $Q := \{x \in \mathbb{R}^n : Ax - d \in L^m\}$, where $A \in \mathbb{Q}^{m \times n}$ has $\text{rank}(A) = m$. The split disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ is arbitrary and gives two sets $Q_1 := \{x \in Q : \pi^T x \leq \pi_0\}$ and $Q_2 := \{x \in Q : \pi^T x \geq \pi_0 + 1\}$.

The main purpose of this section is to reduce the problem of characterizing $\text{conv}(Q_1 \cup Q_2)$ to the problem of characterizing $\text{conv}(S_1 \cup S_2)$, where $S_1 := \{y \in L^m : \delta^T y \leq r_1\}$ and $S_2 := \{y \in L^m : \delta^T y \geq r_2\}$ for some disjunction $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$ on $\mathbb{R}^m$ which will be defined below.

We first characterize when no further inequalities are needed to describe $\text{conv}(Q_1 \cup Q_2)$. The set $\mathcal{L}$ denotes the nullspace of A, and $\mathcal{A}$ denotes the affine set $\mathcal{A} := \{x \in \mathbb{R}^n : Ax = d\} = \bar{x} + \mathcal{L}$, where $\bar{x}$ solves $Ax = d$.

Lemma 2.1 *We have* $\text{conv}(Q_1 \cup Q_2) \neq Q$ *if and only if*

(i) π *is orthogonal to* $\mathcal{L}$*, and*

(ii) $\mathcal{A}$ *lies strictly between the hyperplanes* $\pi^T x = \pi_0$ *and* $\pi^T x = \pi_0 + 1$*.*

Proof. Suppose (i) is *not* satisfied, *i.e.*, $\pi \notin \mathcal{L}^\perp$. Hence there exists $l \in \mathcal{L}$ such that $\pi^T l < 0$. Clearly $\text{conv}(Q_1 \cup Q_2) \subseteq Q$. Let $z \in Q$ be arbitrary. If $\pi^T z \notin]\pi_0, \pi_0 + 1[$ then $z \in \text{conv}(Q_1 \cup Q_2)$, so we assume $\pi^T z \in]\pi_0, \pi_0 + 1[$. Now $\pi^T l < 0$ implies we can choose $\mu^1, \mu^2 > 0$ such that $z^1 := z + \mu^1 l \in Q_1$ and $z^2 := z - \mu^2 l \in Q_2$. Since z is on the line between z^1 and z^2, we get $z \in \text{conv}(Q_1 \cup Q_2)$.

Next suppose (ii) is *not* satisfied. Wlog let $z \in \mathcal{A}$ satisfy $\pi^T z \leq \pi_0$. Clearly $\text{conv}(Q_1 \cup Q_2) \subseteq Q$. Let $w \in Q$ be arbitrary. We can assume $\pi^T w \in]\pi_0, \pi_0 + 1[$, since otherwise $w \in \text{conv}(Q_1 \cup Q_2)$. Define $r := w - z$. We have $\pi^T r > 0$. Furthermore, since $Az = d$ we have $Ar \in L^m$, and since L^m is a cone this gives $\{z + \alpha \cdot r : \alpha \geq 0\} \subseteq Q$. Also, $z \in Q^1$ and $\pi^T r > 0$ implies $\{z + \alpha \cdot r : \alpha \geq 0\} \subseteq \text{conv}(Q_1 \cup Q_2)$. Since w is on this halfline, we get $w \in \text{conv}(Q_1 \cup Q_2)$.

Finally suppose (i) and (ii) are satisfied. We claim $\bar{x} \notin \text{conv}(Q_1 \cup Q_2)$. Suppose, for a contradiction, that $\bar{x} \in \text{conv}(Q_1 \cup Q_2)$. We do not have $\bar{x} \in Q_1 \cup Q_2$ since $\pi^T \bar{x} \in]\pi_0, \pi_0 + 1[$ by (ii). Hence there exists $\lambda \in]0, 1[$, $x^1 \in Q_1$ and $x^2 \in Q_2$ so that $\bar{x} = \lambda x^1 + (1 - \lambda) x^2$. Let $y^1 := Ax^1 - d$ and $y^2 := Ax^2 - d$. We have $0_m = A\bar{x} - d = \lambda y^1 + (1 - \lambda) y^2$, so $-y^1 = \frac{(1-\lambda)}{\lambda} y^2$. Since $y^2 \in L^m$, $\frac{(1-\lambda)}{\lambda} > 0$ and L^m is a cone this gives $-y^1 \in L^m$. We now have $\pm y^1 \in L^m$, and therefore the line $\{\alpha \cdot y^1 : \alpha \in \mathbb{R}\}$ is contained in L^m. Since L^m is pointed, this implies $y^1 = y^2 = 0_m$. However, then $x^1, x^2 \in \mathcal{A}$, and since $\pi^T z = \pi^T \bar{x} \in]\pi_0, \pi_0 + 1[$ for all $z \in \mathcal{A}$ from (i) and (ii), this is a contradiction. □

We now present the reduction. Define a disjunction $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$ on $\mathbb{R}^m$ as follows.

Definition 2.2 (Definition of the disjunction $\delta^T y \leq r_1 \vee \delta^T y \geq r_2$)
The vector $\delta \in \mathbb{R}^m$ is the projection of π onto $\mathcal{L}^\perp$, *i.e.*, we define $\delta = (AA^T)^{-1} A\pi$. The numbers $r_1, r_2 \in \mathbb{R}$ are given by $r_1 := \pi_0 - \delta^T d$ and $r_2 := r_1 + 1$.

Given a disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$, we now argue that a description of $\text{conv}(S^1 \cup S^2)$ gives a description of $\text{conv}(Q^1 \cup Q^2)$(Lemma 2.(iii)). This argument is standard, but important, and therefore proven in the appendix.

Lemma 2.3 *Suppose* $(\pi, \pi_0) \in \mathbb{R}^{n+1}$ *satisfies (i) and (ii) of Lemma 1. Then*

(i) $0 \in]r_1, r_2[$ *and* $\text{conv}(S_1 \cup S_2) \neq L^m$*,*

(ii) $Q_k = \{x \in \mathbb{R}^n : Ax - d \in S_k\}$ *for* $k = 1, 2$*, and*

(iii) $\text{conv}(Q_1 \cup Q_2) = \{x \in \mathbb{R}^m : Ax - d \in \text{conv}(S_1 \cup S_2)\}$.

Observe that Lemma 2.(ii) gives a condition for when a linear inequality suffices to describe $\text{conv}(Q_1 \cup Q_2)$. Indeed, since L^m is a self-dual cone, $\delta \in L^m$ implies $\delta^T z \geq 0$ for all $z \in L^m$. Since $r_1 < 0 < r_2$, this gives $\text{conv}(S_1 \cup S_2) = S_2$ when $\delta \in L^m$. Symmetrically $\text{conv}(S_1 \cup S_2) = S_1$ when $-\delta \in L^m$.

Corollary 2.4 *Suppose* $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ *satisfies (i)-(ii) of Lemma 1.*

(i) If $\delta \in L^m$*, then* $\mathrm{conv}(S_1 \cup S_2) = S_2$ *and* $\mathrm{conv}(Q_1 \cup Q_2) = Q_2$.

(ii) If $-\delta \in L^m$*, then* $\mathrm{conv}(S_1 \cup S_2) = S_1$ *and* $\mathrm{conv}(Q_1 \cup Q_2) = Q_1$.

3 Describing a piece of the boundary of the convex hull

We now describe a part of the boundary of $\mathrm{conv}(S_1 \cup S_2)$, where S_1 and S_2 are as defined in Sect. 2 from a disjunction $\delta^T x \le r_1 \vee \delta^T x \ge r_2$ with $(\delta, r_1, r_2) \in \mathbb{R}^{m+2}$ and $r_1 r_2 < 0$. This will give the inequality needed to describe $\mathrm{conv}(S_1 \cup S_2)$. We assume $\pm\delta \notin L^m$. For simplicity let $C := \mathrm{conv}(S_1 \cup S_2)$. We consider points $x \in \partial C$ each being a convex combination of points $a \in S_1$ and $b \in S_2$ maximizing a linear form $h \in \mathbb{R}^m \setminus \{0_m\}$ over C. These points belong to the set B below.

Definition 3.1 Let $F_k := \{x \in \mathbb{R}^m : \sum_{i=1}^{m-1} x_i^2 = x_m^2 \wedge \delta^T x = r_k\}$ for $k = 1, 2$. We define the set B as follows.

$$B := \{x \in \mathbb{R}^m : \exists (h, a, b, t) \in (\mathbb{R}^m \setminus \{0_m\}) \times F_1 \times F_2 \times \mathbb{R} :$$
$$x = ta + (1-t)b \quad \text{and} \quad h^T a = h^T b$$
$$\dim(\mathrm{span}(h, \nabla L(a), \delta)) \le 2 \quad \text{and} \quad \dim(\mathrm{span}(h, \nabla L(b), \delta)) \le 2 \ \}.$$

Here ∇L denotes the gradient of $x \mapsto x_m^2 - \sum_{i=1}^{m-1} x_i^2$.

Theorem 3.2 *Let* $x \in C$ *with* $r_1 < \delta^T x < r_2$. *If* $x \in \partial C$ *then* $x \in B$.

Proof. Since $r_1 < \delta^T x < r_2$ x must be a convex combination of a point $a \in S_1$ and a point $b \in S_2$. By convexity of L^m we may assume that $a \in H_1$ and $b \in H_2$, where $H_k := \{x \in \mathbb{R}^m : \delta^T x = r_k\}$ for $k = 1, 2$.

Since x is in ∂C, we have $a \in \partial(L^m \cap H_1)$ and $b \in \partial(L^m \cap H_2)$ in the affine spaces H_1 and H_2 respectively. This proves $a_m^2 = \sum_{i=1}^{m-1} a_i^2$ and $b_m^2 = \sum_{i=1}^{m-1} b_i^2$.

Since $x \in \partial C$ and C is convex, there exists an $h \in \mathbb{R}^m \setminus \{0\}$ which as a linear form attains its maximum over C in x. Moving along a line from x towards a or b we stay in C. Therefore, by colinearity of a, b and x, we get $h^T(a - b) = 0$.

Finally, consider the projection $\tilde{h}$ of h to the linear space parallel to H_1. Since h attains its maximum over $C \cap H_1$ at a, the gradient of $a_m^2 - \sum_{i=1}^{m-1} a_i^2$ in the subspace and $\tilde{h}$ are dependent. Hence $\dim(\mathrm{span}(h, \nabla L(a), \delta)) \le 2$. A similar argument for b shows that $\dim(\mathrm{span}(h, \nabla L(b), \delta)) \le 2$. $\square$

Theorem 3.3 *Any point* $x \in B$ *must satisfy the equation* $\sum_{i=1}^{m-1} x_i^2 = x_m^2$ *or*

$$4r_1 r_2(\delta^T x - r_1)(\delta^T x - r_2) + (r_1 - r_2)^2 \Big(\sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2\Big)\Big(\sum_{i=1}^{m-1} x_i^2 - x_m^2\Big) = 0. \qquad (8)$$

These are polynomial equations in x *with coefficients involving* δ, r_1 *and* r_2.

Before proving Theorem 3.3, we show how to deduce the above equations for $m = 3$ in the computer algebra system Singular [10]. In Singular we type:

```
ring r=0,(delta1,delta2,delta3,r1,r2,x1,x2,x3,a1,a2,a3,b1,b2,b3,t,h1,h2,
         h3,A1,A2,B1,B2),dp;
poly f1=a1^2+a2^2-a3^2;
poly f2=a1*delta1+a2*delta2+a3*delta3-r1;
poly g1=b1^2+b2^2-b3^2;
poly g2=b1*delta1+b2*delta2+b3*delta3-r2;
poly K1=-h1+A1*diff(f1,a1)+B1*delta1;
poly K2=-h2+A1*diff(f1,a2)+B1*delta2;
poly K3=-h3+A1*diff(f1,a3)+B1*delta3;
poly L1=-h1+A2*diff(g1,b1)+B2*delta1;
poly L2=-h2+A2*diff(g1,b2)+B2*delta2;
poly L3=-h3+A2*diff(g1,b3)+B2*delta3;
poly R=h1*(a1-b1)+h2*(a2-b2)+h3*(a3-b3);
poly X1=x1-t*a1-(1-t)*b1;
poly X2=x2-t*a2-(1-t)*b2;
poly X3=x3-t*a3-(1-t)*b3;
ideal I=f1,f2,g1,g2,K1,K2,K3,L1,L2,L3,R,X1,X2,X3;
option(prot);
LIB "elim.lib";
ideal J=h1,h2,h3;
ideal K=eliminate(sat(I,J)[1],a1*a2*a3*b1*b2*b3*t*h1*h2*h3*A1*A2*B1*B2);
LIB "primdec.lib";
primdecGTZ(K);
```

This script produces the desired polynomials. We now explain how it works. Each polynomial in the list

```
ideal I=f1,f2,g1,g2,K1,K2,K3,L1,L2,L3,R,X1,X2,X3;
```

encodes a polynomial equation by setting the polynomial equal to zero. These equations arise from Definition 3.1. For simplicity we strengthen the dimension conditions to $h \in \text{span}(\nabla L(a), \delta)$ and $h \in \text{span}(\nabla L(b), \delta)\}$, respectively, and express these by introducing the three unknown coefficients of the two linear combinations as variables A_1, B_1, A_2, B_2. In total we form 14 equations.

The ideal I is the infinite set of polynomial consequences of the 14 polynomials obtained by forming linear combinations of these with polynomials as coefficients. As a subset of I we find the ideal $K \subseteq \mathbb{R}[\delta_1, \ldots, \delta_m, x_1, \ldots, x_m, r_1, r_2]$ containing consequences only involving δ, x, r_1, and r_2. Our computation shows that the ideal K is a principal ideal, *i.e.*, all its elements are polynomial multiples of a single polynomial P. The `eliminate` command computes this polynomial P. The last line of the script factors P into $x_m^2 - \sum_{i=1}^{m-1} x_i^2$ and a 37 term polynomial. It is not obvious that this polynomial gives the formula in Theorem 3.3, but for $m = 3$ the formula can easily be expanded and checked in Singular.

We still need to explain the operation `sat(I,J)` in the script. A priori, the vector h can always be chosen to be 0_3, and hence there would be no consequence for x in terms of δ, r_1 and r_2. To exclude this we *saturate* I wrt. the ideal $J = \langle h_1, h_2, h_3 \rangle$. We refer the reader to [11] for an introduction to elimination and saturation of polynomial ideals.

Proof of Theorem 3.3 We substitute $\delta^T a$ for r_1 and $\delta^T b$ for r_2. It remains to prove (under the assumption $h \neq 0_m$) that $\sum_{i=1}^{m-1} x_i^2 = x_m^2$ or

$$4(\delta^T a)(\delta^T b)(\delta^T x - \delta^T a)(\delta^T x - \delta^T b) + (\delta^T (a-b))^2 (\sum_{i=1}^{m-1} \delta_i^2 - \delta_n^2)(\sum_{i=1}^{m-1} x_i^2 - x_m^2) = 0$$

is a consequence of $x = ta + (1-t)b$, $h^T a = h^T b$, $\dim(\text{span}(h, \nabla L(a), \delta)) \leq 2$, $\dim(\text{span}(h, \nabla L(b), \delta)) \leq 2$, $h \neq 0_m$, $\sum_{i=1}^{m-1} a_i^2 - a_m^2 = 0$ and $\sum_{i=1}^{m-1} b_i^2 - b_m^2 = 0$.

To simplify we make substitutions and work over the complex numbers $\mathbb{C}$. In δ and h we multiply the last coordinate by i (where $i^2 = -1$), and in x, a, b we multiply the last coordinate by $-i$. With these changes our assumptions become

- $h \neq 0_m$ and $h \cdot a = h \cdot b$,
- $\{\delta, h, a\}$ and $\{\delta, h, b\}$ are both linearly dependent sets,
- $x = ta + (1-t)b$,
- $a \cdot a = 0$ and $b \cdot b = 0$.

where for $x, y \in \mathbb{C}^m$ we let $x \cdot y := \sum_{i=1}^{m} x_i y_i$. With this notation we must prove

$$4(\delta \cdot a)(\delta \cdot b)(\delta \cdot x - \delta \cdot a)(\delta \cdot x - \delta \cdot b) + (\delta \cdot (a-b))^2(\delta \cdot \delta)(x \cdot x) = 0. \qquad (9)$$

First assume $\mathrm{span}(h, \delta, a) \neq \mathrm{span}(h, \delta, b)$. In this case h and δ are proportional, implying $\delta \cdot a = \delta \cdot b$, which equals $\delta \cdot x$. Equation (9) follows easily.

Suppose now $\mathrm{span}(h, \delta, a) = \mathrm{span}(h, \delta, b)$. Then a, b and δ are in the same 2-dimensional plane. Suppose that $\delta = ka + lb$ with $k, l \in \mathbb{C}$. We compute the left hand side of (9):

$$\begin{aligned}
&4((ka+lb)\cdot a)((ka+lb)\cdot b)((ka+lb)\cdot(x-a))((ka+lb)\cdot(x-b))+\\
&((ka+lb)\cdot(a-b))^2((ka+lb)\cdot(ka+lb))(x\cdot x)\\
=&4((ka+lb)\cdot a)((ka+lb)\cdot b)((ka+lb)\cdot((t-1)(a-b)))((ka+lb)\cdot(t(a-b)))+\\
&((ka+lb)\cdot(a-b))^2((ka+lb)\cdot(ka+lb))(x\cdot x)\\
=&((ka+lb)\cdot(a-b))^2(4((ka+lb)\cdot a)((ka+lb)\cdot b)(t-1)t+((ka+lb)\cdot(ka+lb))(x\cdot x))\\
=&((ka+lb)\cdot(a-b))^2(4(lb\cdot a)(ka\cdot b)(t-1)t+((2kla\cdot b)(2(ta)\cdot(1-t)b)))=0.
\end{aligned}$$

In the case $\delta \notin \mathrm{span}(a, b)$ we have that a and b are dependent. Wlog $x = c \cdot a$ for some $c \in \mathbb{C}$. Now $x \cdot x = (ca) \cdot (ca) = c^2(a \cdot a) = c^2 0 = 0$. Translated to our original coordinates, we are in the case where $\sum_{i=1}^{m-1} x_i^2 = x_m^2$. □

When $x \in B$ is between the hyperplanes $\delta^T x = r_1$ and $\delta^T x = r_2$ we can exclude one of the cases of Theorem 3.3.

Lemma 3.4 *Suppose $x \in B$, with a and b in Definition 3.1 chosen such that $a_m \geq 0$ and $b_m \geq 0$. Furthermore suppose $r_1 < \delta^T x < r_2$. Then $\sum_{i=1}^{m-1} x_i^2 \neq x_m^2$.*

Proof. Consider the degree two polynomial we get by restricting $\sum_{i=1}^{m-1} x_i^2 - x_m^2$ to the line from a through x to b. This polynomial evaluates to zero in a and b. Since the degree is two it is either the zero polynomial or non-zero between a and b. In the second case we conclude that $\sum_{i=1}^{m-1} x_i^2 \neq x_m^2$. In the first case, every point y on the line passing through a and b satisfies $\sum_{i=1}^{m-1} y_i^2 = y_m^2$. The hypersurface defined by $\sum_{i=1}^{m-1} y_i^2 = y_m^2$ contains only lines passing through the origin. From the inequalities $a_m \geq 0$ and $b_m \geq 0$ it follows that a and b are on the same side of the origin, contradicting $r_1 r_2 < 0, \delta^T a = r_1$ and $\delta^T b = r_2$. □

4 Characterization of the convex hull

The goal of this section is to deduce Theorem 4.4 below.

Lemma 4.1 *Let $f \in \mathbb{R}[x_1, \ldots, x_m]$ be the left hand side of (8). Let $t \mapsto a + bt$ be a parametrization of a line. If $\delta^T b \neq 0$ and $b_m^2 > \sum_{i=1}^{m-1} b_i^2$ then $f(a + tb) \to -\infty$ as $t \to \pm\infty$.*

Proof. The summand $4r_1 r_2(\delta^T x - r_1)(\delta^T x - r_2)$ goes to $-\infty$. The second summand of f is either zero or has the sign of $\sum_{i=1}^{m-1} b_i^2 - b_m^2$ since it eventually will be dominated by $(r_1 - r_2)^2(\sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2)t^2(\sum_{i=1}^{m-1} b_i^2 - b_m^2)$. □

Proposition 4.2 *If $x \in C$ and $r_1 < \delta^T x < r_2$ then*

$$4r_1r_2(\delta^T x - r_1)(\delta^T x - r_2) + (r_1 - r_2)^2(\sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2)(\sum_{i=1}^{m-1} x_i^2 - x_m^2) \leq 0. \qquad (10)$$

Proof. Let $f \in \mathbb{R}[x_1, \dots, x_m]$ be the polynomial on the left hands side of (10). First observe that $f(0_m) > 0$. Suppose (10) did not hold for x. Then $f(x) > 0$. Consider the line starting at 0_m passing through x. On this line we find a point on the boundary of C. By Theorem 3.2 f has value zero here. Furthermore, by Lemma 4.1 the values of f far from the origin are negative. For a degree-two polynomial this is a contradiction. (Note that when applying Lemma 4.1 we assume $\delta^T x \neq 0$. If this is not true, we perturb x and f cannot be positive there). □

Proposition 4.3 *Let x satisfy (10) and $r_1 < \delta^T x < r_2$. If $x_m > 0$ then $x \in C$.*

Proof. The assumptions imply that the first term of (10) is positive. Since $\pm\delta \notin L^m$ (10) gives $x \in L^m$. Choose $\varepsilon > 0$ such that f is positive on an ε-ball around the origin. In an ε-ball around x we choose a point b such that b is in the interior of L^m and $\delta^T b \neq 0$. Consider the line $x + tb$ and the values that f attains on this line. For $t = -1$ we are in the ε-ball where f is positive. For $t \to \pm\infty$ the function goes to $-\infty$ by Lemma 4.1. For some $t_0 > 0$ we get $\delta^T(x + t_0 b) = r_i$ for $i = 1$ or $i = 2$. Furthermore, the Lorentz inequality is satisfied, so that $x + t_0 b$ is in C. As we move towards $t = 0$, f will attain value 0 as we pass the boundary of C. After this f stays positive at least until the $x_m = 0$ hyperplane is reached, where f attains a positive value on the line. We conclude $x \in \text{closure}(C)$.

To prove that $x \in C$, suppose this is not the case. Intersect C with $\{y \in \mathbb{R}^m : y_m - 1 \leq x_m\}$. This intersection is compact because the convex hull of a compact set is compact. We conclude $x \in C$. □

We complete and summarize our result in the following theorem.

Theorem 4.4 *Assume $x_m \geq 0$. Then $x \in C$ if and only if x is in L^m and satisfies*

$$4r_1r_2(\delta^T x - r_1)(\delta^T x - r_2) + (r_1 - r_2)^2(\sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2)(\sum_{i=1}^{m-1} x_i^2 - x_m^2) \leq 0. \qquad (11)$$

Proof. Suppose $x \in C$. Clearly $x \in L^m$ since $C \subseteq L^m$. If $r_1 < \delta^T x < r_2$ then then (11) follows from Proposition 4.2. If $r_1 \geq \delta^T x$ or $\delta^T x \geq r_2$, then the first term on the left hand side of (11) is ≤ 0. The second term is ≤ 0 since $x \in C \subseteq L^m$.

Conversely, suppose $x \in L^m$ and (11) is satisfied. If $r_1 < \delta^T x < r_2$, then $x \in C$ by Proposition 4.3. If not then $x \in C$ by the definition of C. □

5 Conic quadratic representations

We have identified (11) for describing $C = \text{conv}(S_1 \cup S_2)$. However, this inequality is not in conic quadratic form. In this section we give conic quadratic representations of (11). We first consider the special case when $\delta_2 = \ldots = \delta_{m-1} = 0$.

Lemma 5.1 *If we assume that $\delta_2 = \delta_3 = \cdots = \delta_{m-1} = 0$ then*

$$4r_1r_2(\delta^T x - r_1)(\delta^T x - r_2) + (r_1 - r_2)^2(\sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2)(\sum_{i=1}^{m-1} x_i^2 - x_m^2) =$$

$$((r_1+r_2)(\delta_1 x_1+\delta_m x_m)-2r_1 r_2)^2+(r_1-r_2)^2(\delta_1^2-\delta_m^2)(\sum_{i=2}^{m-1} x_i^2)-(r_1-r_2)^2(\delta_m x_1+\delta_1 x_m)^2.$$

In particular, if $\pm\delta \notin L^m$ the polynomial is a conic quadratic form with m terms.

The proof of Lemma 5.1 is simply a sequence of equalities, and it is therefore placed in the appendix. We now give an interpretation of the coefficients in the expression of Lemma 5.1. For simplicity suppose furthermore that $\delta_1 > 0$. Then

- $\delta_1 x_1 + \delta_m x_m = \delta^T x$
- $\delta_1^2 - \delta_m^2 = \sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2$
- $\delta_m x_1 + \delta_1 x_m = \frac{\delta_m}{\sqrt{\sum_{i=1}^{m-1} \delta_i^2}} \begin{pmatrix} x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} \cdot \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_{m-1} \end{pmatrix} + x_m \sqrt{\sum_{i=1}^{m-1} \delta_i^2}$
- $\sum_{i=2}^{m-1} x_i^2$ is the squared norm of the projection of x to $\mathrm{span}(\delta, e_m)^\perp$

Except for the third item, these quantities have geometric meaning. All are invariant under orthonormal linear transformation fixing the last coordinate. Since such transformations preserve the Lorentz cone, the assumption $\delta_2 = \cdots = \delta_{m-1} = 0$ was made without loss of generality, and in general the coefficients of our quadratic equation can be obtained from the right hand sides above.

The sum $\sum_{i=2}^{m-1} x_i^2$ remains a sum of squares after a linear transformation of coordinates. If $\delta_2, \ldots, \delta_{m-1}$ are not all zero, we still want a closed form formula. Let $b_2, \ldots, b_{m-1}$ be an orthogonal basis for $\mathrm{span}(\delta, e_m)^\perp$. Then the squared length of the projection of x to this subspace is given by

$$\frac{(x \cdot b_2)^2}{b_2 \cdot b_2} + \cdots + \frac{(x \cdot b_{m-1})^2}{b_{m-1} \cdot b_{m-1}}.$$

We have reached the following generalization of Lemma 5.1

Lemma 5.2 *Let $b_2, \ldots, b_{m-1}$ be an orthogonal basis for $\mathrm{span}(\delta, e_m)^\perp$. Then*

$$4r_1 r_2(\delta^T x - r_1)(\delta^T x - r_2) + (r_1 - r_2)^2(\sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2)(\sum_{i=1}^{m-1} x_i^2 - x_m^2) =$$

$$((r_1 + r_2)\delta^T x - 2r_1 r_2)^2 + (r_1 - r_2)^2(\delta_1^2 - \delta_m^2) \left(\frac{(x \cdot b_2)^2}{b_2 \cdot b_2} + \cdots + \frac{(x \cdot b_{m-1})^2}{b_{m-1} \cdot b_{m-1}} \right)$$

$$-(r_1 - r_2)^2 \left(\frac{\delta_m}{\sqrt{\sum_{i=1}^{m-1} \delta_i^2}} \begin{pmatrix} x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} \cdot \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_{m-1} \end{pmatrix} + x_m \sqrt{\sum_{i=1}^{m-1} \delta_i^2} \right)^2.$$

Lemma 5.2 gives a general scheme for obtaining conic quadratic forms of inequality (11). We now give a concrete conic quadratic form of (11), which can be directly computed from the data $(\delta, r_1, r_2) \in \mathbb{R}^{m+2}$. Let $x^k := (x_1, \ldots, x_k)$ and $\delta^k = (\delta_1, \ldots, \delta_k)$ be vectors of the first k coordinates of x and δ. Inequality (11) can be written as:

$$((r_1 + r_2)\delta^T x - 2r_1 r_2)^2 - (r_1 - r_2)^2 ||\delta^{m-1}||^2 (x_m + \frac{(\delta^{m-1})^T x^{m-1}}{||\delta^{m-1}||^2} \delta_m)^2$$

$$+ (r_1 - r_2)^2(||\delta^{n-1}||^2 - \delta_m^2)(\sum_{k=2}^{m-1} \frac{||\delta^{k-1}||^2}{||\delta^k||^2}(x_k - \frac{(\delta^{k-1})^T x^{k-1}}{||\delta^{k-1}||^2} \delta_k)^2) \leq 0. \qquad (12)$$

6 Conic quadratic intersection cuts and split cuts

In a linear setting split cuts and intersection cuts are equivalent [1]. We now give an example showing that this is not true in a conic quadratic setting. Consider a mixed integer conic quadratic set $Q_I := \{x \in Q : x_j \in \mathbb{Z} \text{ for } j \in I\}$ with continuous relaxation $Q := \{x \in \mathbb{R}^n : Ax - d \in L^m\}$, where the rows of A are *not* linearly independent. For a spit disjunction $\pi^T x \le \pi_0 \vee \pi^T x \ge \pi_0 + 1$, a *split cut* for Q_I is a valid inequality for $\text{conv}(Q_1 \cup Q_2)$ with $Q_1 = \{x \in Q : \pi^T x \le \pi_0\}$ and $Q_2 = \{x \in Q : \pi^T x \ge \pi_0 + 1\}$ that is *not* valid for Q.

Example 6.1 The conic quadratic set

$$Q := \{(x, y) \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in L^3\} \tag{13}$$

equals $\{(x, y) \in \mathbb{R}^2 : 1 \le 2xy \wedge y > 0\}$ and is shown in Figure 1. Consider the relaxation Q^B of Q obtained from the first and last row of A:

$$Q^B := \{(x, y) \in \mathbb{R}^2 : (x-1)^2 \le (x+y-1)^2 \wedge x+y-1 \ge 0\}.$$

The set Q^B is polyhedral since Q^B the preimage of the 2-dimensional Lorentz cone under a linear map. We may think of Q^B as a relaxation obtained by substituting L^3 with $L^3 + \mathbb{R} \cdot e_2$ in (13). By instead choosing $L^3 + \mathbb{R} \cdot e_1$ we get the relaxation $Q^{\bar{B}}$ of Q obtained from the last two rows of the matrix defining Q. In general, adding any line generated by some $v \in \mathbb{R}^2 \times \{0\}$ to L^3 gives a relaxation of Q. Relaxations for three choices of v are shown in Figure 1. The important observation is that the boundary of such a relaxation is tangent to the boundary of Q *in at most one point.*

Now consider any split disjunction (π, π_0), and suppose the intersection cut is a secant line between two points a, b on the curve $1 = 2xy$. For an intersection cut from a relaxation of the above type to give the same cut, the relaxation must contain *both* a and b in the boundary, which as argued above is impossible. Conic quadratic intersection cuts are therefore not always split cuts.

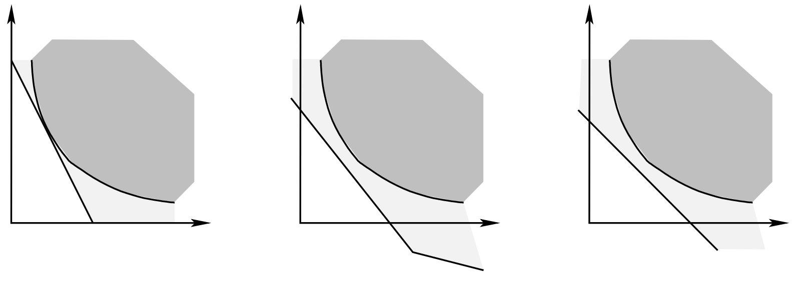

Figure 1: The conic quadratic set Q of (13) and relaxations for $v = (1, 0), (\sqrt{3/4}, 1/2)$ and $(\sqrt{1/2}, \sqrt{1/2})$ respectively.

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7 Appendix: Two proofs

Proof of Lemma 2.3 .

Observe that since $\pi \in \mathcal{L}^\perp$ by Lemma 1.(i), and since $(A^T\delta - \pi)\perp\mathcal{L}^\perp$ from the definition of δ, we have $\delta^T Ax = \pi^T x$ for all $x \in \mathbb{R}^n$.

(i): Lemma 1.(ii) gives $\pi^T z \in]\pi_0, \pi_0 + 1[$ for $z \in \mathcal{A}$ arbitrary. Therefore $0 = \delta^T(Az - d) = (\pi^T z - \pi_0) + r_1$ implies $0 \in]r_1, r_2[$. This also implies $0_m \notin S_1 \cup S_2$, and since 0_m is apex of the pointed cone L^m, we have $0_m \notin \text{conv}(S_1 \cup S_2)$.

(ii): Wlog we only consider $k = 1$. If $z \in Q_1$, then $Az - d \in L^m$ and $\pi^T z \leq \pi_0$, and therefore $\delta^T(Az - d) = (\pi^T z - \pi_0) + r_1 \leq r_1$ which implies $z \in \{x \in \mathbb{R}^n : Ax - d \in S_1\}$. Conversely, if $z \in \{x \in \mathbb{R}^n : Ax - d \in S_1\}$, then $Az - d \in L^m$ and $\delta^T(Az - d) = (\pi^T z - \pi_0) + r_1 \leq r_1$ which implies $z \in Q_1$.

(iii): The inclusion $Q_1 \cup Q_2 \subseteq \{x \in \mathbb{R}^n : Ax - d \in \text{conv}(S_1 \cup S_2)\}$ follows from (ii), and since $\{x \in \mathbb{R}^n : Ax-d \in \text{conv}(S_1 \cup S_2)\}$ is convex, we have $\text{conv}(Q_1 \cup Q_2) \subseteq \{x \in \mathbb{R}^n : Ax - d \in \text{conv}(S_1 \cup S_2)\}$.

For the other inclusion suppose $z \in \{x \in \mathbb{R}^n : Ax - d \in \text{conv}(S_1 \cup S_2)\}$. We write $z = l + \hat{l}$ with $l \in \mathcal{L}$ and $\hat{l} \in \mathcal{L}^\perp$, and since $Az - d \in \text{conv}(S_1 \cup S_2)$, we may write $Az - d = \lambda \cdot y^1 + (1 - \lambda) \cdot y^2$ for some $y^1 \in S_1$, $y^2 \in S_2$ and $\lambda \in [0, 1]$. Also, since $\text{rank}(A) = m$, the linear map $x \mapsto Ax$ maps $\mathcal{L}^\perp$ onto $\mathbb{R}^m$, so we can write $y^1 = A \cdot \hat{l}^1 - d$ and $y^2 = A \cdot \hat{l}^2 - d$, where $\hat{l}^1, \hat{l}^2 \in \mathcal{L}^\perp$. Finally the identity $Az = A\hat{l} = A(\lambda\hat{l}^1 + (1 - \lambda)\hat{l}^2)$ gives $A(\lambda\hat{l}^1 + (1 - \lambda)\hat{l}^2 - \hat{l}) = 0_m$ which implies $\lambda\hat{l}^1 + (1 - \lambda)\hat{l}^2 - \hat{l} \in \mathcal{L} \cap \mathcal{L}^\perp$, and therefore $\hat{l} = \lambda\hat{l}^1 + (1 - \lambda)\hat{l}^2$.

Define $x^1 := l + \hat{l}^1$ and $x^2 := l + \hat{l}^2$. We have $Ax^1 - d = y^1 \in L^m$ and $Ax^2 - d = y^2 \in L^m$. Furthermore, since $y^1 \in S_1$, we have $\delta^T y^1 = \delta^T(Ax^1 - d) = (\pi^T x^1 - \pi_0) + r_1 \leq r_1$ which implies $\pi^T x^1 \leq \pi_0$. Similarly $\pi^T x^2 \geq \pi_0 + 1$. Hence $x^1 \in Q_1$ and $x^2 \in Q_2$, and therefore $z = \lambda x^1 + (1 - \lambda)x^2 \in \text{conv}(Q_1 \cup Q_2)$. □

Proof of Lemma 5.1 We subtract the left hand side from the right hand side and get:

$$4r_1r_2(\delta^T x - r_1)(\delta^T x - r_2) + (r_1 - r_2)^2(\sum_{i=1}^{m-1} \delta_i^2 - \delta_m^2)(\sum_{i=1}^{m-1} x_i^2 - x_m^2)$$

$$-((r_1 + r_2)(\delta_1 x_1 + \delta_m x_m) - 2r_1r_2)^2 - (r_1 - r_2)^2(\delta_1^2 - \delta_m^2)(\sum_{i=2}^{m-1} x_i^2)$$

$$+(r_1 - r_2)^2(\delta_m x_1 + \delta_1 x_m)^2$$

$$= 4r_1r_2(\delta^T x - r_1)(\delta^T x - r_2) + (r_1 - r_2)^2(\delta_1^2 - \delta_m^2)(x_1^2 - x_m^2)$$

$$-((r_1 + r_2)(\delta_1 x_1 + \delta_m x_m) - 2r_1r_2)^2 + (r_1 - r_2)^2(\delta_m x_1 + \delta_1 x_m)^2$$

$$= 4r_1r_2((\delta^T x)^2 - (r_1 + r_2)(\delta^T x) + r_1r_2)$$

$$+(r_1 - r_2)^2((\delta_1^2 - \delta_m^2)(x_1^2 - x_m^2) + (\delta_m x_1 + \delta_1 x_m)^2)$$

$$-((r_1 + r_2)^2(\delta_1 x_1 + \delta_m x_m)^2$$

$$+4r_1^2r_2^2 - 4r_1r_2(r_1 + r_2)(\delta_1 x_1 + \delta_m x_m))$$

$$= 4r_1r_2((\delta^T x)^2 + r_1r_2)$$

$$+(r_1 - r_2)^2(\delta_1^2 x_1^2 + \delta_m^2 x_m^2 - \delta_1^2 x_m^2 - \delta_m^2 x_1^2 + \delta_m^2 x_1^2 + \delta_1^2 x_m^2 + 2\delta_1\delta_m x_1 x_m)$$

$$-((r_1 + r_2)^2(\delta_1 x_1 + \delta_m x_m)^2 + 4r_1^2r_2^2)$$

$$= 4r_1r_2(\delta^T x)^2 + (r_1 - r_2)^2(\delta_1^2 x_1^2 + \delta_m^2 x_m^2 + 2\delta_1\delta_m x_1 x_m)$$

$$-(r_1 + r_2)^2(\delta_1 x_1 + \delta_m x_m)^2$$
$$= (r_1 - r_2)^2(\delta_1 x_1 + \delta_m x_m)^2 - (r_1 - r_2)^2(\delta_1 x_1 + \delta_m x_m)^2 = 0.$$

We notice that $\delta_1^2 - \delta_m^2 > 0$ for $\pm\delta \notin L^m$, which makes all coefficients in our expression have the desired signs. □