TRACE OF CATEGORIES, AND A UNIVERSAL PROPERTY OF THE REPRESENTATION RING OF A FINITE GROUP

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The representation ring $R_{\mathbb{C}}(G)$ of a finite group is the commutative ring with one additive generator for each complex irreducible representation of G. The multiplication is induced by the tensor product operation of representations. One well known universal property of it is that it is the Grothendieck ring K_o of the category of (finite dimensional) complex G-representations (see e.g. [5] 12.1). We are exhibiting here another universal property of the complexification $\mathbb{C} \otimes R_{\mathbb{C}}(G) = FC(G)$ of this ring. The ring $FC(G)$ may be described in more elementary terms as the $\mathbb{C}$-algebra of complex valued central functions $f:G \longrightarrow \mathbb{C}$ on G (see [5] 9.1). Recall that a function $f:G \longrightarrow \mathbb{C}$ is <u>central</u> (or a <u>class function</u>) if

(0.1) $$f(g_1 \cdot g_2) = f(g_2 \cdot g_1) \qquad \forall g_1, g_2 \in G .$$

The universal property we have in mind is that $FC(G)$ becomes a <u>Trace-object</u> for the $\mathbb{C}$-vector space valued hom-functor for G-representations.

We discuss this notion in § 1. In § 2 we prove the assertion about $FC(G)$; § 3 contains some remarks on the notion dual of trace, "center" .

1. Trace objects

The notion of "trace objects for endo-profunctors" on enriched categories has been around at least since 1971, where Lawvere and I jointly investigated some properties of the notion, as well as some examples. Some of these are described in [3] , § 5. Lawvere also calculated the Trace for the category of finite sets (it is a semiring). I calculated the Trace for the category of G-representations on $\mathbb{C}$. (This result was announced in a talk at the University of Chicago, May 1972.). The notion of trace has also been considered by Kelly and Laplaza [1] .

We shall here only describe the trace notion for "identily profunctors", that is, for hom-functors. Suppose $\underline{M}$ is a small category whose hom-functor takes value in a category $\mathcal{M}$ (technically: $\underline{M}$ is enriched over $\mathcal{M}$). By the trace object for $\underline{M}$ we understand the coend (formed in $\mathcal{M}$)

$$Tr(\underline{M}) = \int^{M \in \underline{M}} hom(M,M)$$

together with the "inclusion" maps (morphisms in $\mathcal{M}$)

$$tr_M : hom(M,M) \longrightarrow Tr(\underline{M}) \qquad (\forall M \in \underline{M});$$

see [4] for the notion of coend. For the case where the set-valued hom functor for $\underline{M}$ comes about from the $\mathcal{M}$-valued hom functor via a faithful $\mathcal{M} \longrightarrow$ <u>set</u>, the notion can be described in elementary terms as follows. Call a family of maps in $\mathcal{M}$

$$hom(M,M) \xrightarrow{t_M} T \qquad \forall M \in \underline{M}$$

<u>compatible</u> if for any pair of maps in $\underline{M}$

$$M \underset{h}{\overset{f}{\rightleftarrows}} N$$

we have

(1.1) $$t_M(h \circ f) = t_N(f \circ h) .$$

A trace object for $\underline{M}$ is an object $Tr(\underline{M}) \in \mathcal{M}$ and a compatible family

$$hom(M,M) \xrightarrow{tr_M} Tr(\underline{M}) \qquad \forall\ M \in \underline{M}$$

which is universal : to any compatible family $\{t_M : hom(M,M) \longrightarrow T \mid M \in \underline{M}\}$ there exists a unique $s: Tr(\underline{M}) \longrightarrow T$ so that $t_M = s \circ tr_M \ \forall M \in \underline{M}$. Clearly $Tr(\underline{M})$ together with the tr_M's is unique up to unique isomorphism.

If $\mathcal{M}$ is the category of vector spaces over a field K , and $\underline{M}$ the category of finite dimensional vector spaces over K , the hom sets $hom(M_1,M_2)$ are in a natural way vector spaces, so we have a hom functor into $\mathcal{M}$ and can form $Tr(\underline{M})$.

Proposition 1 The family

(1.2) $$hom(M,M) \xrightarrow{tr_M} K \qquad M \in \underline{M}$$

(where $tr_M(f)$ is the ordinary trace of the endomorphism $f: M \longrightarrow M$) is the trace object in the sense above.

Proof First, "taking trace" is a linear proces, so the maps tr_M do live in $\mathcal{M}$, as required.
Secondly, the property (1.1) is well known for the 'ordinary' trace formation. So (1.2) is a compatible family. To see its universal property, let $\{t_M \mid M \in \underline{M}\}$ be some compatible family with value in T . We claim that for any endomorphism $f: M \longrightarrow M$, we have

(1.3) $$t_M(f) = tr_M(f) \cdot t_K(id_K)$$

so that the $s: K \longrightarrow T$ to be produced is just the one that corresponds to the vector $t_K(id_K) \in T$ .To see (1.3) choose a basis for M , and denote by $proj_i$, $incl_i$ the evident linear maps

$$M \underset{incl_i}{\overset{proj_i}{\rightleftarrows}} K \qquad\qquad i = 1,\dots,\dim(M)$$

If $\{a_{ij}\}$ is the matrix of f with respect to this basis, we have

$$f = \sum_{i,j} a_{ij} \cdot (incl_i \circ proj_j) \;,$$

Thus

$$t_m(f) = \sum_{i,j} a_{ij} \cdot t_M(incl_i \circ proj_j)$$

(by K-linearity of t_M)

$$= \sum_{ij} a_{ij} \cdot t_K(proj_j \circ incl_i)$$

(by (1.1))

$$= \sum_{ij} a_{ij} \cdot t_K(\delta_{ij} \cdot id_K)$$

$$= \sum_{ij} a_{ij} \cdot \delta_{ij} \circ t_K(id_K)$$

$$= \sum_{i} a_{ii} \cdot t_K(id_K)$$

$$= tr_M(f) \cdot t_K(id_K) \;.$$

This prove the proposition. A more elegant argument, due to Brian Day (1972) goes as follows

$$\int^M hom(M,M) \;\cong\; \int^M hom(M,K) \otimes M$$

$$\cong\; K \;,$$

The first isomorphism sign by $hom(M,M) \cong M^* \otimes M$, and the second by density of finite dimensional vector spaces in $\mathcal{M}$.

2. The ring of central functions as a trace object

Let G be a finite group and K a field. Let $\underline{M}$ denote the category of finite dimensional representations of G over K. Let $FC(G)$ denote the ring (K-algebra) of central functions on G with values in K. For each $M \in \underline{M}$ we have a map

(2.1) $$\hom(M,M) \xrightarrow{\chi_M} FC(G)$$

given by

$$f \longmapsto \{g \longmapsto \text{Trace of } f \circ \rho_g\}$$

where $\rho_g : M \to M$ is the action of $g \in G$ on M.

Note that

$$\chi_M(id_M) = \{g \longmapsto \text{Trace } (\rho_g)\} = \text{character of } M .$$

Clearly the χ_M in (2.1) depends in a K-linear way on $f \in \hom(M,M)$. We claim that the family of maps $\{\chi_M \mid M \in \underline{M}\}$ is a compatible family, in the sense of §1. Let $f: M \longrightarrow N$ and $h: N \longrightarrow M$ be morphism in $\underline{M}$. To prove

$$\chi_M(h \circ f) = \chi_N(f \circ h),$$

we have, for any $g \in G$,

$$\chi_M(h \circ f)\ (g) = tr_M(h \circ f \circ \rho_g)$$

$$= tr_M(h \circ (\rho_g \circ f))$$

(since f is G-equivariant)

$$= tr_N((\rho_g \circ f) \circ h)$$

$$= tr_N(f \circ h \circ \rho_g)$$

(since $f \circ h$ is G-equivariant)

$$= \chi_N(f \circ h)\ (g) ,$$

so that $\chi_N(f \circ h) = \chi_M(h \circ f)$, as required.

<u>Theorem</u>. If $K=\mathbb{C}$, then the family of maps (2.1) makes $FC(G)$ into the trace object for $\underline{M}$ (with respect to the $\mathbb{C}$-vector space structure on the hom sets of $\underline{M}$). Alternatively, the family χ_M $(M \in \underline{M})$ is a universal compatible family of $\mathbb{C}$-linear maps.

<u>Proof</u>. Given any other compatible family

$$t_M : \hom(M,M) \longrightarrow T \qquad\qquad M \in \underline{M}$$

of $\mathbb{C}$-linear maps. To define

$$s : FC(G) \longrightarrow T$$

we use the fact (see e.g. [5] Theorem 6) that $FC(G)$ has a $\mathbb{C}$-basis consisting of the finitely many irreducible characters of G . Let χ_N be such. Put

$$s(\chi_N) := t_N(id_N)$$

where N is an irreducible representation with χ_N as character. If N' is another such, $N \cong N'$ in $\underline{M}$, which by the compatibility condition for the t_M's is easily seen to imply the independence of the choice of N . Having defined s on the basis, we extend by linearity, to get s .

It just remains to be proved that, for $M \in \underline{M}$, the triangle

$$\begin{array}{ccc} \hom(M,M) & \xrightarrow{\chi_M} & FC(G) \\ & \searrow{\scriptstyle t_M} & \downarrow{\scriptstyle s} \\ & & T \end{array} \tag{2.2}$$

commutes.

Let the canonical decomposition of M be

$$M = M_1 \oplus \ldots \oplus M_h$$

(see [5] 2.7) so that each M_j is direct sum of mutually isomorphic irreducible representations. If $f: M \longrightarrow M$, it follows from Schur's lemma that f is actually of form $f_1 \oplus \ldots \oplus f_h$, where $f_j : M_j \longrightarrow M_j$. Also, the identity map of M can be decomposed

$$\sum_{j=i}^{h} \mathrm{incl}_j \circ \mathrm{proj}_j ,$$

and $f = \sum \mathrm{incl}_j \circ f_j \circ \mathrm{proj}_j$. Then

$$t_M(f) = t_M(\sum_j \mathrm{incl}_j \circ f_j \circ \mathrm{proj}_j)$$

$$= \sum_j t_M((\mathrm{incl}_j \circ f_j) \circ \mathrm{proj}_j)$$

$$= \sum_j t_{M_j}(\mathrm{proj}_j \circ \mathrm{incl}_j \circ f_j) = \sum_j t_{M_j}(f_j) ,$$

using additivity and (1.1) for the t_M's. Similarly for the χ_M's. Thus, it suffices to prove the commutativity for the case when $M = M_j$ $(j=1,\ldots,h)$, that is, M is "isotypic",

$$M = N \oplus \ldots \oplus N \quad (k \text{ copies, say})$$

with N irreducible. An endomorphism f of M in $\underline{M}$ is thus given by a $k\times k$ matrix with entries from the ring $K' = \mathrm{hom}(N,N)$ which, however, using Schur's lemma ([5], Proposition 4) equals $\mathbb{C}$.

By the same sort of calculation as the one employed in the proof of Proposition 1, we see that

$$(2.3) \qquad t_M(f) = \sum_i a_{ii} \cdot t_N(\mathrm{id}_N) .$$

But this equals, by construction of s

$$(2.4) \qquad \sum_i a_{ii} \cdot s(\chi_N) = \sum_i a_{ii} \cdot s(\chi_N(\mathrm{id}_N)) = s(\sum_i a_{ii} \cdot \chi_N(\mathrm{id}_N)) .$$

Now (2.3) holds also when t is replaced by χ since the χ_M's are also $\mathbb{C}$-linear and compatible. Thus, using (2.3) for χ,

$$\sum_i a_{ii} \cdot \chi_N(id_N) = \chi_M(f)$$

whence the right hand side of (2.4) equals $s(\chi_M(f))$.

This proves $s \circ \chi_M = t_M$. Uniqueness of s with this property is clear, since elements of form $\chi_M(f)$ ($M \in \underline{M}$, $f \in hom(M,M)$ generate the vector space $FC(G)$.

The theorem is proved.

3. <u>The center of a category</u>.

Let $\underline{M}$ and $\mathcal{M}$ be as in §1. We define the <u>center</u> of $\underline{M}$ to the categorical dual notion of the trace of $\underline{M}$, that is,

$$\text{Cent}\ (\underline{M}) = \int_{M \in \underline{M}} hom(M,M) ,$$

taking "end" instead of coend. Thus, $\text{Cent}(\underline{M})$ comes equipped with a compatible family of "projections" in $\mathcal{M}$,

$$\sigma_M : \text{Cent}(\underline{M}) \longrightarrow hom(M,M) ,$$

with $f_* \circ \sigma_M = h^* \circ \sigma_N$ for any $f:M \longrightarrow N$, $h:N \longrightarrow M$, in $\underline{M}$, and is universal with this property.

For the case considered in §2 : $\underline{M}$ = finite dimensional complex representations of a finite group G, $\mathcal{M}$ = complex vector spaces, one can easily see that the center in the sense above is $Z(G)$, the center of the group algebra $\mathbb{C}[G]$, with the σ's

$$\sigma_M : Z(G) \longrightarrow hom(M,M) \qquad , M \in \underline{M} ,$$

given by $\lambda \longmapsto$ (action by λ). Note that action by λ is a morphism in $\underline{M}$ because λ commutes with everything in the group algebra.

I believe one may obtain an alternative proof of the Theorem of §2 by observing : (i) $\underline{M}$ is self-dual (associate to a representation M its "contragredient" representation, M* , [5], 2.1 ex.3); (ii) the dual Z(G)* of Z(G) can be identified with FC(G) in a natural way (cf. [2] , XVIII. Theorem 5) .

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