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CORRIGENDUM AND ADDENDA TO THE PAPER "CONVENIENT VECTOR SPACES EMBED..."

by A. KOCK and G.E. REYES

RÉSUMÉ. Dans cet article, la construction du plongement de la catégorie $\underline{E}$ des "bons" espaces vectoriels dans le topos des "Cahiers" faite dans l'article indiqué dans le titre est corrigée, et le principal résultat de cet article est amélioré en montrant que ce plongement préserve toute la structure cartésienne fermée de la catégorie $\underline{E}$.

The main assertion of [*], namely that convenient vector spaces embed nicely into the Cahiers topos, is correct and follows indeed from the "Weil prolongations" developed in that paper. However, the attempted shortcut which the author of [*] made, in describing a site of definition for the Cahiers topos as a semi-direct product $\underline{f\alpha}\underline{W}$ arising out of Weil prolongation, is an error. The site thus constructed has the right objects, but too few maps.

It is possible to put in the missing maps in the spirit of the semidirect product construction (a "twisted" semidirect product). But since this is quite unelegant, we prefer to give the embedding construction in terms of a more standard description of the site of definition for the Cahiers topos, namely by describing as certain category $\underline{D}$ of C^∞ rings.

Besides correcting the embedding construction of [*], we improve its main result by proving that the embedding preserves all exponentials Y^X of convenient vector spaces (where [*] only dealt with the case where X is finite dimensional).

As site of definition for the Cahiers topos, we take the (dual of) the full subcategory $\underline{D}$ of the category of C^∞ rings of form $C^\infty(R^k)\otimes_\infty W$, where $k \in \mathbf{N}$, and W is a Weil algebra. Thus there is a faithful functor, bijective on objects

$$(1) \qquad i: \underline{f\alpha}\underline{W} \to \underline{D},$$

sending (R^k,W) to $C^\infty(R^k)\otimes_\infty W$.

The main construction in [*] is that of Weil prolongation of a convenient vector space, which takes form of an *action* of $\underline{W}$ on $\underline{E}$ (= category of convenient vector spaces and their smooth, not necessarily linear, maps), i.e., a functor

(2) $$\otimes: \underline{E}\times\underline{W} \to \underline{E},$$

cf. [*] Theorem 3.1.

We shall extend the action (2) to an action by $\underline{D} \supset \underline{W}$,

(3) $$\otimes: \underline{E}\times\underline{D} \to \underline{E}.$$

In fact, we shall ultimately have, for $C = C^\infty(R^k)\otimes_\infty W$,

(4) $$X\otimes C = X\otimes(C^\infty(R^k)\otimes_\infty W) \simeq C^\infty(R^k,X)\otimes W \ ;$$

note that $C^\infty(R^k,X)$ is convenient if X is, so $C^\infty(R^k,X)\otimes W$ makes sense as a convenient vector space.

The right hand of (4) is functorial in $X \in \underline{E}$, but not a priori functorial in the $\underline{D}$-variable $C^\infty(R^k)\otimes_\infty W$; so we shall give a more intrinsic description of $X\otimes C$. If $W = C^\infty(R^\lambda)/I$, we have $C = C^\infty(R^{k+\lambda})/I^*$, where I^* is the ideal in $C^\infty(R^{k+\lambda})$ generated by $I \subset C^\infty(R^\lambda) \subset C^\infty(R^{k+\lambda})$. We put

(5) $$X\otimes C := C^\infty(R^{k+\lambda},X)/I^*(X)$$

(notation of [*] §2).

This is well defined as a vector space, and for trivial reasons functorial in C (a fact that does not depend on the special nature of the ideal I^* that defines C). Also, it is functorial in the X-variable with respect to smooth *linear* maps. We shall make (5) functorial (as a set, to start with) in the X-variable with respect to arbitrary smooth maps.

For this, we shall need the following considerations. In [**], Theorem 2.11, we proved that for $I = m^r \subset C^\infty(R^\lambda)$ (= ideal of functions that vanish to order r at $0 \in R^\lambda$), and for Y convenient, the following conditions on $g: R^\lambda \to Y$ are equivalent:

(i) $$g \in I(Y),$$

(ii) g can be written $g(s) = \Sigma\ k_i(s).\ h_i(s)$,
with $h_i: R^\lambda \to R$ in I, and $k_i: R^\lambda \to Y$ smooth.

These conditions in turn are equivalent to

(iii) $$D^{\alpha}g(0) = 0 \quad \text{for all } \alpha \in A$$

where A is the set of multi-indices in λ variables of degrees $< r$. (The equivalence of (iii) with (i) and (ii) follows from [**], Theorem 2.12, say.)

We prove, for any Weil ideal $I \supset m^r$, and any convenient vector space X:

PROPOSITION 1. Let $f\colon R^{k+\lambda} \to X$. *Then*

(*) $$f^{\wedge}\colon R^{\lambda} \to C^{\infty}(R^k,X) \quad \in I(C^{\infty}(R^k,X))$$

iff

(**) $$f\colon R^{k+\lambda} \to X \quad \in I^*(X).$$

PROOF. For the special case $I = m^r$, this is almost immediate from the above: if $f^{\wedge} \in I(C^{\infty}(R^k,X))$, we have, by (i) $\Rightarrow$ (ii), that

$$f^{\wedge}(s) = \Sigma\, k_i(s).\, h_i(s) \qquad \text{with } h_i \in I,$$

so

$$f(t,s) = \Sigma\, k_i(t,s).\, h_i(s) \qquad \text{with } h_i \in I,$$

and this immediately implies $f \in I^*(X)$. Conversely, if $f \in I^*(X)$, one clearly has $f(t,-) \in I(X)$ for any $t \in R^k$, so

$$D^{\alpha}f(t,0) = 0 \quad \text{for any } \alpha \in A \text{ and } t \in R^k,$$

so $D^{\alpha}f^{\wedge}(0) = 0$ in $C^{\infty}(R^k,X)$, whence by (iii) $\Rightarrow$ (i), $f^{\wedge} \in I(C^{\infty}(R^k,X))$.

The general case $I \supset m^r$ is derived from the special case by some finite matrix calculations, replacing (iii) by some other system of linear equations in the Taylor coefiicients $D^{\alpha}g(0)$: consider the exact sequence of finite dimensional vector spaces

$$0 \to I/m^r \longrightarrow C^{\infty}(R^{\lambda})/m^r \longrightarrow C^{\infty}(R^{\lambda})/I \to 0\ ;$$

we identify it with an exact sequence of form

(6) $$0 \longrightarrow R^B \xrightarrow{\ a\ } R^A \xrightarrow{\ C\ } R^C \longrightarrow 0$$

by identifying $f \in C^\infty(R^\lambda)$ (mod m^r) with the A-tuple f_α, where $f_\alpha = (1/|\alpha|!)D^\alpha f(0)$, and by picking a base for I/m^r consisting of polynomials of degree $< r$:

$$h_i = \Sigma_{\alpha\in A}\ a_{i\alpha}.s^\alpha \ .$$

The $a_{i\alpha}$'s form the entries of the matrix a in (6). The matrix c is constructed in some arbitrary way to make (6) exact. For an $f: R^\lambda \to R$ to belong to I, it is by the exactness necessary and sufficient that the Taylor coefficients f_α satisfy the linear equations

(iii') $$\Sigma_{\alpha\in A}\ f_\alpha.c_{\alpha j} = 0 \quad \text{for all } j \in C.$$

More generally, if $f: R^\lambda \to Y$, and $f \in I(Y)$, then (iii') holds; for it suffices to test with ψ's in Y', and $(\psi\circ f)_\alpha = \psi(f_\alpha)$. (The converse implication also holds.) The exactness of (6) is preserved by tensoring with any vector space Y. This means that if $(y_\alpha)_{\alpha\in A}$ is an A-tuple of vectors in Y, and $\Sigma\ y_\alpha.c_{\alpha j} = 0$ for all j, then there exists a B-tuple $(z_i)_{i\in B}$ of vectors with

$$y = \Sigma_{i\in B}\ z_i.a_i \qquad \text{for all } \alpha \in A,$$

and conversely.

Now consider $f: R^{k+\lambda} \to X$. If $f\hat{}$ satisfies (*), we know that the A-tuple $f\hat{}_\alpha \in C^\infty(R^k,X)$ satisfies $\Sigma_{\alpha\in A}\ f\hat{}_\alpha.c_{\alpha j} = 0$ for all j, and so there exists a B-tuple $z_i \in C^\infty(R^k,X)$ with $f\hat{}_\alpha = \Sigma_i\ z_i.a_{i\alpha}$. Then, with $\equiv$ denoting congruence mod $m^r(C^\infty(R^k,X))$,

(7) $$f\hat{}(s) \equiv \Sigma_\alpha f\hat{}_\alpha.s^\alpha = \Sigma_\alpha\Sigma_i\ z_i a_{i\alpha} s^\alpha = \Sigma_i z_i.\Sigma_\alpha a_{i\alpha}, s_\alpha = \Sigma_i\ z_i.\ h_i(s),$$
with $h_i \in I$.

By what has been already proved for m^r, this implies

$$f(t,s) \equiv \Sigma_i\ z_i(t).\ h_i(s) \quad \text{mod} \ (m^r)^*(X),$$

and the right hand side here is in $I^*(X)$. So $f \in I^*(X)$.

Conversely, if $f \in I^*(X)$, one has $f(t,-) \in I(X)$ for all $t \in R^k$, so the $f(t,-)_\alpha$'s satisfy

$$\Sigma_\alpha f(t,-)_\alpha.c_{\alpha j} = 0 \quad \text{for each } j \in C \text{ and } t \in R^k.$$

The Taylor coefficients $f\hat{}_\alpha \in C^\infty(R^k,X)$ of $f\hat{}$ thus satisfy (iii'), and hence, as in (7) above

$$f\hat{}(s) \equiv \Sigma\ z_i .\ h_i(s) \mod m^r(C^\infty(R^k,X)),$$

with $h_i \in$ I, and this implies $f\hat{} \in I(C^\infty(R^k,X))$.

(Note that, from the very form of the conclusion of Proposition 1, the result will hold, not only when I is a Weil ideal, but also when I is of the form J* with J a Weil ideal.)

Proposition 1 allows us to sharpen a result (Proposition 2.1) of [*].

COROLLARY 2. *Let* I *be a Weil ideal, and* g*:* X → Y *smooth. Let* f_1*,* f_2*:* $R^{k+\lambda} \to X$. *If* $f_1 \equiv f_2 \mod I^*(X)$, *then* $g \circ f_1 \equiv g \circ f_2 \mod I^*(Y)$.

(For a further sharpening, see Proposition 9 below.)

PROOF. By the proposition, $f_1\hat{} \equiv f_2\hat{} \mod I(C^\infty(R^k,X))$. Since

$$g_*: C^\infty(R^k,X) \to C^\infty(R^k,Y)$$

is smooth, we get by Proposition 2.1 of [*] that

$$g_* \circ f_1\hat{} \equiv g_* \circ f_2\hat{} \mod I(C^\infty(R^k,Y)).$$

The function $R^{k+\lambda} \to Y$ corresponding to $g_* \circ f_i\hat{}$ is $g \circ f_i$, so by Proposition 1 again we get

$$g \circ f_1 \equiv g \circ f_2 \mod I^*(Y).$$

The Corollary 2 implies that the construction (5), as a set, depends functorially on X ∈ E. With the obvious functoriality in C, it is in fact a bifunctor

$$(8) \qquad \otimes: \underline{E} \times \underline{D} \to \underline{\text{Sets}}.$$

By Proposition 1, and by $C^\infty(R^{k+\lambda},X) \simeq C^\infty(R^\lambda, C^\infty(R^k,X))$ we get an isomorphism

$$C^\infty(R^{k+\lambda},X)/I^*(X) \simeq C^\infty(R^\lambda, C^\infty(R^k,X))/I(C^\infty(R^k,X))$$

or

$$X \otimes C \simeq C^\infty(R^k,X) \otimes W \qquad (\text{where } W = C^\infty(R^\lambda)/I).$$

This isomorphism is natural in X. The right hand side, as a functor of $X \in \underline{E}$, takes, however, values in $\underline{E}$, by [*]. The isomorphism thus provides the left hand side with the structure of a convenient vector space, functorially in $X \in \underline{E}$. To produce the bifunctor (3) with values in $\underline{E}$, it remains to be seen that if $\gamma: C_1 \to C_2$ is a homomorphism in $\underline{D}$, then the induced map $X\otimes C_1 \to X\otimes C_2$ is smooth. For this, we need

LEMMA 3. *Let $I \subset C^\infty(R^\lambda)$ be a Weil ideal. The canonical map*

$$\pi: C^\infty(R^\lambda,X) \longrightarrow C^\infty(R^\lambda,X)/I(X) = X\otimes W \tag{9}$$

is smooth linear, and has a smooth linear section.

PROOF. It is smooth linear, because "picking Taylor coefficients depends smoothly on f ". To provide a section, pick a linear section σ of $C^\infty(R^\lambda) \to W$, which amounts to picking a finite set of smooth functions $g_j : R^\lambda \to R$ ($j = 1,\ldots, m$, where $m = \dim W$). Then under the identification $X\otimes W = X^m$ (which furnished $X\otimes W$ with a convenient structure, cf. [*]), a smooth linear section of (9) is provided by

$$(x_1, \ldots, x_m) \mapsto \Sigma\, x_j .\, g_j(-).$$

Let I be as in Lemma 3, let $I^* \subset C^\infty(R^{k+\lambda})$ be the ideal it generates, and let $C = C^\infty(R^{k+\lambda})/I^*$, as usual. Then

LEMMA 4. *The canonical map*

$$\pi': C^\infty(R^{k+\lambda},X) \to C^\infty(R^{k+\lambda},X)/I^*(X) = X\otimes C$$

is smooth linear, and has a smooth linear section.

PROOF. By construction of the convenient structure on $X\otimes C$, the map π' participates in a commutative square, with smooth horizontal isomorphisms

$$\begin{array}{ccc} C^\infty(R^\lambda, C^\infty(R^k,X)) & \xrightarrow{\;\cong\;} & C^\infty(R^{k+\lambda},X) \\ \pi \downarrow & & \downarrow \pi' \\ C^\infty(R^k,X)\otimes W & \xrightarrow{\;\cong\;} & X\otimes C\ . \end{array}$$

But the map π is smooth with a smooth section by Lemma 3 (applied to $C^\infty(R^k,X)$). Thus π' has also these properties.

PROPOSITION 5. Let $\gamma: C_1 \to C_2$ *be a morphism in* $\underline{D}$. *Then* $X\otimes\gamma: X\otimes C_1 \to X\otimes C_2$ *is smooth (and linear).*

PROOF. If $C^\infty(R^{k+\lambda}) \to C_1$ and $C^\infty(R^{k'+\lambda'}) \to C_2$ are presentations, γ lifts to an algebra homomorphism $C^\infty(R^{k+\lambda}) \to C^\infty(R^{k'+\lambda'})$, which in turn is induced by a smooth map $g: R^{k'+\lambda'} \to R^{k+\lambda}$. Thus γ sits in a commutative square

$$\begin{array}{ccc} C^\infty(R^{k+\lambda},X) & \xrightarrow{\quad g^* \quad} & C^\infty(R^{k'+\lambda'},X) \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow \\ X\otimes C & \xrightarrow[X\otimes\gamma]{} & X\otimes C' \end{array} ;$$

the top map is thus smooth linear. By Lemma 4, π' has a smooth linear section, and this displays the bottom map as a composite of three smooth linear maps.

This proposition was the last missing piece in providing the construction (5) with the structure of a bifunctor $\underline{E}\times\underline{D} \to \underline{E}$. This bifunctor extends the bifunctor $\underline{E}\times\underline{W} \to \underline{E}$ ("Weil prolongation") of [*] §2; our next task is to extend the "transitivity law", [*], 3.1, that is to produce an isomorphism

(10) $$(X\otimes C_1)\otimes C_2 \simeq X\otimes(C_1\otimes_\infty C_2)$$

natural in $X \in \underline{E}$ and $C_1, C_2 \in \underline{D}$. For C_1, C_2 free algebras, i.e. of form $C^\infty(R^m)$ and $C^\infty(R^n)$, respectively, $C_1\otimes_\infty C_2 = C^\infty(R^{m+n})$. In this case, the construction of (10) is immediate, since

(11) $$C^\infty(R^n, C^\infty(R^m, X)) \simeq C^\infty(R^{n+m},X),$$

(naturally in X).

We note that we have, for $C = C^\infty(R^m)$ a free algebra,

(12) $$X\otimes C = X\otimes C^\infty(R^m) = C^\infty(R^m,X) = X^{R^m}$$

(exponential object). We can therefore get an interchange isomorphism

(13) $$(X\otimes W)\otimes C^\infty(R^m) \simeq (X\otimes C^\infty(R^m,X))\otimes W ;$$

in view of (12), this follows from the isomorphism

(14) $$(X\otimes W)^Y \simeq X^Y\otimes W$$

of [*] (p. 14, line 1). This is natural in X, as well.

But every object C of $\underline{D}$ is of the form $C^\infty(\mathbf{R}^n)\otimes W$, and so the existence of an isomorphism (10), natural in X, follows purely formally from the two special cases: 1) where the C_i's are Weil algebras (this case was proved in [*]), and 2) where the C_i's are free (cf. (11); in conjunction with the interchange isomorphism (13).

We shall now prove that the isomorphism (10) constructed is also natural in C_1 and $C_2 \in \underline{D}$. This is almost clear for the case of free algebras. For algebra homomorphisms $C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^m)$ correspond bijectively to smooth maps $\mathbf{R}^m \to \mathbf{R}^n$; so by (12), the isomorphism (10) is just the isomorphism

$$(X^{R^m})^{R^n} \simeq X^{R^m \times R^n}$$

which is certainly natural in $\mathbf{R}^m$, $\mathbf{R}^n$ (with respect to smooth maps).

For the general case, we first note that (10) is natural with respect to the homomorphisms

$$C^\infty(\mathbf{R}^{k+l}) \to C_1, \quad C^\infty(\mathbf{R}^{k'+l'}) \to C_2$$

which present C_1 and C_2 - simply inspect the construction of (10).

Now the general case can be seen as follows. Let $\gamma_i\colon C_i \to C_i'$ (i = 1, 2) be homomorphisms, let $F_1 \to C_1$, $F_1' \to C_1'$ etc. be the canonical presentations, with the F_i's free. The homomorphism γ_i can be lifted to homomorphisms ϕ_i between the F_i's. There arises, for the given X, a commutative box containing the desired naturality square for (γ_1, γ_2) as bottom, and the already available naturality square for the (ϕ_1,ϕ_2) as top. The sides are commutative, and

$$X\otimes(F_1\otimes_\infty F_2) \to X\otimes(C_1\otimes_\infty C_2)$$

is surjective, being of the form

$$C^\infty(\mathbf{R}^{n+m},X) \to C^\infty(\mathbf{R}^{n+m},X)/(I^*,J^*).$$

So the bottom is commutative. This proves the naturality of (10).

We let C denote the "real" Cahiers topos (in contrast to the "faulty" one of [*]), so $C \hookrightarrow \underline{\text{Sets}}^{D}$. The "real" embedding $\underline{E} \to C$ may now be described by describing the composite

$$\underline{E} \longrightarrow C \longrightarrow \underline{\text{Sets}}^{D},$$

as the functor $J'\colon \underline{E} \to \underline{Sets}^{D}$ which is exponential adjoint to the action (3), or rather, to the composite

$$\underline{E}\times\underline{D} \xrightarrow[\otimes]{} \underline{E} \xrightarrow[\#]{} \underline{Sets}$$

where # is the underlying-set functor.

So, explicitly, if X is a convenient vector space, and $W \in \underline{W}$,

$$J'(X)(C^{\infty}(\mathbf{R}^{k})\otimes_{\infty}W) := X\otimes(C^{\infty}(\mathbf{R}^{k})\otimes_{\infty}W) \simeq C^{\infty}(\mathbf{R}^{k},X)\otimes W.$$

The fact that the values of J' are sheaves is proved as in [*], and since the inclusion $C \to \underline{Sets}^{D}$ preserves exponentials and products, it makes no difference for the statements and proofs we are going to make whether we consider $J'\colon \underline{E} \to C$ or $J'\colon \underline{E} \to \underline{Sets}^{D}$. So we make the statements for C, the proofs for $\underline{Sets}^{D}$.

We can now state and prove Theorem 5.2 of [*] in the form in which it was intended; furthermore, we strengthen it on the issue of exponentiation:

THEOREM 5.2'. *The functor* $J'\colon \underline{E} \to C$ *is full and faithful. It preserves finite products, and it preserves all exponentials.*

PROOF. Some of this can be derived from Theorem 5.2 of [*] by means of the inclusion functor (1), as we shall now indicate. (An alternative approach is to proceed in analogy with the proofs of [*].) First, we have

PROPOSITION 6. *Let* $(\mathbf{R}^{k},W) \in \underline{f\alpha W}$, *and let* $Y \in \underline{E}$. *Then*

$$\hom_{\underline{E\alpha W}}((\mathbf{R}^{k},W),\ (Y,\ \mathbf{R})) \simeq J'(Y)(C^{\infty}(\mathbf{R}^{k})\otimes_{\infty}W).$$

PROOF. An element on the left hand side is by definition a pair consisting of a smooth map $\mathbf{R}^{k} \to Y\otimes W$ and a Weil algebra homomorphism $\mathbf{R} \to W$; the latter is unique, so the data is equivalent to an element of

$$C^{\infty}(\mathbf{R}^{k},Y\otimes W) \simeq C^{\infty}(\mathbf{R}^{k},Y)\otimes W = J'(Y)(C^{\infty}(\mathbf{R}^{k})\otimes_{\infty}W),$$

the isomorphism here because of (14).

From Proposition 6, we get the commutativity of the following diagram of functors (up to isomorphism), where J is the embedding of [*].

$$\begin{array}{ccc} \underline{E} & \xrightarrow{\ J\ } & \underline{Sets}^{(\mathcal{F}\times W)^{op}} \\ & \searrow^{J'} & \uparrow i^{*} \\ & & \underline{Sets}^{\underline{D}} \end{array}$$

Since by Theorem 5.2 of [*], J is full and faithful, and since i^* is faithful (i being bijective on objects) J' is full and faithful. Similarly, since J preserves products by loc. cit., and i^* preserves products and reflects isomorphisms, J' preserves products.

The proof that J' preserves exponentials depends on

PROPOSITION 7. *Let* $A \in \underline{D}$. *Then*

$$J'(X \otimes A) = J'(X)^{A^\wedge}$$

where $A^\wedge$ *is the object in* $\underline{Sets}^{\underline{D}}$ *represented by* A.

PROOF. This is a purely formal consequence of the transitivity of the action $\otimes$: let $B \in \underline{D}$. Then

$$J'(X\otimes A)(B) = \#((X\otimes A)\otimes B) \simeq \#(X\otimes(A\otimes_\infty B)) = J'(X)(A\otimes_\infty B) \simeq J'(X)^{A^\wedge}(B),$$

the last isomorphism by a well known calculation of how to exponentiate by a representable object in a presheaf category whose index category has binary coproducts $\otimes_\infty$. This proves the proposition.

We shall also need the following strengthening of formula (14):

PROPOSITION 8. *For any* X, $Y \in \underline{E}$ *and* $A \in \underline{D}$, *we have*

$$(X\otimes A)^Y \simeq X^Y \otimes A.$$

PROOF. Let $A = C^\infty(R^k)\otimes_\infty W$. Writing $C^\infty(Y, X)$ for X^Y, we then have

$$\begin{aligned} C^\infty(Y,X)\otimes A &\simeq C^\infty(R^k, C^\infty(Y,X))\otimes W \simeq C^\infty(Y, C^\infty(R^k,X))\otimes W \\ &\underset{\text{by (14)}}{\simeq} C^\infty(Y, C^\infty(R^k,X)\otimes W)) \simeq C^\infty(Y, X\otimes A). \end{aligned}$$

This proves the proposition.

Now we consider an arbitrary exponential $X^Y = C^\infty(Y,X)$ in $\underline{E}$. For $A \in \underline{D}$ arbitrary, we then have

$$J'(C^\infty(Y,X))(A) = \#(C^\infty(Y,X)\otimes A) \simeq \#(C^\infty(Y,X\otimes A) \qquad \text{(Prop. 8)}$$

$$\simeq \mathrm{Hom}_F(Y, X \otimes A) \simeq \mathrm{hom}_C(J'Y, J'(X \otimes A)) \simeq \mathrm{hom}_C(J'Y, (J'X)^{A^\wedge}) \qquad \text{(Prop. 7)}$$
$$\simeq \mathrm{hom}_C(A^\wedge, J'X^{J'Y}) \simeq J'X^{J'Y}(A).$$

Thus

$$J'(C^\infty(Y,X)) \simeq J'X^{J'Y},$$

and the last assertion of the theorem is proved.

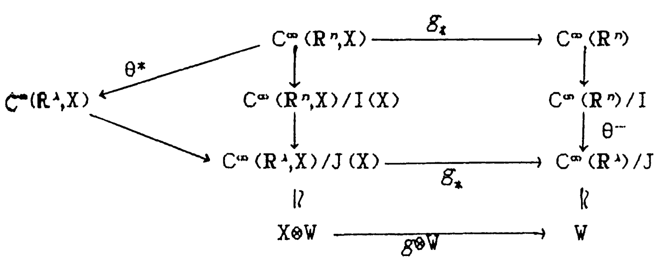
We finish by presenting a generalization of Proposition 2.1 of [*]. Recall that $C^\infty(R^n)$ carries a standard (Frechet-)topology, and recall (from [MR], say) that closed ideals are exactly the "W-determined" or "near-point determined" ones.

PROPOSITION 9. *Let* $I \subset C^\infty(R^n)$ *be any closed ideal. If* $f_1 \equiv f_2$ mod I *(where* f_i: $R^n \to X$), *then we have*

$$g \circ f_1 \equiv g \circ f_2 \mod I$$

for any smooth g: $X \to Y$. (Notation as in [*], §2.)

PROOF. As in [*], it suffices to consider the case Y = R. We must prove $g \circ f_1 - g \circ f_2 \in I \subset C^\infty(R^n)$. Since the ideal I is near-point determined, it suffices (by definition of this phrase) to prove that any algebra homomorphism θ^- from $C^\infty(R^n)/I$ into a Weil algebra W annihilates $(g \circ f_1 - g \circ f_2) + I$. If $W = C^\infty(R^\lambda)/J$, θ^- comes about from some C^∞-algebra map $C^\infty(R^n) \to C^\infty(R^\lambda)$, and thus from a smooth map θ: $R^\lambda \to R^n$. By Proposition 2.1 in [*], we have a commutative diagram



Now f_1 and f_2 go to the same in $C^\infty(R^n,X)/I(X)$. Since θ^- is linear, it follows that $\theta^-(g \circ f_1 - g \circ f_2) = 0$. This proves the proposition.

Note, however, that for a general closed ideal I, we have not (yet) defined $C^\infty(R^n,X)/I(X)$ as a *convenient* vector space, only as a vector space.

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