A coherent theory of sites

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We describe in coherent (= finitary geometric) language a notion of site. For a site T in an elementary topos $\mathcal{S}$, we describe likewise in coherent terms, the notion of model for T in any $\mathcal{S}$-topos $\mathcal{E}$; and we describe an $\mathcal{S}$-topos sh(T) of sheaves on T. This topos classifies T-models (Theorem 12. below).

The coherent formulations of the notions of site and model make base change along inverse image functors of geometric morphisms $q : \mathcal{S} \to \mathcal{S}'$ immediate. In particular, using the classification theorem, one deduces that $sh(q^*T)$ appears as a pull-back of sh(T) along q in the category of toposes (Corollary 13 below).

The notion of site goes back to Grothendieck, cf. [1] Expose 3. A notion of internal site in a topos $\mathcal{S}$ was considered by Lawvere and Tierney and utilized by Diaconescu [2], and many others: namely an internal category $\mathbb{C}$ in $\mathcal{S}$, together with a Lawvere-Tierney topology j on the topos $\mathcal{S}^{\mathbb{C}^{op}}$. Base change for sites, in this context, has been considered in [7] and [8]. A site notion which is expressed in explicit small but higher order terms, has been considered in Johnstone [4] (cf. also [3]), also aiming at base change. Finally, Joyal-Tierney [5], Moerdijk [6] and others have utilized internal sites where the structure is given in terms of certain 'families' of arrows that are considered to be covering, so this is implicitly an 'indexed' notion; and it is not clear to us what the theory of base change for such notions is.

Conventions. All constructions and arguments that take place inside one topos $\mathcal{S}$ are performed as if $\mathcal{S}$ were the category of sets, and with free use of higher order constructions in $\mathcal{S}$. Notation and terminology are as in [3], except that we compose arrows of a category backwards; and so, for a category object $\mathbb{C}$ in $\mathcal{S}$,

$$- x_{C_0} C_1$$

means pulling back along the codomain map $d_1: C_1 \to C_0$, which is the opposite of the convention in [3] p.47 (footnote).

1. Sites.

A site T in an elementary topos $\mathcal{S}$ is defined as a category object $\mathbb{C}$ in $\mathcal{S}$, with some further structure. We don't assume that $\mathbb{C}$ has pull-backs; the usual assumption that 'pull-backs of covers are covers' is replaced by a certain further structure of algebraic nature, called σ; the choice of σ does not affect neither the models nor the sheaves of T.

Consider a category object $\mathbb{C}$ in $\mathcal{S}$

$$C_1 \underset{\underset{id}{\longleftarrow}}{\overset{\overset{d_0}{\longrightarrow}}{\xrightarrow{d_1}}} C_0$$

with composition $\circ$.

<u>Definition 1.</u> A site structure $T = (T,\sigma)$ on $\mathbb{C}$ consists of i) a commutative square

(1.1)
$$\begin{array}{ccc} B & \xrightarrow{\gamma} & C_1 \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle d_1} \\ A & \xrightarrow[\alpha]{} & C_0 \end{array}$$

such that $<\beta,\gamma>: B \to A \times C_1$ is monic (so B may be identitied with a subset of $A \times C_1$); and ii), a map

(1.2)
$$A \times_{C_0} C_1 \xrightarrow{\sigma} A$$

satisfying the conditions

(1.3) $\forall(a,g) : \alpha(\sigma(a,g)) = d_0 g$

(1.4) $\forall(a,g,h) : (\sigma(a,g),h) \in B \Rightarrow \exists g',h' : (a,h') \in B \wedge g \circ h = h' \circ g'$;

where $a \in A$, and where g,h,g',h' are assumed to satisfy the book-keeping condition that

$$\begin{array}{ccc} & g' & \\ & \overline{} & \\ h \Big| & & \Big| h' \\ & g & \\ & \overline{} & \end{array}$$

makes sense. A site is a category object with a site structure.

The heuristics of the definition is that an $a \in A$ is a name for a covering of $\alpha(a) \in C_0$, and B consists of pairs $(a,g) \in A x C_1$ such that g is in the covering named by a (or equivalently, $\beta^{-1}(a)$ is an index set for the covering named by a). So we shall sometimes write $g \in_T a$ for $(a,g) \in B$.

Further, $\sigma(a,g) \in A$ is a name a' for some covering which refines the pull-back along g of the cover of $d_1(g)$ named by a.

The existence of σ implies the property

$$(1.5) \qquad \forall a,g \;\; \exists a' \;\; \forall h \in_T a' \; \exists g',h' : \; (h' \in_T a \wedge \; g{\circ}h = h'{\circ}g')$$

which is a standard way of expressing the desired stability property for the covering notion. However, (1.5) is not a coherent sequent.

The σ only enters, via the property (1.5), in the construction of a certain object L that appears in Proposition 6 below and is not used otherwise.

Since (1.3) and (1.4) are coherent sequents, we get that if (T,σ) is a site structure on $\mathbb{C}$ in $\mathcal{S}$, and $q: \mathcal{S}' \to \mathcal{S}$ is a geometric morphism, $(q^*T, q^*\sigma)$ is a site structure on $q^*\mathbb{C}$ in $\mathcal{S}'$.

Example 2. Let $\lambda: L_0 \to C_0$ have a right action by $\mathbb{C}$, so defining an object L in $\mathcal{S}^{\mathbb{C}^{op}}$. Let $L' \to L_0$ be a subobject, stable under the action. Then we have a site structure, denoted by $T[L' \to L_0]$, on $\mathbb{C}$, given by $A = L_0$, $B = [\![(l,g) \in L_0 \times_{C_0} C_1 = L_1 \mid l{\cdot}g \in L']\!]$, and σ is the action map $L_0 \times_{C_0} C_1 \to L_0$.

We may make L_1 and B into objects over C_0 by $(l,g) \mapsto d_0(g)$, and, as such, they acquire a right action by $\mathbb{C}$, namely by composition. With these structures, B appears in a pull-back diagram in $\mathcal{S}^{\mathbb{C}^{op}}$

$$(1.6) \qquad \begin{array}{ccc} B & \longrightarrow & L_1 \\ \Big| & & \Big| \\ L' & \longrightarrow & L \end{array} .$$

Let in particular L be a subobject of the subobject classifier Ω of $\mathcal{S}^{\mathbb{C}^{op}}$, and assume that true : $1 \to \Omega$ factors through L.

Definition 3. A site structure on $\mathbb{C}$ of the form $T[1 \xrightarrow{\text{true}} L]$ for some such L is called an ideal site structure on $\mathbb{C}$.

2. Models for sites.

If $\mathcal{E}$ is an $\mathcal{S}$-topos in virtue of a geometric morphism $p: \mathcal{E} \to \mathcal{S}$, we let $\text{Funct}_{\mathcal{S}}(\mathbb{C},\mathcal{E})$ denote the category of internal covariant functors from $\mathbb{C} \in \mathcal{S}$ to $\mathcal{E}$, that is, the category of objects $\mu: M \to p^*C_0$ in $\mathcal{E}$ equipped with a left action by the category object $p^*\mathbb{C}$ in $\mathcal{E}$. If $r : \mathcal{E}' \to \mathcal{E}$ is a geometric morphism over $\mathcal{S}$, r^* induces a functor r^*: $\text{Funct}_{\mathcal{S}}(\mathbb{C},\mathcal{E}) \to \text{Funct}(\mathbb{C},\mathcal{E}')$. Sometimes one writes $M : \mathbb{C} \to \mathcal{E}$ for such internal functors, and $\mathbb{C} \xrightarrow{M} \mathcal{E} \xrightarrow{r^*} \mathcal{E}'$ for $r^*(M)$.

We shall consider full subcategories $\text{Flat}_{\mathcal{S}}(\mathbb{C},\mathcal{E})$, and, given a site structure T on $\mathbb{C}$, $\text{Mod}_{\mathcal{S}}(T,\mathcal{E})$,

$$\text{Mod}_{\mathcal{S}}(T,\mathcal{E}) \subseteq \text{Flat}_{\mathcal{S}}(\mathbb{C},\mathcal{E}) \subseteq \text{Funct}_{\mathcal{S}}(\mathbb{C},\mathcal{E}).$$

The subcategory $\text{Flat}_{\mathcal{S}}(\mathbb{C},\mathcal{E})$ is the full subcategory of **flat** internal functors $M : \mathbb{C} \to \mathcal{E}$ in the standard sense [2], [3] (meaning that the internal category of 'elements of M ' is inverse filtering, or equivalently that a certain functor $- \otimes_{\mathbb{C}} M$, see (3.1) below, is left exact). To describe the subcategory $\text{Mod}_{\mathcal{S}}(T,\mathcal{E})$ we need the following

Construction 4. Let T, as in (1.1), be a site structure on $\mathbb{C}$, and let $M \in \text{Funct}_{\mathcal{S}}(\mathbb{C},\mathcal{E})$. We define a subobject

$$T(M) \subseteq p^*A \times_{C_0} M \tag{2.1}$$

by

$$T(M) := [\![(a,m) \mid \exists g \in p^*C_1 \ \exists m' \in M : \ g \cdot m' = m \ \wedge \ (a,g) \in p^*B]\!],$$

where the book-keeping conditions are $\mu(m') = d_0(g)$ and $\mu(m) = \alpha(a) = d_1(g)$, and where $\cdot$ denotes the action of $p^*\mathbb{C}$ on M.

Because of the coherence of the formula describing T(M),

$$r^*(T(M)) = T(r^*(M)) \tag{2.2}$$

whenever $r : \mathcal{E}' \to \mathcal{E}$ is a geometric morphism over $\mathcal{S}$.

Definition 5. If T is a site structure on $\mathbb{C}$, $\mathrm{Mod}_{\mathcal{S}}(T,\mathcal{E})$ is the full sub- category of $\mathrm{Funct}_{\mathcal{S}}(\mathbb{C}, \mathcal{E})$ consisting of such M which are flat, and where (2.1) is an equality.

If $r : \mathcal{E}' \to \mathcal{E}$ is as above, $r^* : \mathrm{Funct}_{\mathcal{S}}(\mathbb{C},\mathcal{E}) \to \mathrm{Funct}_{\mathcal{S}}(\mathbb{C},\mathcal{E}')$ is known to restrict to a functor $\mathrm{Flat}_{\mathcal{S}}(\mathbb{C},\mathcal{E}) \to \mathrm{Flat}_{\mathcal{S}}(\mathbb{C}, \mathcal{E}')$, and by (2.2) it further restricts to a functor $r^*: \mathrm{Mod}_{\mathcal{S}}(T,\mathcal{E}) \to \mathrm{Mod}_{\mathcal{S}}(T,\mathcal{E}')$.

Proposition 6. For any site structure T' on $\mathbb{C}$, there exists an ideal site structure $T = T[1 \longrightarrow L]$ $(L \subseteq \Omega \in \mathcal{S}^{\mathbb{C}^{op}})$ such that for any $\mathcal{S}$-topos $\mathcal{E}$, $\mathrm{Mod}_{\mathcal{S}}(T',\mathcal{E}) = \mathrm{Mod}_{\mathcal{S}}(T,\mathcal{E})$.

Proof. Let A',B',α' etc. denote the data which enter into the site structure T'. For $u \in C_0$, define $L(u)$ to consist of the maximal crible on u plus all cribles which contain one of the coverings

$$[\![\, g \in C_1 \mid g \in_{T'} a \,]\!]$$

(recall that $g \in_{T'} a$ means $(a,g) \in B'$), for some a with $\alpha'(a) = u$).

The existence of σ' then gives that

$$L_0 := \coprod_{u \in C_0} L(u)$$

with its natural map $\lambda: L_0 \to C_0$, is stable under the right action of $\mathbb{C}$ on Ω, thus defines a subobject L of Ω in $\mathcal{S}^{\mathbb{C}^{op}}$; and it clearly contains true: $1 \to \Omega$. Thus we have an ideal site structure $T = T[1 \to \Omega]$ on $\mathbb{C}$. To argue that T' and T have the same models, we introduce two auxiliary notions, order and refinement :

Let T, as in (1.1), be an arbitrary site structure on $\mathbb{C}$. An **order** on T is a partial order on A and a given map $t : C_0 \to A$, satisfying the book-keeping condition $a_1 \le a_2 \Rightarrow \alpha(a_1) = \alpha(a_2)$, and the conditions

$$\forall a_1,a_2, g : \quad a_1 \le a_2 \Rightarrow (\, g \in_T a_1 \Rightarrow g \in_T a_2 \,).$$

and

$$\forall g : \quad g \in_T t(d_1(g)).$$

A commutative diagram

(2.3)
$$\begin{array}{ccccc} B' & \xrightarrow{\gamma'} & B & \longrightarrow & C_1 \\ | & & |\,\beta & & | \\ A' & \xrightarrow{\gamma} & A & \longrightarrow & C_0 \end{array}$$

where the right hand square is T, and where the total square is a site structure T', is said to exhibit T as a refinement of T' if

(2.4) $$\forall a' \in A' \; \forall g \in_T \gamma(a') \; \exists h \in_{T'} a' \; \exists h' : \; g = h \circ h'$$

and

(2.5) $$\forall a \in A : \; (a = t(\alpha(a)) \; \vee \; \exists a' \in A' : \; \gamma(a') \leq a).$$

Lemma 7. If the ordered site T is a refinement of a site T', then for any $\mathcal{S}$-topos $\mathcal{E}$, $\mathrm{Mod}_{\mathcal{S}}(T,\mathcal{E}) = \mathrm{Mod}_{\mathcal{S}}(T',\mathcal{E})$.

Proof. Since the notions site, order, refinement and model are all formulated in coherent language, it suffices to consider the case $\mathcal{E} = \mathcal{S}$. So assume M is a model of T, and let $(a',m) \in A' \times_{C_0} M$. Since M is a T-model, $(\gamma(a'),m) \in T(M)$, so there exists (g,m_1) with $g \in_T \gamma(a')$ and $g \cdot m_1 = m$. By (2.4), there exists $h \in_{T'} a'$ and h' with $g = h \circ h'$. So $h \cdot h' \cdot m_1 = m$, and thus h and $h' \cdot m_1$ witness that $(a',m) \in T'(M)$. Conversely, assume M is a T'-model, and let $(a,m) \in A \times_{C_0} M$. Either $a = t(\alpha(a))$, and then $(a,m) \in T(M)$ in virtue of $\mathrm{id}_{\alpha(a)} \in_T t(\alpha(a))$; or there exists a' with $\gamma(a') \leq a$. Then since M is a T'-model there exists $g \in_{T'} a'$ and $m_1 \in M$ with $g \cdot m_1 = m$. By commutativity of the left hand square in (2.3), $g \in_T \gamma(a')$, and since $\gamma(a') \leq a$, $g \in_T a$. So $g \cdot m_1 = m$ witnesses $(a,m) \in T(M)$. This proves the lemma.

Returning, then, to a site structure T', as in the Proposition, and the ideal site $T = T[1 \rightarrow L]$ constructed out of T', we note that the A-part of T is L, which, as a subobject of Ω, carries a natural order relation in $\mathcal{S}^{\mathbb{C}^{op}}$, so that L as an object L_0 in $\mathcal{S}$ carries a partial order satisfying the conditions for an order on a site with $t = \mathrm{true}$. With this order T refines T'; for $a' \in A'$, we let $\gamma(a') \in L_0$ be the crible generated by the covering named by a'; then (2.4) will hold, To define γ', let $(a',g) \in B'$, where $g : u \rightarrow v$ is an arrow in $\mathbb{C}$. Then

$$g \in_T \gamma(a') \in L(\alpha(a)) = L(v) \subseteq \Omega(v)$$

and so $\mathrm{id}_u \in g^{-1}(\gamma(a'))$, and so $g^{-1}(\gamma(a'))$ is the maximal crible on u. (In the notation of Example 2, this would read $\gamma(a') \cdot g$ belongs to the subobject L' $(= \mathrm{true} : 1 \rightarrow L)$). So $(\gamma(a'),g)$ belongs to the B for the site $T = T[1 \rightarrow L]$, and this defines γ'. So $T[1 \rightarrow L]$ refines T', and, by the lemma, Proposition 6 follows.

3. The topos of sheaves for a site.

A sheaf for a site T, as in (1.1), will be an internal functor $\mathbb{C}^{op} \to \mathcal{S}$ with certain properties, but we do not attempt to describe these properties explicitly; rather, we describe in non-explicit terms, a Lawvere-Tierney topology j(T) on $\mathcal{S}^{\mathbb{C}^{op}}$, such that the category sh(T) of sheaves for T is the category of sheaves for j(T).

Recall that the internal Yoneda functor $Y : \mathbb{C} \to \mathcal{S}^{\mathbb{C}^{op}}$ is given by $\langle d_1, d_0 \rangle : C_1 \longrightarrow p^*C_0$, with left action by $p^*\mathbb{C}$ given by composition (where $p : \mathcal{S}^{\mathbb{C}^{op}} \to \mathcal{S}$ is the canonical geometric morphism).

Definition 8. A site structure on $\mathbb{C}$ has the crible property if it satifies

$$\forall a,g,h : \ g \in_T a \ \Rightarrow \ g \circ h \in_T a$$

where $\alpha(a) = d_1(g)$ and $d_1(h) = d_0(g)$.

Any site of the form $T[L' \to L]$ (Example 2) has the crible property.

Note that

$$A \times_{C_0} C_1 \longrightarrow C_1 \xrightarrow{d_0} C_0$$

equipped with a right action by $\mathbb{C}$ given by composition, defines an object in $\mathcal{S}^{\mathbb{C}^{op}}$ which may be identified with $p^*A \times_{p^*C_0} C_1$; and T has the crible property iff $B \longrightarrow A \times_{C_0} C_1$ is stable under the action; and under the identification mentioned, it is easily seen that

Proposition 9. If T has the crible property, then the subobject

$$T(Y) \subseteq p^*A \times_{p^*C_0} C_1$$

equals

$$B \subseteq A \times_{C_0} C_1.$$

Let $\mathcal{H} \subseteq \mathcal{S}^{\mathbb{C}^{op}}$ be a subtopos with $r^* : \mathcal{S}^{\mathbb{C}^{op}} \to \mathcal{H}$ as reflection functor.

Proposition 10. If $L' \to L$ is a mono in $\mathcal{S}^{\mathbb{C}^{op}}$ which is inverted by r^*, then $r^*(Y)$ $(= r^* \circ Y)$ is a model for the site structure $T[L' \to L]$.

Proof. This follows from Proposition 9 and the pull-back diagram (1.6).

Recall that a flat $M : \mathbb{C} \to \mathcal{E}$ ($\mathcal{E}$ an $\mathcal{S}$-topos in virtue of $p : \mathcal{E} \to \mathcal{S}$) has a classifying geometric morphism $\overline{M} : \mathcal{E} \to \mathcal{S}^{\mathbb{C}^{op}}$, whose inverse image functor $\overline{M}^*$ we denote $-\otimes_{\mathbb{C}}M$. For $L \in \mathcal{S}^{\mathbb{C}^{op}}$ it may be described as the coequalizer

$$(3.1) \qquad p^*L_0 \times_{p^*C_0} p^*C_1 \times_{p^*C_0} M \rightrightarrows p^*L \times_{p^*C_0} M \longrightarrow L \otimes_{\mathbb{C}} M$$

the two parallel maps are induced by the actions.

With such M, and with $L' \to L$ in $\mathcal{S}^{\mathbb{C}^{op}}$, we have

Proposition 11. If M is a model in $\mathcal{E}$ for the site structure $T[L' \to L]$, then $L' \otimes_{\mathbb{C}} M \longrightarrow L \otimes_{\mathbb{C}} M$ is an isomorphism.

Proof. The map in question is monic, since $-\otimes_{\mathbb{C}}M$ is left exact by flatness of M. To prove it epi, it suffices, by coherence of the notions involved, to consider the case $\mathcal{E} = \mathcal{S} = \underline{\text{Sets}}$.

Consider an arbitrary element x in $L \otimes_{\mathbb{C}} M$. By (3.1), it is represented by an element $(a,m) \in L \times_{C_0} M$. Since M is a model for T, $T(M) = L \times_{C_0} M$, so there exists (g,m') so that $g \cdot m' = m$ and $(a,g) \in B$. This means by construction of $T[L' \to L]$ that $a \cdot g \in L'$. Then $(a \cdot g, m') \in L' \times_{C_0} M$ represents an element x' in $L' \otimes_{\mathbb{C}} M$. But

$$(a \cdot g, m') \equiv (a, g \cdot m') = (a,m)$$

where $\equiv$ denotes the equivalence relation on $L \times_{C_0} M$ which defines $L \otimes_{\mathbb{C}} M$, by (3.1). Thus x' maps to x, witnessing the surjectivity of the map in question, and this proves the Proposition.

Let T be a site structure on $\mathbb{C}$, and let $T[1 \xrightarrow{\text{true}} L]$ be some ideal site structure on $\mathbb{C}$ which has the same models as T, using Proposition 6. Let j be the smallest Lawvere-Tierney topology on $\mathcal{S}^{\mathbb{C}^{op}}$ for which $\text{true} : 1 \to L$ is dense; such exists, by [3].

Theorem 12. The $\mathcal{S}$-topos $\text{sh}(j) \xrightarrow{r} \mathcal{S}^{\mathbb{C}^{op}} \longrightarrow \mathcal{S}$ is the classifying topos for T-models, with $\mathbb{C} \xrightarrow{Y} \mathcal{S}^{\mathbb{C}^{op}} \xrightarrow{r^*} \text{sh}(j)$ as generic model. In particular, sh(j) only depends on T and may be written sh(T). So we have an equivalence of categories

$$(3.2) \qquad \text{Mod}_{\mathcal{S}}(T,\mathcal{E}) \simeq \text{Top}_{\mathcal{S}}(\mathcal{E},\text{sh}(T))$$

with $q : \mathcal{E} \longrightarrow \text{sh}(T) = \text{sh}(j)$ corresponding to the model $q^* \circ r^* \circ Y$.

Proof. Since $\mathcal{S}^{\mathbb{C}^{op}}$ classifies flat functors (cf. [3]), it suffices to prove that for a flat functor $M : \mathbb{C} \to \mathcal{E}$, $-\otimes_{\mathbb{C}}M : \mathcal{S}^{\mathbb{C}^{op}} \to \mathcal{E}$ factors across r^* iff M is a model for T. If M is a model, $\text{true} \otimes_{\mathbb{C}} M : 1 \otimes_{\mathbb{C}} M \to L \otimes_{\mathbb{C}} M$ is iso, by Proposition 11, so $-\otimes_{\mathbb{C}}M$ inverts the monic which defines the topology j, so factors across r^*. Conversely, since r^* inverts $1 \to L$, $r^* \circ Y$ is a model for T, by Proposition 10. So if $- \otimes_{\mathbb{C}} M$ factors across r^*, as $- \otimes_{\mathbb{C}} M = q^* \circ r^*$, say, then $M = q^*(r^* \circ Y)$, and since $r^* \circ Y$ is a model, M is. This proves the Theorem.

If $q : \mathcal{S}' \to \mathcal{S}$ is a geometric morphism, we get a functor

$$\Sigma_q : \mathrm{Top}_{\mathcal{S}'} \longrightarrow \mathrm{Top}_{\mathcal{S}} \quad ,$$

namely the one which to an $\mathcal{S}'$-topos $p : \mathcal{H} \to \mathcal{S}'$ associates the $\mathcal{S}$-topos $\mathcal{H} \xrightarrow{p} \mathcal{S}' \xrightarrow{q} \mathcal{S}$. Also, for a site structure T on $\mathbb{C}$ in $\mathcal{S}$, we get a site structure q^*T on $q^*\mathbb{C}$ in $\mathcal{S}'$, by coherence of the site notion. We evidently have

$$\mathrm{Funct}_{\mathcal{S}'}(q^*\mathbb{C}, \mathcal{H}) \simeq \mathrm{Funct}_{\mathcal{S}}(\mathbb{C}, \Sigma_q \mathcal{H})$$

and inspecting the Construction 4 and Definition 5 (and the notion of flatness), we get that this restricts to an equivalence

$$(3.3) \qquad \mathrm{Mod}_{\mathcal{S}'}(q^*T, \mathcal{H}) \simeq \mathrm{Mod}_{\mathcal{S}}(T, \Sigma_q \mathcal{H}).$$

In particular, combining this with Theorem 12, we have the equivalence (natural in $\mathcal{H} \in \mathrm{Top}_{\mathcal{S}'}$)

$$\mathrm{Top}_{\mathcal{S}'}(\mathcal{H}, \mathrm{sh}(q^*T)) \simeq \mathrm{Mod}_{\mathcal{S}'}(q^*T, \mathcal{H}) \simeq \mathrm{Mod}_{\mathcal{S}}(T, \Sigma_q \mathcal{H}) \simeq \mathrm{Top}_{\mathcal{S}}(\Sigma_q \mathcal{H}, \mathrm{sh}(T)),$$

proving

<u>Corollary 13.</u> The following diagram is a pull-back (in the usual lax sense) in the category of toposes

$$\begin{array}{ccc} \mathrm{sh}(q^*T) & \longrightarrow & \mathcal{S}' \\ \Big| & & \Big| \, q \\ \mathrm{sh}(T) & \longrightarrow & \mathcal{S} \end{array}$$

the left hand vertical geometric morphism classifies the q^*T-model in $\Sigma_q(\mathrm{sh}(q^*T))$ corresponding to the generic T-model in $\mathrm{Mod}_{\mathcal{S}'}(q^*T, \mathrm{sh}(q^*T))$ under (3.3).

4. Examples

We shall consider two examples: site structures on the terminal category 1 of $\mathcal{S}$, and the canonical site structure on any frame X (= locale) in $\mathcal{S}$.

A site structure T on 1 is given by a commutative square

$$\begin{array}{ccc} B & \longrightarrow & 1 \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle id} \\ A & \longrightarrow & 1 \end{array}$$

with β mono; a σ always exists, namely the identity map on A. This site has the crible property.

By Proposition 6, this site has the same models as a suitable ideal site $T[1 \to L]$ where $L \subseteq \Omega$, and the Lawvere-Tierney topology associated to $T[1 \to L]$ is the smallest for which $1 \to L$ is dense. In the present case, we shall prove that this topology may also be described as the smallest one for which $\beta : B \to A$ is dense.

There is, up to isomorphism. exactly one flat internal functor $M: 1 \to \mathcal{E}$ (where $\mathcal{E}$ is an $\mathcal{S}$-topos in virtue of $q : \mathcal{E} \to \mathcal{S}$), namely $q^* \circ Y$ where $Y : 1 \to \mathcal{S}$ is the (internal) Yoneda functor. By Proposition 9

$$T(M) = T(q^*Y) = q^*(T(Y)) = q^*B$$

so the unique flat functor $1 \to \mathcal{E}$ is a T-model iff $q^*B = q^*A$, iff q^* inverts $\beta : B \to A$. (This example is implicit in [7] §1.)

We next consider an internal frame X in $\mathcal{S}$. Let Idl(X) denote the set of lower sets $X' \subseteq X$ (i.e. subsets satisfying $x_2 \leq x_1 \in X' \Rightarrow x_2 \in X'$). Let $A = \mathrm{Idl}(X)$, and let $B = \left\{ (x,X') \mid x \in X' \in \mathrm{Idl}(X) \right\}$. We have a commutative square

$$\begin{array}{ccc} B & \xrightarrow{\ \gamma\ } & \leq \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle d_1} \\ A & \xrightarrow[\ \sup\ ]{} & X \end{array}$$

(where $\leq$ is the set of $(x_1,x_2) \in X \times X$ with $x_1 \leq x_2$) with $\beta(x,X') = X'$, $\gamma(x,X')=(x,\sup X')$. We define $\sigma : A \times_X \leq \to A$ by

$$\sigma(X', (y,\sup X')) = \left\{ x\wedge y \mid x\in X' \right\}$$

The distributivity of $\wedge$ over sup guarantees validity of (1.3); (1.4) is obvious.

The site thus defined may be denoted T(X). Then sh(T(X)) = sh(X) in the sense of [5].

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