ON DOUBLE DUALIZATION MONADS

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If \( \mathcal{V} \) is a symmetric monoidal closed category in the sense of [2, Section III], and \( D \) is any object in it, the functor

\[ - \otimes D : \mathcal{V} \to \mathcal{V}^{\text{op}} \]

(where \( \otimes \) denotes the inner hom-functor of \( \mathcal{V} \)) is a (strong) left adjoint for

\[ - \otimes D : \mathcal{V}^{\text{op}} \to \mathcal{V}, \]

and so gives rise to a strong monad \( (- \otimes D) \otimes D \) on \( \mathcal{V} \). If \( T = T, \eta, \mu \) is any other strong monad on \( \mathcal{V} \), then giving a \( T \)-algebra structure on \( D \) is equivalent to giving a transformation of monads

(0.1)

\[ \tau : T \to (- \otimes D) \otimes D \]

(Theorem 3.2 below). So the monad \( (- \otimes D) \otimes D \) plays a role analogous to that of the ring of endomorphisms, \( \text{End}(A) \), of an abelian group: giving a \( A \)-module structure \( \xi \) on \( A \) is the same as giving a ring-homomorphism

\[ \tau : A \to \text{End}(A). \]

We shall exploit this fact to define when two structures (for two possibly different monads) on a single object commute. In particular, we call an algebra commutative if the structure of the algebra commutes with itself. In the last section we prove that all algebras for a monad are commutative if and only if the monad is commutative in the sense of [4].

The notation and setting in the present paper is like in [4] and [5] which in turn is almost like in [2]. It should be possible to read this paper on the basis of knowledge of the concepts from [2] alone, although we shall need a few Lemmas and Definitions from [4] and [5]. Since the inner hom-functor \( \text{hom}\mathcal{V}(-,-) \) (here denoted by \( \otimes \) between the arguments) in this paper appears "iterated", it is convenient to avoid brackets by the convention

\[ X \otimes Y \otimes Z = (X \otimes Y) \otimes Z. \]

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I owe credit to F. W. Lawvere for many interesting discussions. In
particular, he called my attention to the existence of a \( \tau \) like (0.1) which
of course is fundamental for the present point of view.

1. Description of the monads.

Let \( D \) be an object in \( \mathcal{V} \), fixed in what follows. Then we have a con-
travariant functor \(- \downarrow D\) from \( \mathcal{V}_0 \) to itself. The category \( \mathcal{V} \) is a \( \mathcal{V} \)-category,
and we can thus form the dual \( \mathcal{V} \)-category \( \mathcal{V}^* \) by the recipe in Section
III.2 of [2]; and, according to Section III.6 of the same paper, \(- \downarrow D\) is
the functor part of a \( \mathcal{V} \)-functor \( R = RD \),

\[
R: \mathcal{V}^* \to \mathcal{V}.
\]

We use the notation \( RD_{BA} \) (from [2]) for the strength of the functor \( R \),
\[
RD_{BA}: A \downarrow B \to (B \downarrow D) \downarrow (A \downarrow D).
\]

The dual \( \mathcal{V} \)-functor \( R^* \) (formed according to Proposition III.2.2, p. 514
of [2]),

\[
R^*: \mathcal{V} = \mathcal{V}^{**} \to \mathcal{V}^*,
\]
is left adjoint to \( R \) in the strong sense, meaning that there exist \( \mathcal{V} \-
natural \)isomorphisms

\[
\varphi_{A,B}: AR^* \downarrow A \to A \downarrow BR,
\]
(\( \downarrow \) denoting the hom-functor for \( \mathcal{V}^* \), and \( \mathcal{V} \)-naturality meaning that
axiom VN of [2, p. 466], is satisfied). The construction of \( \varphi_{A,B} \) is as the
composite arrow in

\[
AR^* \downarrow A = B \downarrow (A \downarrow D) \xrightarrow{p_{BAD}^{-1}} (B \otimes A) \downarrow D
\]

\[
\xrightarrow{c \downarrow 1}
\]

\[
A \downarrow BR = A \downarrow (B \downarrow D) \xleftarrow{p_{ABD}} (A \otimes B) \downarrow D,
\]

where \( c \) is the symmetry of \( \mathcal{V} \), and \( p \) is the fundamental isomorphism
for monoidal closed categories [2, p. 475]. According to Theorem III.7.4,
p. 543, of [2], \( c \) and \( p \) are \( \mathcal{V} \)-natural, and so the composite (1.4) is \( \mathcal{V} \-
natural \), by Theorem I.10.2, p. 466, of [2].

For our purposes, it is more convenient to restate the adjointness in
terms of the front- and end-adjunctions. If one applies the underlying-
functor \( V \) to (1.3), one gets an adjointness between the functors.

\[
\begin{array}{ccc}
\mathcal{V}_0 & \xleftarrow{- \downarrow D} & \mathcal{V}_0^{opp} \\
\mathcal{- \downarrow D} & \xrightarrow{} & \mathcal{- \downarrow D}
\end{array}
\]
the front adjunction \( y_A \) is then the value of the set mapping \( q_{A, AR^*} \) on the element

\[
1_{AR^*} = 1_A \otimes_D \in (AR^* \otimes_A AR^*)V.
\]

From Proposition III.7.9, p. 548, of [2], one directly infers

**Proposition 1.1.** The transformation \( y_A \) is \( \mathcal{V} \)-natural in \( A \) (and also \( \mathcal{V} \)-natural in \( D \) in the extraordinary sense defined in [2, III.5]).

The end-adjunction \( \varepsilon \) for the adjointness is, by inspecting (1.4), seen also to be \( y_A \), this time considered as a morphism in \( \mathcal{V}_0^{\text{opp}} \) from \( A \otimes D \otimes D \) to \( A \). The diagram stating that \( \varepsilon \) is \( \mathcal{V} \)-natural in \( A \) is then, by replacing \( X \otimes Y \) by \( Y \otimes X \), exactly the diagram stating that \( y_A \) is \( \mathcal{V} \)-natural in \( A \). So \( \varepsilon_A \), too, is \( \mathcal{V} \)-natural in \( A \), by the proposition.

By Theorem I.10.7, p. 469, of [2], one gets a hypercategory \( \mathcal{V}_* \)

(=associative 2-dimensional category) by taking the objects to be \( \mathcal{V} \)-categories, morphisms (arrows) to be \( \mathcal{V} \)-functors, and hypermorphisms (2-cells) to be \( \mathcal{V} \)-natural transformations. Since \( y \) and \( \varepsilon \) are \( \mathcal{V} \)-natural, it follows that they make \( R^* \) and \( R \) adjoint arrows in \( \mathcal{V}_* \). Adjoint arrows in any hypercategory give, by composition, rise to a monad in that hypercategory. So we have the following proposition (which of course is also easy to prove by direct methods, without appeal to hypercategories):

**Proposition 1.2.** The data

\[
\begin{align*}
&\begin{cases}
\alpha : \mathcal{V}_0 \to \mathcal{V}_0^* \\
A \otimes B \xrightarrow{R_{BA}^D} (B \otimes D) \otimes (A \otimes D) \xrightarrow{R_{A \otimes D, B \otimes D}^D} [(A \otimes D) \otimes D] \otimes [(B \otimes D) \otimes D]
\end{cases}
\end{align*}
\]

(i) \( y_A : A \to (A \otimes D) \otimes D \)

(ii) \( y_A \delta_D \delta 1 : (A \otimes D) \otimes D \to (A \otimes D) \otimes D \)

define a \( \mathcal{V} \)-monad on \( \mathcal{V} \), that is, (i) defines a \( \mathcal{V} \)-functor \( T : \mathcal{V} \to \mathcal{V} \), and (ii) and (iii) define \( \mathcal{V} \)-natural transformations

\[
\eta : 1 \to T, \quad \mu : T.T \to T,
\]

so that the following (usual) equations hold for any \( A \in \mathcal{V}_0^* \):

\[
\begin{align*}
(1.6) \quad & \eta_A T.\mu_A = 1_{AT} = \eta_{AT}.\mu_A \\
(1.7) \quad & \mu_A T.\mu_A = \mu_{AT}.\mu_A.
\end{align*}
\]

**Remark 1.3.** Applying the right adjoint \( - \otimes D \) to an object \( B \) in \( \mathcal{V}_0^{\text{opp}} \),
and to the end-adjunction \( y_B \) for \( B \), gives, by general theory [3], an algebra for the monad \( - \star D \star D \).

2. The twisting.

It is convenient to "express" the symmetry \( c: A \otimes B \to B \otimes A \) assumed for \( \mathcal{V} \) in terms of \( \star \) alone. For any objects \( X, Y, Z \) in \( \mathcal{V} \), we have a composite isomorphism which we denote \( TW_{X,Y,Z} \),

\[
\begin{align*}
X \star (Y \star Z) & \xrightarrow{TW_{X,Y,Z}} Y \star (X \star Z) \\
\downarrow & \\
(X \otimes Y) \star Z & \xrightarrow{c \star 1} (Y \otimes X) \star Z.
\end{align*}
\]

It is clearly "involuntary": \( TW_{X,Y,Z} \cdot TW_{Y,X,Z} = 1 \). Furthermore, using Theorem III.7.4, p. 543, in [2], we see that \( TW \) is \( \mathcal{V} \)-natural in each variable. So it is also natural in the ordinary sense in all three variables.

Applying the "underlying" functor \( V \) to \( TW_{X,Y,Z} \) gives a set mapping

\[
Tw_{X,Y,Z}: \mathcal{V}_0(X, Y \star Z) \to \mathcal{V}_0(Y, X \star Z)
\]

(in the sequel subscripts will often be omitted) which again is natural in all three variables.

The definition of \( y_A^D \) can be rephrased in terms of the \( Tw \)-operation. We have

\[
Tw(A \star D \xrightarrow{1} A \star D) = A \xrightarrow{y_A^D} (A \star D) \star D.
\]

Also, the transformation \( \lambda \), which is fundamental to [5] and to this paper can be defined in terms of \( Tw \). Let \( T, st \) be a \( \mathcal{V} \)-functor from \( \mathcal{V} \) to itself (see [2, p. 444]). Then define for any pair \( A,B \) of objects of \( \mathcal{V} \) a morphism

\[
\lambda_{A,B}: (A \star B)T \to A \star B T
\]

by

\[
\lambda_{A,B} = Tw(A \xrightarrow{y_A^B} (A \star B) \star B \xrightarrow{st} (A \star B)T \star BT).
\]

This is just a restatement of the definition of \( \lambda \) from [5]. In particular, \( \lambda_{A,B} \) is \( \mathcal{V} \)-natural in both variables.

There is a description of the \( Tw \)-operation in terms of morphisms already present in \( \mathcal{V}_0 \). Let \( t: X \to Y \star Z \) be a morphism. Then \( Tw(t) \) is the composite morphism
(2.2) \[ Y \xrightarrow{u} X \triangleleft (Y \otimes X) \xrightarrow{1 \triangleleft c} X \triangleright (X \otimes Y) \xrightarrow{1 \triangleright (t \otimes 1)} X \triangleright ((Y \triangleleft Z) \otimes Y) \xrightarrow{1 \triangleright \text{ev}} X \triangleright Z, \]

where \( u \) and \( \text{ev} \) are the front- and end-adjunctions for the adjointness of the type

\[ \neg \otimes A \vdash A \triangleright \neg \]

(called \( u \) and \( t \), respectively, in [2, p. 477]).

3. Double dualization monads and algebras.

Let \( T = (T, \text{st}, \eta, \mu) \) be a \( \mathcal{V} \)-monad on the \( \mathcal{V} \)-category \( \mathcal{V} \), that is, \( T, \text{st} \)

\[ 1_{\mathcal{V}} \xrightarrow{\eta} T \xleftarrow{\mu} T.T \]

satisfying the (usual) equations (1.6) and (1.7). Recall [3] that a \( T \)-algebra is a pair \( (X, \xi) \), where \( X \in \mathcal{V}_0 \) and \( \xi : XT \to X \) satisfies the unit- and associative laws

(3.1) \[ \eta_{X \cdot \xi} = 1, \quad \xi T \cdot \xi = \mu_{X \cdot \xi}; \]

a \( T \)-homomorphism \( (X, \xi) \to (X', \xi') \) is a morphism \( f : X \to X' \) so that \( fT \cdot \xi' = \xi \cdot f \).

Let \( (D, \delta) \) be a \( T \)-algebra. Construct a transformation \( \tau : T \to (- \triangleright D) \triangleright D \) by putting \( \tau_A \) equal to the composite

(3.2) \[ AT \xrightarrow{y_A \cdot PT} ((A \triangleright D) \triangleright D)T \xrightarrow{\lambda_A \cdot \delta_{D,D}} (A \triangleright D) \triangleright DT \xrightarrow{1 \triangleright \delta} (A \triangleright D) \triangleright D, \]

where \( \lambda \) is defined as in (2.1). Since \( y \) and \( \lambda \) are \( \mathcal{V} \)-natural in both variables, \( \tau_A \) is \( \mathcal{V} \)-natural in \( A \) (and natural in the extraordinary sense with respect to \( (D, \delta) \)).

**Proposition 3.1.** The morphisms \( \tau_A \) from (3.2) constitute a transformation of \( \mathcal{V} \)-monads.

**Proof.** We have argued that \( \tau_A \) is \( \mathcal{V} \)-natural in \( A \). It remains to be proved that "\( \tau \) commutes with the \( \eta \)'s and \( \mu \)'s". To prove

\[ \eta_A \cdot \tau_A = y_A \]

means proving commutativity of the outer diagram in
The left square commutes by naturality of \( \eta \), the middle triangle by Lemma 1.6 in [5], and the right-hand triangle commutes by (3.1). To prove \( \mu_A \cdot \tau_A = (\tau_A)T \cdot \tau_A \wedge D \wedge D \cdot (y_A \wedge D \wedge 1) \) means proving commutativity of the outer diagram in

The top square commutes by naturality of \( \mu \); then comes a pentagon, and it commutes by Lemma 1.6 in [5]. The two next squares are commutative by naturality of \( \lambda \) and (3.1), respectively. The bottom right square obviously commutes; the bottom left square (reading upwards) again commutes by naturality of \( \lambda \). Finally

\[
(y_A \wedge D \wedge D)T \cdot (y_A \wedge D \wedge 1)T = 1
\]
comes from one of the standard equations between front- and end-adjunctions

\[
y_{(A \wedge D)R} \cdot (e_A \wedge D)R^* = 1
\]
for the fundamental adjointness (1.3). This proves Proposition 3.1.

We can perform a converse construction. If
(3.3) \[ \tau: T \to (\neg \land D) \land D \]

is a monad transformation, \(D\) can be endowed with a structure \(\delta\). This follows from the obvious observation that \(D\) carries a canonical algebra structure \(\Delta\) for the \((\neg \land D) \land D\) monad. For, let \(I\) be the unit object which is part of the data for a closed category. Then \(D \cong I \land D\), and so

\[ y_I \land 1: I \land D \land D \land D \to I \land D \]

is an algebra structure on \(I \land D\). Transporting it to \(D\) by means of \(I \land D \cong D\) defines the structure \(\Delta\) on \(D\). An easy (and well-known) argument gives that if \(\tau: T \to S\) is a monad transformation and \(X, \xi\) an \(S\)-algebra, then

\[ XT \xrightarrow{\tau_X} XS \xrightarrow{\xi} X \]

makes \(X\) into a \(T\)-algebra. So in particular, with the \(\tau\) of (3.3),

(3.4) \[ DT \xrightarrow{\tau_D} (D \land D) \land D \xrightarrow{\Delta} D \]

makes \(D\) into a \(T\)-algebra.

**Theorem 3.2.** There is a one-to-one correspondence between \(T\)-algebra structures \(\delta\) on \(D\) and \(V\)-monad transformations from \(T\) to the double dualization monad for \(D\).

The theorem will follow from the above observations together with

**Proposition 3.3.** The passage from \(\tau\) to \(\delta\) and conversely describes a 1-1 correspondence between the set of maps \(\delta: DT \to D\) and the set of \(V\)-natural transformations \(T \to (\neg \land D) \land D\).

**Proof.** Let us start with a map \(\delta: DT \to D\). Then \(\tau_\Delta\) is given by (3.2). The clockwise composite in the following diagram is then the map \(DT \to D\) constructed out of \(\tau\):

\[ DT \xrightarrow{yT} (D \land D) \land D)T \xrightarrow{\lambda} D \land D \land DT \xrightarrow{1 \land \delta} D \land D \land D \]

\[ (I \land D)T \xrightarrow{yT} (I \land D) \land D \land D)T \xrightarrow{\lambda} I \land D \land D \land DT \xrightarrow{1 \land \delta} I \land D \land D \land D \]

\[ (I \land D)T \xrightarrow{yT} I \land DT \xrightarrow{1 \land \delta} I \land D \]

\[ (i_D^{-1})T \xrightarrow{\lambda} DT \xrightarrow{\delta} D. \]
In the first row the first diagram commutes by naturality of \( y \), the second by naturality of \( \lambda \), and the third one for obvious reasons. In the next row, the commutativity of the "triangle" is one of the adjunction equations for the fundamental adjointness; the first square commutes by naturality of \( \lambda \), and the second one for obvious reasons.

Finally, the bottom "triangle" commutes by Lemma 1.7 in [5], and the bottom square by naturality of \( i \). The counterclockwise composite in the diagram is \( \delta \). Conversely, if a \( \mathcal{V} \)-functor transformation \( \tau : T \to (- \triangleleft D) \triangleright D \) is given, we have to show that we get \( \tau \) back again when applying the two processes. It can be done directly, but is quite elaborate. Instead, we shall derive from (the Eilenberg–Kelly version of) Yoneda's lemma, that the process leading from \( \tau \) (assumed to be \( \mathcal{V} \)-natural) to \( \delta \) is in fact one-to-one, onto; our process leading from \( \delta \) to \( \tau \) is then the inverse, since we already have seen that it is a one-sided inverse.

First notice that the \( Tw \)-operation can be used to establish a bijection (also called \( Tw \))

\[
\mathcal{V} \cdot \text{Nat}(G, (-)F \triangleright D) \cong \mathcal{V} \cdot \text{Nat}(F, (-)G \triangleright D),
\]

where \( \mathcal{A} \) is a \( \mathcal{V} \)-category and

\[
F : \mathcal{A} \to \mathcal{V} \quad \text{and} \quad G : \mathcal{A}^* \to \mathcal{V}
\]

are \( \mathcal{V} \)-functors. For, let a \( \tau \) be given on the left; construct \( \hat{\tau} \) on the right by putting

\[
\hat{\tau}_x = (\tau_x)Tw_{XG, XF, D}, \quad X \in |\mathcal{A}|.
\]

Then \( \hat{\tau}_x \) is \( \mathcal{V} \)-natural if and only if

(3.5) \[
(\hat{\tau}_x)_I : I \to XF \triangleright (XG \triangleright D)
\]

is \( \mathcal{V} \)-natural in \( X \), by Lemma III.7.8, p. 547, of [2]. But (3.5) can be described as the composite

\[
I \xrightarrow{(\tau_x)_I} XG \triangleright (XF \triangleright D) \xrightarrow{Tw_{XG, XF, D}} XF \triangleright (XG \triangleright D),
\]

which as a composite of \( \mathcal{V} \)-natural transformations is \( \mathcal{V} \)-natural. From the involutary property of \( Tw \) it follows that the established correspondence is one-to-one, onto.

In particular, we have an isomorphism

(3.6) \[
\mathcal{V} \cdot \text{Nat}(T, (- \triangleleft D) \triangleright D) \xrightarrow{Tw} \mathcal{V} \cdot \text{Nat}(- \triangleleft D, (-)T \triangleright D),
\]
but for the right-hand side here we can use Eilenberg–Kelly’s Yoneda lemma (Theorem I.8.6, p. 457, in [2]) to get a bijection \( \Gamma' \), displayed as the first arrow in

\[
\begin{align*}
\mathcal{V}^2\text{-Nat}(\neg \hat{\cdot} D, (-)T \hat{\cdot} D) & \xrightarrow{\Gamma} \mathcal{V}_0^2(I, DT \hat{\cdot} D) \\
& \overset{\cong}{\xrightarrow{T_w}} \mathcal{V}_0^2(DT, I \hat{\cdot} D) \\
& \overset{\cong}{\xrightarrow{\mathcal{V}_0(1, i_D^{-1})}} \mathcal{V}_0^2(DT, D).
\end{align*}
\]

(3.7)

Proposition 3.3 then follows from

**Lemma 3.4.** The bijection (3.6) followed by the bijection (3.7) sends \( \tau \) to the structure \( \tau_{D, \Delta} \) displayed in (3.4).

**Proof.** By (2.2), \( \hat{\tau}_D \) is the composite from \( D \hat{\cdot} D \) to the lower right-hand corner in the diagram (3.8), p. 160.

The whole clockwise composite is \( (\hat{\tau}) \Gamma' \). The whole counter-clockwise composite is, again by (2.2),

\[
(\tau_{D, \Delta, i_D} 1)T_w = (\tau_{D, \Delta} i_D)T_w.
\]

But (3.8) on p. 160 commutes by naturality of \( u \) and \( c \) and (extraordinary) naturality of \( \text{ev} \). So

\[
(\tau)T_w \Gamma' = (\hat{\tau}) \Gamma' = (\tau_{D, \Delta} i_D)T_w.
\]

Applying \( T_w \) to this equation and multiplying on the right by \( i_D^{-1} \) gives the equality claimed in the lemma.

**Remark 3.5.** Since \( (B \hat{\cdot} D, y_B \cdot 1) \) is an algebra for the monad \( - \hat{\cdot} D \hat{\cdot} D \) (Remark 1.3), Proposition 3.1 implies that there exists a transformation of \( \mathcal{V}^-\)-monads

\[
\zeta: \neg \hat{\cdot} D \hat{\cdot} D \to \neg (B \hat{\cdot} D) \hat{\cdot} (B \hat{\cdot} D).
\]

We can give a simple direct description of \( \zeta \). Consider

\[
\begin{align*}
X \hat{\cdot} D \hat{\cdot} D & \xrightarrow{L_B} B \hat{\cdot} (X \hat{\cdot} D) \hat{\cdot} (B \hat{\cdot} D) \\
& \xrightarrow{T W \hat{\cdot} 1} \\
& X \hat{\cdot} (B \hat{\cdot} D) \hat{\cdot} (B \hat{\cdot} D).
\end{align*}
\]

(3.9)

It is clearly \( \mathcal{V}^-\)-natural in \( X \). So to see that (3.9) actually is \( \zeta_X \), it suffices, by Proposition 3.3, to see that it gives rise to the structure \( y_B \hat{\cdot} 1 \) on \( B \hat{\cdot} D \), (since \( y_B \hat{\cdot} 1 \) was used to define \( \zeta \)). To see this is fairly easy, using \( \Pi.3.20 \), p. 480, and axioms CC 1 and CC 2 of [2].
\[
\begin{align*}
I & \xrightarrow{j_D} D \otimes D \\
& \xrightarrow{u} DT \otimes ((D \otimes D) \otimes DT) \\
& \xrightarrow{1 \otimes c} DT \otimes (DT \otimes (D \otimes D)) \\
& \xrightarrow{1 \otimes (\tau_D \otimes 1)} \\
DT \otimes (I \otimes DT) & \xrightarrow{1 \otimes c} DT \otimes (DT \otimes I) \\
& \xrightarrow{1 \otimes (\tau_D \otimes 1)} DT \otimes ((D \otimes D \otimes D) \otimes I) \\
& \xrightarrow{1 \otimes (1 \otimes j_D)} DT \otimes ((D \otimes D \otimes D) \otimes (D \otimes D)) \\
& \xrightarrow{1 \otimes ev} \\
& \xrightarrow{1 \otimes ev} DT \otimes (I \otimes D \otimes I) \\
& \xrightarrow{1 \otimes ev} DT \otimes D.
\end{align*}
\]
4. Commuting structures.

In this section, we define and study the notion of commutation of structures on an object (with respect to two, possibly different, monads). There are two approaches to this (they can be shown to be equivalent). The one to be used here, defines the notion in terms of \( \otimes \), and gives the desired results in a fairly straightforward way. The other describes commutativity in terms of \( \land \); this is the approach of a preliminary draft [6] of the paper, and is much more complicated. Of course, one might define the notion of symmetric closed category without mentioning \( \otimes \), in which case the \( \land \)-method could be applied, the \( \otimes \)-method not.

Recall [4] that the functor part \( T \) a \( \mathcal{V} \)-monad \( T \) on \( \mathcal{V} \) has two canonical closed (or monoidal) structures

\[
\psi^T, \tilde{\psi}^T : AT \otimes BT \to (A \otimes B)T.
\]

**Definition 4.1.** Let \( T_0, T_1, \) and \( S \) be \( \mathcal{V} \)-monads on \( \mathcal{V} \), and let \( \tau_i : T_i \Rightarrow S, i = 0, 1, \) be \( \mathcal{V} \)-monad transformations. We say that \( \tau_0 \) commutes with \( \tau_1 \) if the following diagram commutes for all \( A, B \in |\mathcal{V}_0| \):

\[
\begin{array}{c}
AT_0 \otimes BT_1 \\
\downarrow \psi^T \\
AS \otimes BS \\
\downarrow \tilde{\psi}^S
\end{array}
\xrightarrow{(\tau_0)_A \otimes (\tau_1)_B}
\begin{array}{c}
(AS \otimes BS) \\
\downarrow \psi^S \\
(A \otimes B)S
\end{array}
\]

\( T_i \) being the functor part of \( T_i \), \( S \) the functor part of \( S \).

**Proposition 4.2.** The notion of structures commuting is symmetric.

**Proof.** This follows since \( c \cdot \psi^S_{B,A} = \tilde{\psi}^S_{A,B}(c)S \) by definition of \( \tilde{\psi} \) in terms of \( \psi \), where again \( c \) denotes the symmetry.

**Proposition 4.3.** The notion of commutation is stable under left and right composition, that is, if

\[
T_i' \xrightarrow{\sigma_i} T_i \xrightarrow{\tau_i} S \xrightarrow{\sigma} S'
\]

are transformations of monads, \( i = 0, 1, \) and \( \tau_0 \) commutes with \( \tau_1 \), then \( \varphi_0 \cdot \tau_0 \cdot \sigma \) commutes with \( \varphi_1 \cdot \tau_1 \cdot \sigma \).

**Proof.** It is obvious for composing on the left. For composition by \( \sigma \) on the right, the result follows if we know that the diagram below (and the similar diagram with \( \tilde{\psi} \) instead of \( \psi \)) commutes:

\[
\begin{array}{c}
AS \otimes BS \\
\downarrow \sigma_A \otimes \sigma_B
\end{array}
\xrightarrow{\psi^S}
\begin{array}{c}
(AS \otimes BS) \\
\downarrow \sigma_A \otimes \sigma_B
\end{array}
\xrightarrow{\tilde{\psi}^S}
\begin{array}{c}
(AS' \otimes BS') \\
\downarrow \sigma_A \otimes \sigma_B
\end{array}
\xrightarrow{\psi^{S'}}
\begin{array}{c}
(A \otimes B)S' \\
\downarrow \sigma_A \otimes \sigma_B
\end{array}.
\]

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But these commutativities are immediate, using Lemma 1.1 in [4] together with the definition of ψ and ψ̃.

One might thus define the notion of Freyd tensor product of ℶ-monads on ℶ. Call \( \tilde{τ}_i : T_i \Rightarrow T_0 \otimes T_1, i = 0, 1 \), a tensor product of monads if

1) \( \tilde{τ}_0 \) and \( \tilde{τ}_1 \) are commuting ℶ-monad transformations, and

2) to any other pair \( τ_i : T_i \Rightarrow S \) of commuting ℶ-monad transformations, there exists a unique ℶ-monad transformation \( σ : T_0 \otimes T_1 \Rightarrow S \) with \( \tilde{τ}_i σ = τ_i, i = 0, 1 \). (If such a \( T_0 \otimes T_1 \) exists, it is essentially unique.)

Recall [4] that a monad \( T \) was termed commutative if \( ψ T = ψ T^T \). In the terminology of Definition 4.1 this then just says that the identity transformation on \( T \) commutes with itself. So from Proposition 4.3, one derives

**Proposition 4.4.** If \( T \) is a commutative ℶ-monad, then any ℶ-monad transformation

\[ τ : T \Rightarrow S \]

commutes with itself, \( S \) being an arbitrary ℶ-monad.

**Definition 4.5.** Let \( T_0, T_1 \) be ℶ-monads on ℶ, and let an object \( D \) have \( T_i \)-structure \( δ_i : DT_i \Rightarrow D, i = 0, 1 \). Let \( τ_i : T_i \Rightarrow -D \otimes D \) be the corresponding ℶ-monad transformations. Then \( δ_0 \) is said to commute with \( δ_1 \) if \( τ_0 \) and \( τ_1 \) commute, and \( D, δ_0 \) is called a commutative \( T_0 \)-algebra if \( τ_0 \) commutes with itself.

By Propositions 4.2 and 4.3 we get

**Proposition 4.6.** The notion of commuting structures \( δ_0, δ_1 \) on an object \( D \) is symmetric. Further, it is stable under left composition, that is, if \( ϕ_i : T'_i \Rightarrow T_i, i = 0, 1 \), are ℶ-monad transformations, and \( δ_0, δ_1 \) commute (as above), then the \( T_0' \) (resp. \( T_1' \)) structures on \( D \)

\[ (ϕ_i)_D : DT'_i \Rightarrow D \]

commute.

Let \( δ_0, δ_1 \) and \( τ_0, τ_1 \) be as in Definition 4.5, and let \( B \) be an arbitrary object. By Remark 3.5 we have a ℶ-monad transformation

\[ ζ : -D \otimes D + D \Rightarrow -D \otimes (B \otimes D) + (B \otimes D), \]

and therefore, by composition, ℶ-monad transformations

\[ (4.3) \quad τ_i ζ : T_i \Rightarrow -D \otimes (B \otimes D) + (B \otimes D), \quad i = 0, 1, \]

which by Theorem 3.2 means that we have a \( T_i \)-structure on \( B \otimes D \) "induced by \( δ_i' \), \( i = 0, 1 \). If \( δ_0 \) commutes with \( δ_1 \), that is, \( τ_0 \) commutes
with \( \tau_1 \), then by Proposition 4.3, the two transformations in (4.3) also commute, so that we have

**Proposition 4.7.** If the \( T_0 \)-structure \( \delta_0 \) on \( D \) commutes with the \( T_1 \)-structure \( \delta_1 \) on \( D \), then also the induced structures on \( B \triangleleft D \) commute.

One may describe the induced structure on \( B \triangleleft D \) directly as the composite

\[
(B \triangleleft D)T \xrightarrow{\lambda} B \otimes DT \xrightarrow{1 \triangleleft \delta} B \triangleleft D.
\]

The proof of this fact involves a medium sized diagram. We omit it.

The complete link between the notions of commuting structures, commuting \( \mathcal{V} \)-monad transformations, and commutative \( \mathcal{V} \)-monads is given in

**Theorem 4.8.** Let \( T \) be a \( \mathcal{V} \)-monad on \( \mathcal{V} \). Then the following three statements are equivalent:

(i) every algebra \( (D, \delta) \) for \( T \) is commutative;
(ii) every \( \mathcal{V} \)-monad transformation with domain \( T \) commutes with itself;
(iii) \( T \) is commutative in the sense of [4].

**Proof.** Proposition 4.4 establishes (iii) \( \Rightarrow \) (ii), and (ii) \( \Rightarrow \) (i) is trivial in view of Definition 4.5. Finally assume (i) for the algebra \( (D, \delta) = ((A \otimes B)T, \mu_{A \otimes B}) \), and let \( \tau \) be the corresponding \( \mathcal{V} \)-monad transformation \( T \Rightarrow (\Delta \triangleleft D) \triangleleft D \). Denote the two closed structures on \( (\Delta \triangleleft D) \triangleleft D \) by \( \psi^{DD}, \tilde{\psi}^{DD} \), respectively. Then by assumption, the clockwise composite in the following diagram commutes:

\[
\begin{array}{ccc}
AT \otimes BT & \xrightarrow{\tau_{A \otimes B}} & (A \triangleleft (A \otimes B)T) \triangleleft ((B \triangleleft (A \otimes B)T) \otimes (A \triangleleft (A \otimes B)T)) \\
\psi^T & \downarrow & \psi^{DD} \\
\tilde{\psi}^T & & \tilde{\psi}^{DD} \\
(A \otimes B)T & \xrightarrow{\tau_{A \otimes B}} & (A \otimes B) \triangleleft (A \otimes B)T \\
& \downarrow & \downarrow 1 \triangleleft \eta \triangleleft 1 \\
& (A \otimes B) \triangleleft (A \otimes B) \triangleleft (A \otimes B)T & \xrightarrow{j \triangleleft 1} \\
& & I \triangleleft (A \otimes B)T
\end{array}
\]
Hence, by (4.2), the left hand composite commutes, provided $\ast$ commutes. But using the definition of $\tau$ (and writing $E$ for $A \otimes B$), $\ast$ is the outer diagram of

\[
\begin{array}{cccccccc}
E T & \xrightarrow{(y^{ET})^T} & (E \otimes ET \otimes ET) T & \xrightarrow{\lambda} & E \otimes ET \otimes ET^2 & \xrightarrow{1 \otimes \mu} & E \otimes ET \otimes ET \\
\downarrow{y^{ET}} & & \downarrow{(1 \otimes \eta) \otimes 1)^T} & & \downarrow{\lambda} & & \downarrow{1 \otimes \mu} \\
(E \otimes E \otimes E) T & \xrightarrow{(1 \otimes \eta_E)^T} & (E \otimes E \otimes ET) T & \xrightarrow{\lambda} & E \otimes E \otimes ET^2 & \xrightarrow{1 \otimes \mu} & E \otimes E \otimes ET \\
\downarrow{\lambda} & & \downarrow{1 \otimes \eta_E^T} & & \downarrow{(1 \otimes \eta) \otimes 1} & & \downarrow{1 \otimes \mu} \\
E \otimes E \otimes ET & \xrightarrow{j \otimes 1} & I \otimes ET.
\end{array}
\]

Diagrams with no number commute by naturality. The diagram (1) commutes by extraordinary naturality of $y$, and (2) commutes by a monad law. Finally, (3) commutes using Lemma 1.7 in [5], naturality of $\lambda$, and the equation

(4.4) \quad y. j \otimes 1 = i. \]

To prove (4.4), apply $Tw$ to it and use naturality of $Tw$ with respect to $j$. Since $Tw(y) = 1_{D \otimes D}$, the left hand side gives just $j$. To see that

$Tw(i) = j$,

apply (II.3.15), (II.3.17), and Proposition III.1.1 of [2].

Let us finally, without proof, state how the notion of commutation of structures can be defined in terms of the $\otimes$-structure. Let

$\delta_0 : DT_0 \to D, \quad \delta_1 : DT_1 \to D$

be $T_0$ and $T_1$-structures on $D$. Then $\delta_0$ and $\delta_1$ commute in the sense of Definition 4.5 if and only if the following diagram commutes for all $X$:

\[
\begin{array}{cccccccc}
st_1 & \xrightarrow{(X \otimes D)T_1 \otimes DT_1} & (X \otimes D)T_1 \otimes D & \xrightarrow{1 \otimes \delta_1} \\
\downarrow{\tau_0} & & \downarrow{(X \otimes D)T_1 \otimes D} & & \downarrow{\lambda_1 \otimes 1} \\
XT_0 & \xrightarrow{1 \otimes \delta_1 \otimes 1} & X \otimes DT_1 \otimes D & \xrightarrow{\lambda_1 \otimes 1} \\
\end{array}
\]
where \( \tau_0 \) is the monad transformation associated to \( \delta_0 \), and \( st_1 \) is the strength of \( T_1 \). Note that the equalizer of the square (if it exists) is the "subobject of homomorphisms from \( X \otimes D \) to \( D \)" (with respect to the \( T_1 \)-structure \( \delta_1 \) on \( D \) and the structure induced by \( \delta_1 \) on \( X \otimes D \)). The "subobject of homomorphisms" was used by Bunge in [1] and Linton in [8] to define the \( \mathcal{V} \)-category of algebras for a \( \mathcal{V} \)-monad, and by the author in [5] to define the closed category of monads for a commutative \( \mathcal{V} \)-monad on \( \mathcal{V} \). For the special case of the equalizer for the square diagram above, one may even prove that it defines a submonad of \( - \otimes D \otimes D \), "the dual of \( T_1 \) with respect to \( D \);" compare also [7].

REFERENCES


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