Preface to second edition of "<u>Differential Geometry Without Real Numbers</u>", November 1980.

The first printing of the complete book (Jan. 1979) went out of stock around Jan. 1980, and there seems still to be a need of it, at least until I finish the book I am presently writing for the Cambridge University Press. So I decided to reprint it almost unchanged, except that the following paragraphs of the First Printing

§24 C^k-manifolds and k-monads

§25 Tangent bundles of C^k-manifolds

§26 C^2-manifolds satisfy the Axiom W

<u>are replaced</u> by an Appendix, taken from my article "Formal manifolds and synthetic theory of jet bundles", Cahiers de Top. et Géom. Diff. 21 (1980), 227–246.

Also, §10 on differential forms should have been replaced; better treatments exist now, partly in Inger Holgaard's "speciale" (who use the same notion of form, but a different description of the exterior derivative), partly in Kock: Differential Forms in Synthetic Differential Geometry, Aarhus Preprint Series 1978/79 No. 28, and Kock, Reyes and Veit, Forms and Integration in Synthetic Differential Geometry, Aarhus Preprint Series 1979/80 No. 31. We give even a different notion of form (compared with the other one, also in Inger Holgaard's speciale).

On p. 29, a misprint has been corrected. Else, the notes are in their original, very rough (in particular: sometimes incomplete, sometimes incorrect) form. Please have this in mind when reading!

Anders Kock.

<u>Differential Geometry</u>

<u>without</u>

<u>real numbers</u>

Anders Kock.

We give here a theory of some of the differential geometric aspects of space and other "manifolds." The theory is axiomatic, and does not even presuppose elementary calculus. It is furthermore inconsistent, if the reasoning one applies is that of "metaphysical" logic, but not when only "positive reasoning" and "constructive constructions" are used. The precise meaning of this can be explained by saying that we are talking about objects in some topos, but one which is not the category of <u>abstract</u> sets. In this way, our reasoning can be tested within the framework of standard mathematical rigour; see e.g. [1], [2], [3], [6].

However, in these notes, the method we shall use is, that we do not <u>argue</u> that the reasoning we apply is of the "positive, constructive" kind; we just <u>assert</u> it. In this sense, all our proofs are incomplete, but their completion is a pure technical matter of categorical logic. A partial argument is that we nowhere use reasoning via "law of excluded middle", or "by reductio ad absurdum". Also, the reader will not find the word "not" in these notes.*

The philosophy that such kind of differential geometry is possible is due to Lawvere [4]. His program was realized by him, Reyes, Wraith, myself, and others, see e.g. [1], [2], [3], [6].

* except ...

1 The line A

We start by stating some properties which *the line* has. We call the line A. We choose two distinct points on it and call them 0 and 1. It is now a property of geometric constructions that we can *add* and *multiply* ("adding" by putting line segments next to each other, and "multiplying" essentially by construction of "fourth proportional"). We can also subtract. It is an old experience (and a Theorem in Euclidean synthetic geometry, as well) that A with this structure becomes a *commutative ring*.

In the classical *analytical* model for geometry, A becomes identified with the *field* $\mathbb{R}$ of real numbers (which in turn is an arithmetic construct, basing itself on the "actual infinity, the set $\mathbb{N}$ of natural numbers). However convenient this identification, it is not a necessary aspect of our thinking on geometry. Hjelmslev [1922] even maintained that "$A = \mathbb{R}$" was *false* in "the geometry of reality," because it implies that any two distinct points in the plane $A \times A$ is connected by a *unique* line; and Hjelmslev knew from teaching technical drawing that if the points are very close, the connecting line is not unique.

Also in the present theory, A is not a field. If it were, the theory would collapse. For, we consider the set $D \subseteq A$ defined by

(1.1) $$D = \{ d \in A \mid d^2 = 0 \}.$$

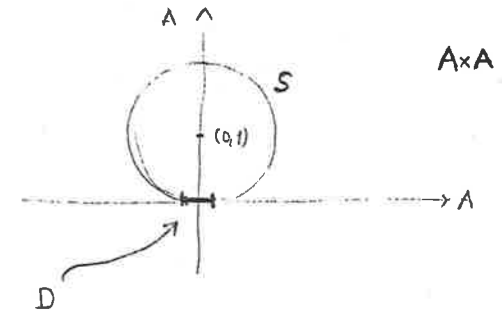
Elements in D are called *infinitesimals*; they are the keys in the theory. If A were a field, 0 would be the only infinitesimal, and there would be no theory.

Joyal has pointed out the following motivation for D. Pythagoras' theorem is a fact about distance measuring in physical space. So in the plane $A \times A$, the set of points at distance 1 from $(0,1)$ is

$$S = \{ (x,y) \mid x^2 + (y-1)^2 = 1 \}.$$

The intersection of this circle with the line A (viewed as the x-axis in $A \times A$) is precisely D:

(1.2)



So D is so small a portion of the line that is also part of the unit circle. It is a unity of opposites, the "curved" and the "straight."

Other small geometric sets that will be of importance is the set $D \times D \subseteq A \times A$, as well as

$$D(2) \subseteq A \times A$$

defined by

(1.3) $$D(2) = \{ (d_1, d_2) \mid d_1^2 = 0 \wedge d_2^2 = 0 \wedge d_1 \cdot d_2 = 0 \}.$$

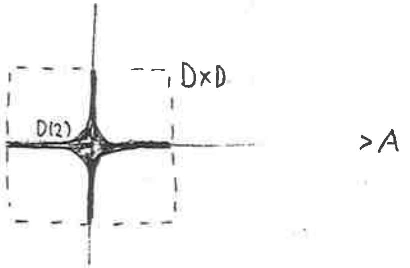
Clearly, $D(2) \subseteq D \times D$. Also, $D(2)$ contains the two axes of $D \times D$,

(1.4) $$\{(d,0) \mid d \in D\} \quad \text{and} \quad \{(0,d) \mid d \in D\}.$$

but it contains more: it contains in particular the diagonal

(1.5) $$\{(d,d) \mid d \in D\}.$$

Alternatively, we have three embeddings $D \longrightarrow D(2)$, namely ι_1: $d \mapsto (d,0)$, ι_2: $d \mapsto (0,d)$, and Δ: $d \mapsto (d,d)$ respectively. Wraith has suggested drawing the following picture of $D(2)$:



where the "thickening" around $(0,0)$ is meant to indicate that $D(2)$ contains more than the two axes (1.4). From the picture, one may think that the "thickening" around $(0,0)$ is not large enough to receive the diagonal, but note 1) there is no largest element in D 2) metric concept should not be applied to infinitesimals.

We more generally have

$$D(n) \subseteq A^n \qquad n = 1, 2, 3, \dots$$

defined by

$$D(n) = \{(d_1, \dots, d_n) \mid d_i \cdot d_j = 0 \quad \forall i,j = 1, \dots, n\}.$$

We list some algebraic properties of infinitesimals:

(1.6) $$a \in A \wedge d \in D \implies a \cdot d \in D.$$

(1.7) $$(d_1, d_2) \in D(2) \implies d_1 + d_2 \in D$$

For,

$$(d_1 + d_2)^2 = d_1^2 + d_2^2 + 2 \cdot d_1 \cdot d_2 = 0,$$

using (1.3). But note that $(d_1, d_2) \in D \times D$ is not sufficient to have $d_1 + d_2 \in D$.

(1.8) $$a \in A \wedge b \in A \wedge d \in D \implies (a \cdot d, b \cdot d) \in D(2).$$

(1.9) $$a \in A \wedge b \in A \wedge (d_1, d_2) \in D(2) \implies (a \cdot d_1, b \cdot d_2) \in D(2)$$

2. Differential calculus

Till now, we have made no assumptions except that we have a commutative ring A. We shall now assume

<u>Axiom 1</u> Every map $g: D \longrightarrow A$ is of the form

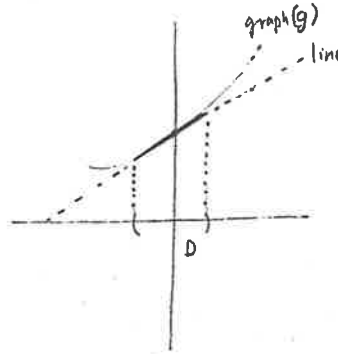
(2.1) $$\boxed{d \longmapsto c + b \cdot d \qquad \forall d \in D}$$

for <u>unique</u> c and b in A.

If this holds, we say A is of <u>line</u> type, [1]. See also §3.

The geometric heuristics is simple. A function of form $x \longmapsto c + b \cdot x$ has a straight line for its graphical picture. The axiom then says that D is <u>so small</u> that the graph of any map $D \to A$ is contained in such a straight line, but also <u>so large</u> that this line is unique:

(2.2)



Clearly, the c occurring in (2.1) has to be $g(0)$. On the other hand, the b occurring in (2.1) indicates the <u>slope</u> of the line in (2.2), and thus can be used as a definition of $g'(0)$.

More generally, let $B \subseteq A$, and let $B' \subseteq B$ be defined by

$$B' = \{ a \in B \mid a+d \in B \quad \forall d \in D \}.$$

Then, for any $f : B \longrightarrow A$ we define its <u>derivative</u> $f' : B' \longrightarrow A$ by putting, for $a \in B'$,

$$\boxed{\begin{array}{l} f'(a) = \text{that unique } b \text{ such that} \\ f(a+d) = f(a) + b \cdot d \qquad \forall d \in D \end{array}}$$

(More precisely, consider $g : D \longrightarrow A$ defined by

$$g(d) = f(a+d) \qquad \forall d \in D;$$

the b given by (2.1) for this g is then $f'(a)$.).

Alternatively, $f'(a)$ is defined as the unique element which makes the following Taylor formula hold:

(2.3) $$f(a+d) = f(a) + d \cdot f'(a)$$

Before proceeding, let us note that for all b_1 and b_2 in A

(2.4) $$(\forall d \in D : d \cdot b_1 = d \cdot b_2) \Rightarrow b_1 = b_2$$

("universally quantified d's may be cancelled"). For, the function $g : D \longrightarrow A$ given by $d \mapsto d \cdot b_1$ $(= d \cdot b_2)$ is uniquely of form $d \mapsto c + d \cdot b$; here c must be 0 and b must be b_1, by uniqueness. Similarly $b = b_2$, so $b_1 = b_2$.

For simplicity, the following formulae are stated and proved for functions defined on the whole of A. (note $A' = A$).

<u>Proposition 2.1</u> Let $f_1, f_2 : A \longrightarrow A$. Then

$$(f_1 + f_2)' = f_1' + f_2'.$$

<u>Proof.</u> Let $a \in A$ be arbitrary. $(f_1+f_2)'(a)$ is then determined by

(2.4) $$(f_1+f_2)(a+d) = (f_1+f_2)(a) + d \cdot (f_1+f_2)'(a) \qquad \forall d \in D$$

But

$$\begin{aligned}(f_1+f_2)(a+d) &= f_1(a+d) + f_2(a+d) \\ &= f_1(a) + d \cdot f_1'(a) + f_2(a) + d \cdot f_2'(a)\end{aligned}$$

(by using Taylor on f_1 and f_2)

(2.5) $$= (f_1+f_2)(a) + d \cdot (f_1'(a) + f_2'(a))$$

for all $d \in D$. Subtract the right hand sides in (2.4) and (2.5) and cancel the universally quantified d.

<u>Proposition 2.2</u> (Leibniz rule). Let $f_1, f_2 : A \longrightarrow A$. Then

$$(f_1 \cdot f_2)' = f_1' \cdot f_2 + f_1 \cdot f_2'.$$

<u>Proof</u> Let $a \in A$ be arbitrary. $(f_1 \cdot f_2)'(a)$ is then determined by

(2.6) $(f_1 \cdot f_2)(a+d) = (f_1 \cdot f_2)(a) + d \cdot (f_1 \cdot f_2)'(a) \quad \forall d \in D.$

But

$$(f_1 \cdot f_2)(a+d) = f_1(a+d) \cdot f_2(a+d)$$
$$= (f_1(a) + d \cdot f_1'(a)) \cdot (f_2(a) + d \cdot f_2'(a))$$

(by using Taylor on f_1 and f_2).

(2.7) $$= f_1(a) \cdot f_2(a) + d \cdot (f_1'(a) \cdot f_2(a) + f_1(a) \cdot f_2'(a)) + d^2 \cdot f_1'(a) \cdot f_2'(a).$$

The last term cancels because $d^2 = 0$. Then subtract the right hand sides in (2.6) and (2.7) and cancel the universaly quantified d.

Proposition 2.3. (Chain rule). Let $f, g : A \to A$. Then

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Proof Let $a \in A$ be arbitrary. $(f \circ g)'(a)$ is then determined by

(2.8) $(f \circ g)(a+d) = (f \circ g)(a) + d \cdot (f \circ g)'(a) \quad \forall d \in D.$

But

$$(f \circ g)(a+d) = f(g(a+d))$$
$$= f(g(a) + d \cdot g'(a))$$

by applying Taylor to g. But now $d \cdot g'(a) \in D$, so that we can next apply Taylor to f to get

(2.9) $$= f(g(a)) + d \cdot g'(a) \cdot f'(g(a))$$

Subtract the right hand sides in (2.8) and (2.9) and cancel the universally quantified d.

Proposition 2.4 Let $i : A \to A$ denote multiplication by some fixed $b \in A$. Then $i'(a) = b \quad \forall a \in A$.

Proof $i'(a)$ is determined by

$$i(a+d) = i(a) + d \cdot i'(a) \qquad \forall d \in D.$$

So

$$b \cdot (a+d) = b \cdot a + d \cdot i'(a) \qquad \forall d \in D$$

but also

$$b(a+d) = b \cdot a + d \cdot b \qquad \forall d \in D$$

Subtract and cancel the universally quantified d.

Corollary The derivative of the identity map is 1. in any $a \in A$.

3. Tangent bundle.

Let M be any set. A map $A \to M$ is a (parametrized) curve in M (since A is the line). A map $t : D \to M$ may be thought of as a very short piece of a parametrized curve; so short that it is straight – in other words, it is a tangent-vector to M. It is attached at the point $t(0) \in M$.

Let M^D denote the set of all such maps $D \to M$. This set comes equipped with a map $\pi : M^D \to M$ given by $t \longmapsto t(0)$. We call this data the *tangent bundle* of M, since it consists of the totality M^D of all tangent vectors to M, together with the information π, "the base point map." The subset of M^D consisting of $t \in M^D$ with $\pi(t) = m$ is called the *tangent space of M at m* and denoted $(M^D)_m$; so

$$(M^D)_m = \pi^{-1}(m)$$

$$= \{ t : D \to M \mid t(0) = m \}.$$

We ask for conditions on M that will imply that each of its tangent spaces is a "vector space" (meaning here: an A-module).

We say M is *infinitesimally linear* ([3], [6]), if for any $m \in M$ and any $t, s \in (M^D)_m$, there exists a unique $\ell : D(2) \to M$ whose restrictions to the two axes equal t and s, respectively

(3.1) $$\ell \circ i_1 = t \quad \wedge \quad \ell \circ i_2 = s ;$$

(We also require a similar condition for $D(3)$: given $t_1, t_2, t_3 \in (M^D)_m$, there is a unique $k : D(3) \to M$ with $k \circ i_j = t_j$ $(j = 1, 2, 3)$ where the $i_1 : D \to D(3)$ is given by $d \mapsto (d, 0, 0)$, and similarly for i_2, i_3)

If M is infinitesimally linear, and $t, s \in (M^D)_m$, we define

$$t + s \in (M^D)_m$$

by putting $t + s = \ell \circ \Delta$ where ℓ is as in (3.1); or,

$$(t+s)(d) = \ell(d,d) \qquad \forall d \in D.$$

with ℓ as in (3.1). Clearly $(t+s)(0) = m$.

Also, if $t \in (M^D)_m$ and $a \in A$, define $a \cdot t$ by

$$(a \cdot t)(d) = t(a \cdot d) \qquad \forall d \in D$$

(using (1.6)).

<u>Proposition 3.1</u> With this structure, $(M^D)_m$ becomes an A-module.

<u>Proof</u> We first prove $+$ associative. Given three tangent vectors t_1, t_2, t_3 at $m \in M$. By the condition involving $D(3)$, we get $k : D(3) \to M$ restricting to t_j on the j'th axis of $D(3)$.

To form the $\ell : D(2) \to M$ restricting to t_1 and t_2 on the two axes of $D(2)$ we may (and must) take the composite

$$D(2) \xrightarrow{i_{12}} D(3) \xrightarrow{k} M$$
$$(d_1, d_2) \longmapsto (d_1, d_2, 0).$$

Thus $t_1 + t_2 = \ell \circ \Delta$, i.e.

$$(t_1 + t_2)(d) = k(d, d, 0) \qquad \forall d \in D.$$

Then clearly the function $D(2) \to M$ given by

$$(d, d') \longmapsto k(d, d, d')$$

restricts to $t_1 + t_2$, and t_3, respectively, on the two axes of $D(2)$, so that

$$((t_1 + t_2) + t_3)(d) = k(d, d, d) \qquad \forall d \in D.$$

Similarly, we find the same expression for $(t_1 + (t_2 + t_3))(d)$. This proves associativity.

To prove commutativity, note that if $\ell : D(2) \to M$ restricts to t and s, then $\ell \circ \tau$ restricts to s and t (where $\tau : D(2) \to D(2)$ is given by $(d_1, d_2) \mapsto (d_2, d_1)$). Thus

$$(t+s)(d) = l(d,d) = (l \circ \tau)(d,d) = (s+t)(d) \quad \forall d \in D.$$

To prove $(a+b)\cdot t = a\cdot t + b\cdot t$ for $a, b \in A$, $t \in (M^D)_m$, consider the $l : D(2) \longrightarrow M$ given by

$$l(d_1, d_2) = t(a\cdot d_1 + b\cdot d_2), \qquad \forall [d_1, d_2] \in D(2)$$

(note that the right hand side is defined, because $a\cdot d_1 + b\cdot d_2 \in D$ because of (1.6) and (1.7)). This l restricts to $a\cdot t$ and $b\cdot t$, respectively, on the axes of $D(2)$, so that

$$(a\cdot t + b\cdot t)(d) = l(d,d) = t(a\cdot d + b\cdot d) = t((a+b)\cdot d)$$
$$= ((a+b)\cdot t)(d) \qquad \forall d \in D,$$

which proves the desired distributivity. To prove $a\cdot(t+s) = a\cdot t + a\cdot s$, consider the $l : D(2) \longrightarrow M$ given by

$$l(d_1, d_2) = \bar{l}(a d_1, a\cdot d_2)$$

where $\bar{l} : D(2) \longrightarrow M$ is the unique map restricting to t and s. (the right hand side being defined, in virtue of (1.9)). Then

$$(a\cdot t + a\cdot s)(d) = l(d,d) = \bar{l}(a\cdot d, a\cdot d)$$
$$= (\bar{l} \circ \Delta)(a\cdot d) = (t+s)(a\cdot d) = (a\cdot(t+s))(d)$$

for all $d \in D$, proving the other distributivity law. The associativity law for multiplication by scalars is obvious: $(a\cdot b)\cdot t = a\cdot(b\cdot t)$. This proves the Proposition.

Note that the 'zero' element in $(M^D)_m$ is the map $D \longrightarrow M$ given by $d \longmapsto m$ $\forall d \in D$, and that the additive inverse $-t$ of t is given by

$$(-t)(d) = t(-d) \qquad \forall d \in D.$$

<u>Remark</u> The reader may use similar technique to prove that if $f : M \longrightarrow N$ is a map between infinitesimally linear objects, and t and $s \in (M^D)_m$, then $f \circ (t+s) = (f \circ t) + (f \circ s)$.

It is not true that A being of line type implies that A is infinitesimally linear (or conversely). We shall usually assume both these things for A. However, just assuming A of line type yields that its tangent bundle has a fibrewise A-module structure. For, the line type axiom may be expressed by saying that the map α:

$$A \times A \xrightarrow{\alpha} A^D$$
$$(c, b) \longmapsto [d \mapsto c + b\cdot d]$$

is invertible. The diagram

$$\begin{tikzcd} A \times A \arrow[rr, "\alpha"] \arrow[dr, "proj_1"'] & & A^D \arrow[dl, "\pi"] \\ & A & \end{tikzcd}$$

commutes, so that α establishes, for fixed $c \in A$, a bijection between

$$\{(c, b) \mid b \in A\} \quad \text{and} \quad (A^D)_c .$$

But the former is in bijective correspondence with A, thus carries an A-module structure, which we then can transport to an A-module structure on $(A^D)_c$, by means of α. The addition for this A-module structure on $(A^D)_c$ may be described

$$(3.2) \qquad (d \longmapsto c + b_1\cdot d) + (d \longmapsto c + b_2\cdot d)$$
$$= (d \longmapsto c + (b_1+b_2)\cdot d).$$

If now further A is infinitesimally linear, we may add the left hand sides in (3.2) according to the general recipe for infinitesimally linear objects; we have to find the (unique) $\ell : D(2) \longrightarrow A$ restricting to

$$d \longmapsto c + b_1 \cdot d \quad \text{and} \quad d \longmapsto c + b_2 \cdot d,$$

respectively. But we may (and must) take ℓ to be given by

$$(d_1, d_2) \longmapsto c + b_1 \cdot d_1 + b_2 \cdot d_2 ,$$

and the sum is then computed as $\ell \circ \Delta$, that is

$$d \longmapsto c + b_1 \cdot d + b_2 \cdot d ,$$

which agrees with the right hand side in (3.2).

If V is an A-module, we shall say that it is <u>Euclidean</u> if the map α

$$V \times V \xrightarrow{\alpha} V^D$$

given by

$$(\underline{u}, \underline{v}) \longmapsto [d \mapsto \underline{u} + d \cdot \underline{v}]$$

is bijective. So A itself (as well as A^n for any n) is Euclidean. The argument above given for A immediately extends to give also

<u>Proposition 3.2</u> If V is a Euclidean module, which is also infinitesimally linear, the tangent space $(V^D)_{\underline{u}}$ for any $\underline{u} \in V$ carries two A-module structures, one by being identified with V via α, the other by infinitesimal linearity. These two structures agree.

For an infinitesimally linear M, $(M^D)_m$ is an A-module, but not necessarily Euclidean.

<u>Example 3.3</u> Let $c \in D$. Prove that the tangent space to D at c, $(D^D)_c$ can be identified with the set

$$\{ b \in A \mid 2 b \cdot c = 0 \} \tag{3.3}$$

(which is an A submodule of A (an ideal)).

The factor 2 in (3.3) is, for obvious geometric reasons, superfluous: the axiom "2 is invertible in A" is true in the intended interpretation (physical space") of our theory.

<u>Proposition 3.4</u> If M and N are infinitesimally linear, then so is $M \times N$, as well as, for any two maps $f, g : M \longrightarrow N$, the set

$$E(f, g) := \{ m \in M \mid f(m) = g(m) \}$$

<u>Proof</u> Given tangent vectors t and s at $(m,n) \in M \times N$, so $t : D \to M \times N$, and similarly s; $t(0) = s(0) = (m,n)$. Write $t(d) = (t_1(d), t_2(d))$ $\forall d$, with $t_1 : D \to M$ and $t_2 : D \longrightarrow N$; similarly $s = (s_1, s_2)$. By infinitesimal linearity of M, there exist $\mu : D(2) \longrightarrow M$ with

$$\mu \circ i_1 = t_1$$
$$\mu \circ i_2 = s_1$$

and by infinitesimal linearity of N, there exists $\nu : D(2) \to N$ with

$$\nu \circ i_1 = t_2$$
$$\nu \circ i_2 = s_2$$

Then the map $\ell : D(2) \longrightarrow M \times N$ given by

$$(d_1, d_2) \longmapsto (\mu(d_1, d_2), \nu(d_1, d_2))$$

restricts to $(t_1,t_2)=t$ and $(s_1,s_2)=s$ on the two axes of $D(2)$; this proves the required existence. The uniqueness proof for l is similar.

Now let t, and s be tangent vectors to $E(f,g)$. at $m\in E(f,g)$. Thus $t: D\to E(f,g)$, $s: D\to E(f,g)$. Now $E(f,g)\subseteq M$. Viewing t and s as maps $D\longrightarrow M$, they are both tangent vectors at $m\in M$, and by infinitesimal linearity of M, there exists a unique $l: D(2)\longrightarrow M$ restricting to t and s, respectively. We just have to prove that l factors through $E(f,g)\subseteq M$, so we have to prove

(3.4) $$f\circ l = g\circ l.$$

But both sides in this equations are maps $D(2)\to N$, and they have the same restriction to the first axis of $D(2)$, because

$$f\circ l\circ i_1 = f\circ t = g\circ t = g\circ l\circ i_1$$

the middle equality sign because $t: D\to M$ factors through $E(f,g)$. – Similarly for the second axis of $D(2)$. Thus, by the uniqueness assertion in the infinitesimal-linearity assumption on N, we conclude that the two sides in (3.4) are equal.

<u>Corollary 3.5</u> If A is infinitesimally linear, then so is D, $D(2)$, $D\times D$, as well as, for instance, $S^1 = \{(x,y)\in A\times A \mid x^2+y^2=1\}$.

4. <u>Vector fields and infinitesimal transformations</u>

A <u>vector field</u> on M is a rule X which to each $m\in M$ associates a tangent vector $X(m)$ at m. Thus X is a map
$$X: M\longrightarrow M^D$$
with the property $\pi\circ X = id_M$ ("a cross section of π"). Since $X(m)\in M^D$, it makes sense to write $X(m)(d)$ for any $d\in D$; $X(m)(0)=m$.

An <u>infinitesimal transformation</u> on M is a map
$$\xi: M\times D\longrightarrow M$$
such that $\xi(m,0)=m \quad \forall m\in M$.

There is a natural bijective correspondence between vector fields and infinitesimal transformations: given a vector field $X: M\longrightarrow M^D$, define $\xi: M\times D\longrightarrow M$, by
$$\xi(m,d) = X(m)(d) \qquad \forall m\in M\ \forall d\in D,$$
and similarly the other way round. Sometimes we identify X and ξ notationally.

The way to visualize a vector field on an M (say, the surface of the earth) is

(4.1)

The way to think about an infinitesimal transformation $\xi: M\times D\longrightarrow M$ is as a <u>flow</u> that flows for a short (infinitesimal) span of time:
$$\xi(m,d)\in M$$
is the point to where m has flown after the elapse of d "seconds" of time. The correspondence

between X and ξ is: the vector field X generates a flow ξ by prescribing at each $m \in M$ a direction (and speed) $X(m)$ in which m should start moving. Conversely, the flow ξ moves each point m in a definite direction $X(m)$.

(In differential geometry, the total flow of a vector field X or the corresponding infinitesimal transformation ξ is a map

$$\bar{\xi} : M \times A \longrightarrow M$$

prescribing a flow that flows for ever (for every $t \in A$, $\bar{\xi}(m,t)$ is given) and which restricts to ξ on $M \times D$. Furthermore, one asks that for all $t_1, t_2 \in A$

(4.2) $$\bar{\xi}(m, t_1 + t_2) = \bar{\xi}(\bar{\xi}(m, t_1), t_2).$$

Such $\bar{\xi}$ lack (still) in our theory.)

We now assume that M is infinitesimally linear.

Proposition 4.1 Let ξ be an infinitesimal transformation on M. Then for any $m \in M$ and any $(d_1, d_2) \in D(2)$, we have

(4.3) $$\xi(m, d_1 + d_2) = \xi(\xi(m, d_1), d_2).$$

Note that the left hand side is defined, since $d_1 + d_2 \in D$, by (1.7). Also note that (4.3) is "contained in" (4.2).

Proof Consider $l : D(2) \longrightarrow M$ defined by the expression on the right hand side in (4.3). It restricts to the map

$$d \longmapsto \xi(m, d)$$

on each of the axes of $D(2)$. But so does the map $D(2) \longrightarrow M$ defined by the left hand side of (4.3) which thus also, by the uniqueness assertion in infinitesimal linearity, equals l. This proves (4.3).

Corollary 4.2 $\xi(\xi(m, d), -d) = m \quad \forall d \in D$.

More generally, if $X : M \longrightarrow M^D$ is the vector field corresponding to ξ,

$$(a \cdot X(m) + b \cdot X(m))(d) = \xi(\xi(m, a \cdot d), b \cdot d)$$

for all $a, b \in A$, $d \in D$.

Corollary 4.3 For each $d \in D$, the map $m \longmapsto \xi(m, d)$ permutes the elements of M (= is bijective).

The set $Vect(M)$ of all vector fields on an infinitesimal linear M form an A-module in an evident way. If $X : M \longrightarrow M^D$ and $Y : M \longrightarrow M^D$ are two vector fields, then the map $M \longrightarrow M^D$ given by

$$m \longmapsto X(m) + Y(m)$$

is a vector field, because $X(m) + Y(m) \in (M^D)_m$ since $X(m)$ and $Y(m)$ do. Similarly for multiplication by scalars. Note that, if ξ is the infinitesimal transformation corresponding to X, then the transformation

(4.4) $$(m, d) \longmapsto \xi(m, -d)$$

corresponds to $-X$, but, on the other hand, by Corollary 4.2 (for fixed d), (4.4) is the inverse of the permutation $m \longmapsto \xi(m,d)$.

5. Lie bracket

For fixed $d \in D$, the permutation $m \mapsto \xi(m,d)$ is called an infinitesimal transformation belonging to the infinitesimal transformation ξ. Such fixed-d-infinitesimal transformations may be composed qua permutations of M, but a composite of such may fail to be a fixed-d-infinitesimal transformation.

We consider the commutator of two fixed-d infinitesimal transformations. Let ξ and η be infinitesimal transformations on M, and let d_1 and d_2 be two fixed elements in D.

We consider for any $m \in M$, the element in M

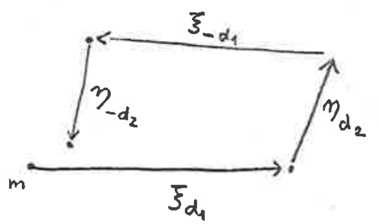
(5.1) $$\eta(\xi(\eta(\xi(m,d_1),d_2),-d_1),-d_2)$$

which by the remarks at the end of the preceding § is precisely $\underbrace{\text{the}}_{\text{action on } m \text{ of the}}$ group theoretic commutator

(5.2) $$\beta^{-1}\circ\alpha^{-1}\circ\beta\circ\alpha$$

with $\alpha = \xi(-,d_1)$, $\beta = \eta(-,d_2)$. The "process" described in (5.1) can be visualized as the "circuit"

(5.3)



writing ξ_{d_1} for $\xi(-,d_1)$, etc.

If one in (5.1) puts $d_2 = 0$, it follows from $\eta(n,0) = n$ $\forall n \in M$ and from Corollary 4.2 that (5.1) equals m. Similarly $d_1 = 0$ yields m. Thus, if either of the parameters d_1, d_2 vanishes, the circuit (5.3) becomes closed. We may express this: the deviation (between start- and end-point in (5.3)) vanishes with vanishing d_1 or d_2, or, heuristically "is proportional to both d_1 and d_2" – thus should be a function of their product $d_1 \cdot d_2$. This is a consequence of the following assumption on M, [6]:

Axiom W M has the property that for any function $t: D\times D \longrightarrow M$ with

$$t(d,0) = t(0,0) = t(0,d) \qquad \forall d \in D,$$

there is a unique $\tau: D \longrightarrow M$ with

$$t(d_1,d_2) = \tau(d_1\cdot d_2) \qquad \forall (d_1,d_2)\in D\times D.$$

If M satisfies this, we define a new infinitesimal transformation $[\xi,\eta]$ on M by putting

$$[\xi,\eta](m,d) = \tau(d)$$

where $\tau: D \longrightarrow M$ is that unique tangent vector for which for all $(d_1,d_2)\in D\times D$

$$\tau(d_1\cdot d_2) = \eta(\xi(\eta(\xi(m,d_1),d_2),-d_1),-d_2)$$

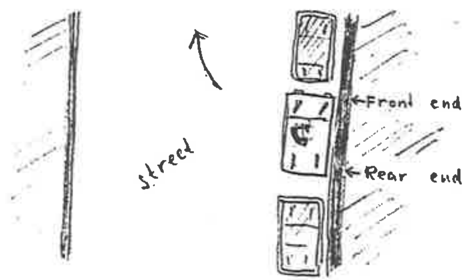
(= right hand side in (5.1)).

Of course, via the bijective correspondence between infinitesimal transformations ξ and vector fields X, we may also define a vector field $[X,Y]$ on M for any given vector fields X and Y on M.

It can be proved (see § below) that the A-module Vect(M) of vector fields on M in this way becomes a <u>Lie-algebra</u> over A.

Circuits (5.3), and the fact that they may be non-closed, is important in practice:

<u>Example</u> (Nelson [ ]). Consider a car. Let M be its configuration space. (It is a 4-dimensional manifold: 2 parameters describe the position of the center of the car on the parking lot, 1 parameter describes the direction of the main axis of the car, and 1 parameter describes the angle the front wheels form with the main axis). There are two distinguished infinitesimal transformations "drive" and "steer," because in any given position, the driver can perform these two functions. Now to get out of a narrow parking lot,



the driver C performs a circuit "turn steering wheel to the right; drive backward; turn steering wheel left; drive forward" (and maybe he has to repeat this circuit 3 or 4 times). After completing a circuit, the car is (fortunately) not in the same position (configuration) as it was before. So the Lie product [steer, -drive] gets the car out.

Of course, this was a qualitative consideration. The purpose of a mathematical theory (like the present) is to provide

a quantitative one. We shall later derive a quantitative consideration (the correct one) for another physical phenomenon: the Foucault pendulum.

6. <u>When one vector field admits another</u>

In this §, M and A are infinitesimally linear, and satisfy Axiom W; X and Y are vector fields on M with ξ and η the corresponding infinitesimal transformations. We say that two points m and n in M are <u>X-neighbours</u> if there exist $d \in D$ with

$$\xi(m,d) = n.$$

If such d's are unique whenever they exist, then we say that the vector field X is <u>proper</u>.

We say X <u>admits</u> Y (or η), if for each $d_1 \in D$, the permutation $\eta(d_1,-) : M \longrightarrow M$ preserves the X-neighbour-relation.

Heuristically, "integral curves for X are built by keeping going to X-neighbours," whence the heuristic of "admitting" is: "$\eta(d_1,-)$ permutes the integral curves for X (viewed as unparametrized subsets of M). In fact, if we call a set of form

$$\{\xi(m,d) \mid d \in D\}$$

for an <u>infinitesimal</u> <u>integral</u> <u>curve</u> <u>for</u> X (or ξ), then if X admits Y, $\eta(d_1,-)$ permutes the <u>infinitesimal</u> integral curves for X.