Also, we say that the circuit (5.3) is an X-<u>trapezium</u> if the end-point r is X-neighbour to the start-point m:

(6.1)
$$\begin{tikzcd} & q \arrow[d, "\eta_{-d_2}"] & & p \arrow[ll, "\xi_{-d_1}"'] \\ m & r \arrow[rr, dashed, "\xi_{d_1}"'] & & n \arrow[u, "\eta_{d_2}"'] \end{tikzcd}$$

<u>Proposition 6.1</u> Assume that X is a <u>proper</u> vector field (meaning each $X(m) : D \to M$ is injective). Then the following conditions are equivalent

(i) X admits the infinitesimal transformation η, i.e., each $\eta(d,-)$ preserves the X-neighbour-relation.

(ii) $[X,Y] = f \cdot X$ for some $f : M \to A$ (necessarily unique)

(iii) The circuits (5.3) formed from ξ and η are trapezia.

<u>Proof</u>. Assume (i). We refer to (6.1) for notation. Since p and q are X-neighbours, then so is n and r. Since X is a proper vector field, there exists a <u>unique</u> $\delta \in D$ with

$$\xi(n, \delta) = r;$$

δ depends on d_1 and d_2 (as well as on m, which we keep fixed throughout). So write $\delta = \delta(d_1, d_2)$. It is easy to see that the expression

(6.2)
$$\delta(d_1, d_2) + d_1$$

describes a function $D \times D \longrightarrow A$ with value 0 on the two axes, and hence by Axiom W for A, we have

(6.3)
$$\delta(d_1, d_2) + d_1 = \psi(d_1 \cdot d_2).$$

for some (unique) $\psi : D \to A$, with $\psi(0) = 0$. Since A is of line type

$$\psi(d) = b \cdot d$$

for some unique b, depending only on $m \in M$. Thus (6.3) reads:

$$\delta(d_1, d_2) + d_1 = b \cdot d_1 \cdot d_2 .$$

From this equation we gain the further information that $(\delta(d_1, d_2), d_1) \in D(2)$, so that $\xi(m, \delta(d_1, d_2) + d_2)$ is defined. We therefore can write (using Proposition 4.1)

$$r = \xi(n, \delta(d_1, d_2)) = \xi(\xi(m, d_1), \delta(d_1, d_2))$$

$$= \xi(m, d_1 + \delta(d_1 d_2))$$

$$= \xi(m, b \cdot d_1 \cdot d_2) = X(m)(b \cdot d_1 \cdot d_2),$$

but, on the other hand, by definition of $[X,Y]$

$$r = [X,Y](m)(d_1 \cdot d_2).$$

Thus by the uniqueness assertion in W.-Axiom for M,

$$X(m)(b \cdot d) = [X,Y](m)(d) \qquad \forall d \in D,$$

or

$$(b \cdot X)(m) = [X,Y](m)$$

where b depends on m. Writing $b = f(m)$, we obtain (ii).

Assume (ii). Then

$$X(m)(f(m) \cdot d_1 \cdot d_2) = [X,Y](m, d_1 \cdot d_2) = r$$

so that $f(m)\cdot d_1 \cdot d_2 \in D$ witnesses that m and r are X-neighbours.

Finally assume (iii). Let p and q be an arbitrary pair of X-neighbours, $q = \xi(p, -d_1)$, say. We have to prove that for arbitrary $d_2 \in D$, $\eta(-, -d_2)$ takes p and q into an X-neighbour pair. Using again the notation from diagram (6.1), we thus have to prove that r and n are X-neighbours. By assumption, m and r (which form beginning and end of a circuit) are X-neighbours,

$$\xi(m, d') = r$$

for some $d' \in D$, which is unique since X is proper. Thus we may write

$$d' = d'(p, d_1, d_2)$$

(note that in the present part of the proof, m varies with d_1 and d_2, whereas p is fixed). It is easy to see (using Corollary 4.2) that $d'(p, d_1, d_2)$ is zero if d_1 or d_2 is. Hence, by Axiom W,

$$d' = \varphi(p, d_1 \cdot d_2)$$

for some unique $\varphi(p, -) : D \longrightarrow D$ taking 0 to 0. Thus by line type axiom, $\varphi(p, d) = b(p) \cdot d \quad \forall d \in D$

Thus

(6.3) $$d' = d'(p, d_1, d_2) = b(p) \cdot d_1 \cdot d_2$$

Now

$$\xi(\xi(n, -d_1), d') = r$$

but from (6.3) follows that $(-d_1, d') \in D(2)$, so that our last equation can be rewritten (using Proposition 4.1):

$$\xi(n, -d_1 + d') = r \qquad (-d_1 + d' \in D);$$

proving that n and r are X-neighbours, as desired.

This proves the Proposition. The proposition is very close to Theorem 9 in Lie [5].

7. Directional derivative

If $X : M \longrightarrow M^D$ is a vector field on an object M and $f : M \longrightarrow V$ is a function with values in a Euclidean module (typically in the ring A itself), we may form the <u>directional derivative of f in direction</u> X, which is again a function $M \longrightarrow V$, denoted $X(f)$. It is defined as follows. For $m \in M$, $X(f)(m) \in V$ is defined as follows. Consider the map $D \longrightarrow V$ given by

$$d \longmapsto f(X(m)(d)).$$

Since V is Euclidean, this map is uniquely of form

$$d \longmapsto \underline{c} + d \cdot \underline{b}$$

with $\underline{c}$ and $\underline{b}$ in V. Clearly $\underline{c} = f(m)$; on the other hand, $\underline{b}$ contains new information, and we define

$$X(f)(m) := \underline{b}.$$

Thus

(7.1) $$f(X(m)(d)) = f(m) + d \cdot X(f)(m) \qquad \forall d \in D$$

in fact defines $X(f)(m)$. We call it the Taylor formula; it generalizes the Taylor formula (2.3), because differentiation $f \mapsto f'$ of functions $f : A \longrightarrow A$ appears as a special case, namely putting $M = V = A$, and taking X to be the vector field $\hat{+}$

$$A \xrightarrow{\hat{+}} A^D$$

given by

$$a \longmapsto [d \mapsto a+d].$$

Returning to the general case, and assuming Axiom W and infinitesimal linearity for M, we now prove:

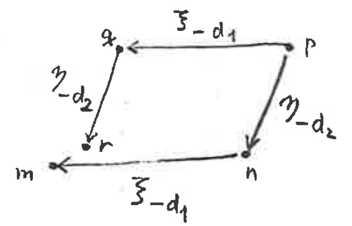
<u>Proposition 7.1</u> For vector fields X, Y on M, and functions $f: M \to V$, (V Euclidean), we have

$$X(Y(f)) - Y(X(f)) = [X,Y](f).$$

<u>Proof</u> Let, as usual,

$$\xi: M \times D \longrightarrow M \quad , \quad \eta: M \times D \longrightarrow M$$

denote the infinitesimal transformations corresponding to X and Y respectively. Consider the circuit, for fixed $(d_1, d_2) \in D \times D$,



(essentially the same as (6.1)). We consider $f(r) - f(m)$:

$$f(r) = f(q) - d_2 \cdot Y(f)(q)$$
$$= f(p) - d_1 \cdot X(f)(p) - d_2 \cdot Y(f)(q)$$

$$(7.2) \left\{ \begin{aligned} f(m) &= f(n) - d_1 \cdot X(f)(n) \\ &= f(p) - d_2 \cdot Y(f)(p) - d_1 \cdot X(f)(n) \end{aligned} \right.$$

using Taylor (7.1) for f four times; so

$$(7.3) \quad \begin{aligned} f(r) - f(m) &= d_1 \cdot \big(X(f)(n) - X(f)(p)\big) \\ &\quad + d_2 \cdot \big(Y(f)(p) - Y(f)(q)\big) \\ &= -d_1 \cdot d_2 \cdot Y(X(f))(p) + d_2 \cdot d_1 \cdot X(Y(f))(p) \end{aligned}$$

On the other hand, $[X,Y]$ was defined so that

$$[X,Y](m)(d_1 \cdot d_2) = r,$$

whence, by Taylor (7.1), applied now to the vector field $[X,Y]$,

$$(7.4) \quad f(r) - f(m) = d_1 \cdot d_2 \cdot [X,Y](f)(m).$$

Now, (7.2) holds for any function f, in particular also for $[X,Y](f)$, and it shows that the difference in values of a function taken in p and in m is of form $d_1 \cdot x + d_2 \cdot y$ and so is annihilated by $d_1 \cdot d_2$. Therefore, the right hand side of (7.4) equals $d_1 \cdot d_2 \cdot [X,Y](f)(p)$. Comparing this new form of (7.4) with (7.3) yields for all $d_1 \in D$, $d_2 \in D$, that

$$d_1 \cdot d_2 \cdot \big(X(Y(f))(p) - Y(X(f))(p)\big) = d_1 \cdot d_2 \cdot [X,Y](f)(p),$$

and cancelling the universally quantified d's one at a time, we get

$$X(Y(f))(p) - Y(X(f))(p) = [X,Y](f)(p),$$

which proves the Proposition.

The following Proposition is proved almost like Proposition 2.1 and Proposition 2.2, and we omit the proof.

Proposition 7.2. For any vector field X on M,

$$X(f_1+f_2) = X(f_1) + X(f_2),$$

and

$$X(\varphi \cdot g) = X(\varphi)\cdot g + \varphi \cdot X(g),$$

where $f_1, f_2, g : M \to V$ (V Euclidean), and where $\varphi : M \to A$.

Assuming that the Euclidean module V is infinitesimally linear, we can further prove

Proposition 7.3 For two vector fields X and Y on M, we have, for any $f : M \to V$,

$$(X+Y)(f) = X(f) + Y(f)$$

Proof Let $m \in M$ be given. Let $\ell : D(2) \to M$ be that map which restricts to $X(m)$ and $Y(m)$ on the axes of $D(2)$. Then $\forall d \in D$, by Taylor,

(7.5) $$d \cdot (X+Y)(f)(m) = \underbrace{f((X+Y)(m)(d))}_{-f(m)} = f(\ell(d,d)) - f(m).$$

On the other hand, by Taylor (twice),

(7.6) $$d \cdot (X(f)+Y(f))(m) = f(X(m)(d)) - f(m) + f(Y(m)(d)) - f(m)$$

We claim that these two expressions are equal, whence the result follows by cancelling the universally quantified d. To see this equality, consider the two functions $D(2) \to V$ given by, respectively,

$$f \circ \ell - f(m),$$

and

$$(d_1,d_2) \mapsto f(X(m)(d_1)) + f(Y(m)(d_2)) - f(m) - f(m)$$

They have the same restrictions to the axes of $D(2)$, namely $d \mapsto d \cdot X(f)(m)$ and $d \mapsto d \cdot Y(f)(m)$, and hence, by infinitesimal linearity of V, are equal. Hence they have equal restriction to $\Delta : D \to D(2)$, but these restrictions are precisely (7.5) and (7.6), respectively.

The following Proposition is essentially trivial, and its proof omitted.

Proposition 7.4 Let X be a vector field on M, $\varphi : M \to A$ and $f : M \to V$ arbitrary functions (V a Euclidean module). Then

$$(\varphi \cdot X)(f) = \varphi \cdot (X(f)).$$

8. Integrals of vector fields

A vector field X on a manifold M poses two kinds of integration problems. The one is to find flows for the vector field, i.e. maps $M \times A \longrightarrow M$ extending the given $\xi : M \times D \longrightarrow M$ (perhaps only defined on some subset B between D and A). (Analytically, this is a problem of ordinary differential equations.) Geometrically, the problem is to find parametrized curves on M whose speed vectors are the given field vectors. – The other integration problem is to find integrals: functions $f : M \longrightarrow A$ that are constant along the field vectors, – or the closely related problem of finding subsets $N \subseteq M$ having the property that the field vector $X(m)$ is tangent to N if $m \in N$. Typically, the flows of X, but now viewed as unparametrized subsets of M are such N's. (Analytically, this is a problem of partial differential equations.)

Here is a description of this notion of 'integral of a vector field'.

Proposition 8.1 Given a vector field X on M and a function $f : M \longrightarrow A$. Then the following conditions are equivalent, and if they are satisfied, we say that f is an integral of X:

(i) $X(f) \equiv 0$ as a function $M \longrightarrow A$

(ii) f is "constant along the infinitesimal flows of X" meaning that for each $m \in M$, the map

$$D \xrightarrow{X(m)} M \xrightarrow{f} A$$

is constant.

(iii) if m and n are X-neighbours (§6), then

$$f(m) = f(n)$$

(iv) for each "level set" $f^{-1}(a)$ $(a \in A)$ and each $d \in D$, the set $f^{-1}(a)$ is stable under the fixed-d infinitesimal transformation ξ_d.

(v) if $t : D \longrightarrow M$ is a vector of the field X (i.e. $t = X(t(0))$), then t is a tangent vector of the set $f^{-1}(f(t(0)))$, i.e., the field vectors are tangent to the level sets of f.

Proof. Assume (i). To prove (ii), we have for any $d \in D$

$$f(X(m)(d)) = f(m) + d \cdot X(f)(m) = f(m),$$

the first equality by "Taylor" (7.1), the second by assumption. That (ii) implies (iii) is trivial. Assume (iii). To prove (iv), let $a \in A$ and $m \in f^{-1}(a)$. We must prove

$$\xi_d(m) \in f^{-1}(a)$$

or

$$f(\xi(m,d)) = a.$$

But m and $\xi(m,d)$ are X-neighbours and $f(m) = a$. Assume (iv). We must prove that t maps into $f^{-1}(f(m))$, where m denotes $t(0)$. But for any $d \in D$

$$t(d) = \xi_d(m)$$

since t is a vector of the field. Since $m \in f^{-1}(f(m))$, then so does $\xi_d(m)$, by assumption, so $f(t(d)) = f(m)$, or $t(d) \in f^{-1}(f(m))$. Finally, assume (v). We have for any $m \in M$ and $d \in D$ that

$$f(X(m)(d)) = f(\xi_d(m)) = f(m)$$

since $f^{-1}(f(m))$ is stable under ξ_d. But the left hand side of this equation is, by "Taylor" (7.1) equal to $f(m) + d \cdot X(f)(m)$ so that $d \cdot X(f)(m) = 0$. Since this holds for all d, we get $X(f)(m) = 0$. Since this holds for all m, $X(f) \equiv 0$. This proves the Proposition.

If N is a subset of M, we consider (for fixed $d \in D$) the set $\xi_d(N) = \{ m \in M \mid \exists n \in N \text{ with } \xi_d(n) = m\}$. Assume M is infinitesimally linear. Then ξ_{-d} is inverse of ξ_d, by Corollary 4.2, so that one gets

$$\xi_d(N) = (\xi_{-d})^{-1}(N) = \{m \in M \mid \xi_{-d}(m) \in N\}$$

Assume now $f: M \longrightarrow A$ (it works also for $f: M \longrightarrow V$ with V an arbitrary Euclidean A-module), and let us consider the case where N is a level set $f^{-1}(\underline{v})$:

$$\begin{aligned} \xi_d(f^{-1}(\underline{v})) &= \{m \in M \mid \xi_{-d}(m) \in f^{-1}(\underline{v})\} \\ &= \{m \in M \mid f(X(m)(-d)) = \underline{v}\} \\ &= \{m \in M \mid f(m) - d \cdot X(f)(m) = \underline{v}\}, \end{aligned}$$

the last equality by "Taylor" (7.1). Thus

$$\xi_d(f^{-1}(\underline{v})) = (f - d \cdot X(f))^{-1}(\underline{v}), \tag{8.1}$$

in words: "transforming level sets of f an amount d along X yields a level set of $f - d \cdot X(f)$."

To formulate the following Theorem (= Theorem 7 in Lie [5]), we need the following technical restriction on functions $f: M \longrightarrow A$, or $f: M \longrightarrow V$, V a Euclidean A-module. We say f is <u>regular</u> if the image $f(M) \subseteq V$ is <u>subeuclidean</u> [2], meaning $\forall \underline{v}, \underline{w} \in V$, $\forall d \in D$

$$\underline{v} \in f(M) \Rightarrow \underline{v} + d \cdot \underline{w} \in f(M)$$

As level sets we shall only count sets of form $f^{-1}(\underline{v})$ for $\underline{v} \in f(M) \subseteq V$. (There is no harm in counting the empty set $\emptyset$ as a level set also. But for reasons hinted at in the introduction, we do not want to commit ourselves to

$$\forall \underline{v} \in V : \underline{\text{either}}\ \underline{v} \in f(M)\ \underline{\text{or}}\ f^{-1}(\underline{v}) = \emptyset.$$

This is the reason for insisting that only sets $f^{-1}(\underline{v})$ for $\underline{v} \in f(M)$ are level sets. They are in particular "non-empty", or, as we must formulate positive things positively, <u>inhabited</u>.) We can now formulate Lie's Theorem 7. M is assumed infinitesimally linear.

<u>Proposition 8.2</u> Let $f: M \longrightarrow V$ be regular, and X a vector field on M. Then the following three conditions are equivalent:

(i) for each d, ξ_d permutes the level sets of f meaning: if $N \subseteq M$ is a level set, then so is $\xi_d(N)$;

(ii) for each d and each level set N of f, $\xi_d(N)$ is contained in some level set of f;

(iii) there exists a map $g: f(M) \longrightarrow V$ with
$$X(f) = g \circ f.$$

Clearly, g is unique if it exists, since $f: M \to f(M)$ is surjective.

Proof. Clearly (i) $\Rightarrow$ (ii). We prove (ii) $\Rightarrow$ (i). By assumption

(8.2) $\quad \xi_d(f^{-1}(\underline{v})) \subseteq f^{-1}(\underline{w}) \quad$ for some $\underline{w} \in f(M)$.

For this $\underline{w}$,

(8.3) $\quad \xi_{-d}(f^{-1}(\underline{w})) \subseteq f^{-1}(\underline{v}') \quad$ for some $\underline{v}' \in f(M)$.

Thus

$$f^{-1}(\underline{v}) = \xi_{-d}(\xi_d(f^{-1}(\underline{v}))) \subseteq \xi_{-d}(f^{-1}(\underline{w})) \subseteq f^{-1}(\underline{v}').$$

Since $f^{-1}(\underline{v})$ is inhabited ("non-empty"), we conclude $\underline{v} = \underline{v}'$. Then (8.3) says $\xi_{-d}(f^{-1}(\underline{w})) \subseteq f^{-1}(\underline{v})$, and applying ξ_d (and Corollary 4.2), we get the converse implication in (8.2).

To prove (i) $\Rightarrow$ (iii). For any $\underline{v} \in f(M)$ and any $d \in D$, we have by assumption

(8.4) $\quad \xi_d(f^{-1}(\underline{v})) = f^{-1}(\underline{w})$.

for some $\underline{w} \in f(M)$ (unique because the set (8.4) is inhabited). Write $\underline{w} = G(\underline{v}, d)$. For fixed $\underline{v}$, G is a map $D \longrightarrow f(M)$. Since $f(M) \subseteq V$, and V is Euclidean, this map is uniquely of form $d \mapsto \underline{r} + d \cdot \underline{s}$; $\underline{r}$ clearly must be $\underline{v}$. If we now record the dependence of $\underline{s}$ on $\underline{v}$, by writing $\underline{s} = g(\underline{v})$, we get

$$\underline{w} = \underline{v} + d \cdot g(\underline{v})$$

for that $\underline{w}$ which satisfies (8.4). Rewrite (8.4) as follows:

$$\xi_d(f^{-1}(\underline{v})) = f^{-1}(\underline{v} + d \cdot g(\underline{v})) \qquad \forall \underline{v} \in f(M) \ \forall d \in D.$$

or

(8.5) $\quad f^{-1}(\underline{v}) = \xi_d^{-1}(f^{-1}(\underline{v} + d \cdot g(\underline{v})))$.

Let $m \in M$ be arbitrary, let $\underline{v} = f(m)$. By (8.5), we have $m \in \xi_d^{-1}(f^{-1}(\underline{v} + d \cdot g(\underline{v})))$, or

(8.6) $\quad f(\xi_d(m)) = \underline{v} + d \cdot g(\underline{v})$

On the other hand, the left hand side here is (by (7.1)):

(8.7) $\quad f(\xi_d(m)) = f(X(m)(d)) = f(m) + d \cdot X(f)(m)$

Comparing (8.6) with (8.7) and using $f(m) = \underline{v}$, we get, cancelling the universally quantified d,

$$X(f)(m) = g(\underline{v}) = g(f(m)) \qquad \forall m \in M$$

which proves (iii).

Finally assume (iii). We have, for $\underline{v} \in f(M)$ and $d \in D$,

$$\xi_d(f^{-1}(\underline{v})) = (f - d \cdot X(f))^{-1}(\underline{v})$$
$$= (f - d \cdot (g \circ f))^{-1}(\underline{v}),$$

by (8.1) and assumption. Since $f(M)$ is subeuclidean, $\underline{v} + d \cdot g(\underline{v}) \in f(M)$. We claim that

$$(f - d \cdot (g \circ f))^{-1}(\underline{v}) \subseteq f^{-1}(\underline{v} + d \cdot g(\underline{v}))$$

which will prove (ii), since the right hand side here is a level set for f. So let $m \in$ left hand side, i.e.

$$f(m) - d\cdot g(f(m)) = \underline{v},$$

so

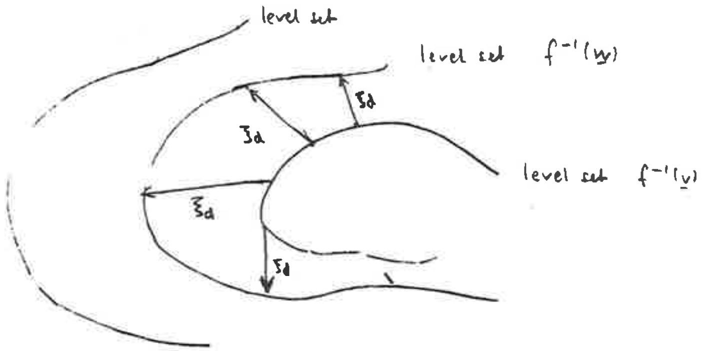
$$f(m) = \underline{v} + d\cdot g(f(m)).$$

Substituting this expression for $f(m)$ into the right hand side occurrence of $f(m)$, we get

$$f(m) = \underline{v} + d\cdot g(\underline{v} + d\cdot g(f(m)))$$
$$= \underline{v} + d\cdot[g(\underline{v}) + d\cdot g'(\underline{v})],$$

by Taylor's formula (g' denoting directional derivative of g in direction $g(f(m))$). Multiplying out and using $d^2 = 0$, we get $f(m) = \underline{v} + d\cdot g(\underline{v})$, or $m \in f^{-1}(\underline{v} + d\cdot g(\underline{v}))$, as claimed. The Proposition is proved.

The picture is (for $M = A\times A$)



9. Frobenius' Theorem

The spirit of the following theorem is like that of Proposition 6.1 in that it deals with relationship between the analytic operation of forming Lie bracket of two vector fields, and geometric properties of the "circuits" (5.3)

We continue to assume that M and A are infinitesimally linear and satisfy axiom W.

Proposition 9.1 Given two independent* vector fields X and Y on M. Then the following two conditions are equivalent:

(i) $[X,Y] = f\cdot X + g\cdot Y$ for suitable $f, g: M \to A$

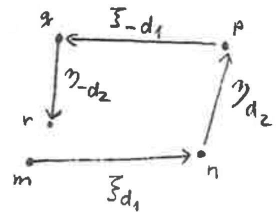
(ii) the circuits (5.3) are plane, meaning that

$$r = \eta(\xi(m,\delta_1),\delta_2)$$

for suitable $(\delta_1,\delta_2) \in D(2)$. (Here m and r denote beginning- and end-point of the circuit, respectively.)

Proof Assume (i), and consider the circuit (5.3):

(9.1)=(5.3)



Then

(9.2) $\quad r = [X,Y](m)(d_1\cdot d_2) = (f(m)\cdot X(m) + g(m)\cdot Y(m))(d_1\cdot d_2)$

We now shall prove

Lemma 9.2 Consider $\ell : D(2) \to M$ given by

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* "independent" means that for any $m\in M$, the map $D\times D \to M$ given by $(\delta_1,\delta_2) \mapsto \eta(\xi(m,\delta_1),\delta_2)$ is monic.

$$\ell(\delta_1,\delta_2) = \eta(\xi(m, f(m)\cdot\delta_1), g(m)\cdot\delta_2) \qquad \forall(\delta_1,\delta_2)\in D(2)$$

Then

$$(f\cdot X + g\cdot Y)(m)(d) = \ell(d,d) \qquad \forall d\in D.$$

Proof. The ℓ given is easily seen to restrict to $f(m)\cdot X(m)$ and $g(m)\cdot Y(m)$ on the axes of $D(2)$, whence the result by the definition of how one adds tangent vectors (§3).

By the Lemma, (9.2) now yields

$$\begin{aligned} r &= \ell(d_1\cdot d_2, d_1\cdot d_2) \\ &= \eta(\xi(m, f(m)\cdot d_1\cdot d_2), g(m)\cdot d_1\cdot d_2) \end{aligned}$$

This proves (ii), with $\delta_1 = f(m)\cdot d_1\cdot d_2$ and $\delta_2 = g(m)\cdot d_1\cdot d_2$.

Conversely, assume (ii),

$$\eta(\xi(m,\delta_1),\delta_2) = r \qquad \text{for some } (\delta_1,\delta_2)\in D(2).$$

They are unique, by independence of ξ,η. Thus, recording the dependence of δ_1, δ_2, and r on the parameters $(d_1,d_2)\in D\times D$, we may write

(9.3) $$\eta(\xi(m, \delta_1(d_1,d_2)), \delta_2(d_1,d_2)) = r(d_1,d_2).$$

Clearly

(9.4) $$\delta_1(d,0) = \delta_1(0,d) = 0$$

and similarly for δ_2 (since in these cases, $r = m$). So, by Axiom W for A, there exist functions $\beta_i : D \to A$ $(i=1,2)$ with

(9.5) $$\delta_i(d_1,d_2) = \beta_i(d_1\cdot d_2) \qquad \forall(d_1,d_2)\in D\times D$$

Again, by (9.4), $\beta_i(0) = 0$ $(i=1,2)$, so that, by Axiom 1 for A, there exist unique $b_1, b_2 \in A$ with

$$\beta_i(d) = b_i\cdot d \qquad \forall d\in D.$$

Thus, (9.5) reads

$$\delta_i(d_1,d_2) = b_i\cdot d_1\cdot d_2 \qquad \forall(d_1,d_2)\in D\times D$$

Till now, m was kept fixed. If we vary m, b_1 and b_2 will vary. Call the functions $M \to A$ thus defined f and g, respectively, i.e.

$$b_1 = f(m) \qquad b_2 = g(m),$$

so that (9.3) reads

$$\begin{aligned} r(d_1,d_2) &= \eta(\xi(m, f(m)\cdot d_1\cdot d_2), g(m)\cdot d_1\cdot d_2) \\ &= (f\cdot X + g\cdot Y)(m)(d_1\cdot d_2), \qquad \forall d_1,d_2\in D\times D \end{aligned}$$

the last equality by Lemma 9.2. On the other hand, by definition of Lie bracket,

$$r(d_1,d_2) = [X,Y](m)(d_1\cdot d_2) \qquad \forall d_1,d_2\in D\times D$$

Comparing these two last equations, we get

$$(f\cdot X + g\cdot Y)(m)(d_1\cdot d_2) = [X,Y](m)(d_1\cdot d_2) \qquad \forall d_1,d_2\in D\times D$$

and, using then the uniqueness assertion in Axiom W for M, we conclude from this the equality (i). This proves the Proposition.

10. Differential forms

If M is infinitesimally linear, a *differential* p-form ω on M is a law which to each $m \in M$ associates a map

$$\omega_m : \underbrace{(M^D)_m \times \dots \times (M^D)_m}_{p \text{ copies}} \longrightarrow A$$

which is p-linear (over A) and alternating.

If $f: M \longrightarrow A$ is a function, we get a 1-form df on M by considering

$$M^D \xrightarrow{f^D} A^D \xrightarrow{\gamma} A;$$

its restriction to $(M^D)_m$ is then A-linear (essentially by Remark on p. 12, and the evident A-linearity of γ $A^D \cong A \times A \xrightarrow{proj_2} A$).

Under suitable hypotheses on M, we shall construct exterior differentiation, a process leading from p-forms to $p+1$-forms, in a fairly geometric way. In order not to do to much multilinear algebra (which is as technical in our setting as in the classical), we consider only the case $p = 1$. A more serious restriction is that M must be "locally Euclidean" in the sense that each tangent space $(M^D)_m$ is a Euclidean A-module. Finally, our definition depends on existence of a "connection" on M: a way to transport one tangent vector along another one

(see Kock & Reyes: Connections in Formal Differential Geometry) - but I conjecture, and coordinate calculations confirm, that all affine connections on M yield the same exterior differentiation process. However, here we avoid these notions and problems by looking only at the special case $M = V$ where V is an infinitesimally linear Euclidean A-module.

A tangent vector $\tau : D \longrightarrow V$ at $\underline{m} \in V$ is then of form $d \longmapsto \underline{m} + d \cdot \underline{t}$ $\forall d \in D$, for some unique $\underline{t} \in V$, called the principal part of τ, denoted $\underline{t} = \gamma(\tau)$. (The γ above for A is a special case.)

To define a 2-form $d\omega$ on V when a 1-form ω on V is given: let $\underline{m} \in V$, and let τ and σ be tangent vectors at $\underline{m}$. We must define

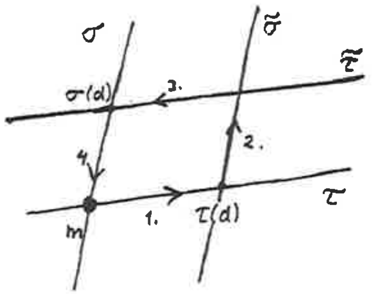
$$(d\omega)_m (\tau, \sigma) \quad \in A.$$

Let $\underline{t}$ and $\underline{s}$ be the principal parts of τ and σ, respectively. Form for given $d \in D$ the following "infinitesimal curve integral of ω":

$$(10.1) \qquad \omega_m(\tau) + \omega_{\tau(d)}(\tilde{\sigma}) - \omega_{\sigma(d)}(\tilde{\tau}) - \omega_m(\sigma)$$

where $\tilde{\sigma}$ is the tangent vector at $\tau(d)$ defined by $\tilde{\sigma}(\delta) = \tau(d) + \delta \cdot \underline{s}$, and similarly $\tilde{\tau}(\delta) = \sigma(d) + \delta \cdot \underline{t}$ ($\tilde{\sigma}$ is "σ transported d units paralel along τ" ; similarly, $\tilde{\tau}$ is "τ transported d units along σ." Here is the picture:

(10.2)



Note that (10.1) takes plus the terms corresponding to the tangents marked 1. and 2., and minus the terms corresponding to 3. and 4. This explains the arrowheads.

The element (10.1) in A depends on $d \in D$, so (10.1) represents a map $\varphi : D \longrightarrow A$. Its value on $0 \in D$ is clearly 0, so that, by Axiom 1, $\varphi(d) = d \cdot b$ for some unique $b \in A$. This element b is declared to be $(d\omega)_m(\tau, \sigma)$. Thus, the defining relation for $(d\omega)_m(\tau,\sigma)$ is

$\forall d \in D$: $d \cdot (d\omega)_m(\tau, \sigma) =$

(10.3)
$$\omega_m(\tau) + \omega_{\tau(d)}(\tilde{\sigma}) - \omega_{\sigma(d)}(\tilde{\tau}) - \omega_m(\sigma).$$

If we identify a tangent vector with its principal part, $\tau = \tilde{\tau} = \underline{t}$, and $\sigma = \tilde{\sigma} = \underline{s}$, ~~so that in this case~~ the defining relation (10.3) reads:

$\forall d \in D$:

(10.4)
$$d \cdot (d\omega)_m(\underline{t}, \underline{s}) = \omega_m(\underline{t}) + \omega_{m + d\cdot\underline{t}}(\underline{s}) - \omega_{m + d\cdot\underline{s}}(\underline{t}) - \omega_m(\underline{s})$$

In words, (and abuse of notation):

$$(d\omega)_m(\underline{t}, \underline{s}) = \frac{1}{d} \text{ (value of } \omega \text{ around boundary of } \underline{t}, \underline{s} \text{ - parallellogram of size } d \times d)$$

which is an infinitesimal form of (a special case of) Gauss-Green-Stokes Theorem.

The notion "differential df of a function $f : V \to A$" can be seen in the same light

$$df_{\underline{m}}(\underline{t}) = \frac{1}{d} \text{ (value of } f \text{ on the boundary of the } \underline{t}\text{-line of length } d)$$

which is an abuse-form of

(10.5)
$$d \cdot df_{\underline{m}}(\underline{t}) = f(\underline{m} + d \cdot \underline{t}) - f(\underline{m})$$

which is true because of the definition of df.

Since $\omega_m(-\underline{s}) = -\omega_m(\underline{s})$ for all $\underline{m}$ and $\underline{s}$, it is clear that (10.4) changes sign if we interchange $\underline{t}$ and $\underline{s}$, so that $d\omega_{\underline{m}}$ is alternating. To prove that it is bilinear: first note that by Proposition 3.2, the identification of tangent vectors at $\underline{m}$ with their principal parts preserve addition and multiplication by scalars from A. Also, the map $\varphi : V \longrightarrow A$ given by $\underline{u} \mapsto \omega_{\underline{m}+\underline{u}}(\underline{s}) - \omega_{\underline{m}}(\underline{s})$ has value 0 at $\underline{0}$. Compose it with the map $A \longrightarrow V$ given by $a \mapsto a \cdot \underline{t}$. For $d \in D$, Taylor developing the composite gives

$$\omega_{\underline{m} + d\cdot\underline{u}}(\underline{s}) - \omega_{\underline{m}}(\underline{s}) = d \cdot D_{\underline{u}}\varphi(u) = d \cdot d\varphi_{\underline{0}}(\underline{u}).$$

Here, $D_{\underline{u}}$ means "directional derivative in direction $\underline{u}$", and the last equality is well known (in our context, the proof is Prop. 3.1 in [2]). But $d\varphi_{\underline{0}}(\underline{y})$ depends linearly on $\underline{y}$. In particular, we conclude

$$\omega_{\underline{m}+d\cdot(\underline{t}_1+\underline{t}_2)}(\underline{s})$$

$$= \omega_{\underline{m}+d\cdot\underline{t}_1}(\underline{s}) + \omega_{\underline{m}+d\cdot\underline{t}_2}(\underline{s}) - \omega_{\underline{m}}(\underline{s}).$$

If we use this, and the linearity of $\omega_{\underline{m}}$ and of $\omega_{\underline{m}+d\cdot s}$, we immediately compute that

$$d\cdot(d\omega)_m(\underline{t}_1+\underline{t}_2, \underline{s}) = d\cdot(d\omega_{\underline{m}}(\underline{t}_1, \underline{s}) + d\omega_m(\underline{t}_2, \underline{s}))$$

for all $d \in D$. Cancelling the universally quantified d, we get additivity of $d\omega_{\underline{m}}(-,-)$ in the first variable. The other verifications are similar.

We can immediately verify $d(df) = 0$. For, substituting (10.5) for ω in (10.4) we get a sum with 8 terms, two for each corner in the parallellogram (10.2), and at each corner, the two terms cancel.

Using the remark p. 12 that "any map $M \to N$ between infinitesimally linear objects induces a fibrewise A-linear map f^D between the tangent bundles $M^D \to N^D$, we can to each p-form ω on N construct a p-form $f^*\omega$ on M in a canonical way. Also, given a p-form ω and a q-form θ, the definition of a p+q form $\omega \wedge \theta$ is as in standard multilinear algebra.

11. Unparametrized curves, and line elements.

The notion (parametrized) 'curve': $A \to M$, and its infinitesimal little brother 'tangent vector' $D \to M$, considered in §3, are dynamical notions; thus, a tangent vector not only indicates a direction of motion, but also its speed. (See picture 4.1).

Now we deal with the pure-geometric notion of unparametrized curve in M, which is just a subset of M, with certain properties, and its infinitesimal little brother: 'tangent-direction' or 'line element'. We begin with the latter.

Let us (in analogy with terminology of §6) call a tangent vector $t : D \to M$ proper if t is an injective map. A tangent-direction or line element at $m \in M$ is an equivalence class of proper tangent vectors at m under the equivalence relation $\equiv$ which may equivalently be described by

(i) $t \equiv s$ if there exists a bijective $\rho : D \to D$ with $\rho(0) = 0$ such that $t \circ \rho = s$

(ii) $t \equiv s$ if there exist an invertible $b \in A$ with $t(b\cdot d) = s(d)$ $\forall d \in D$.

(The equivalence of (i) and (ii) follows easily from the fundamental Axiom 1 in §2) It is clear that ρ (or b) are unique if they exist, due to the assumption that t (or s) is injective. Clearly, also, ρ is just 'multiplication by b'. So, t and s are <u>equivalent</u> if the one results from the other by a <u>reparametrization</u> ρ of D (with $\rho(0)=0$):

$$\begin{tikzcd} D \arrow[rr, "\rho", "\cong"'] \arrow[dr, "s"'] & & D \arrow[dl, "t"] \\ & M & \end{tikzcd}$$

Note that if $s \equiv t$, then the image $t(D) \subseteq M$ equals $s(D)$. (Under suitable hypotheses on M, it can even be proved that that subset uniquely determines the equivalence class.)

The point $t(0) = s(0)$ is called the <u>base point</u> of the line element $\{t\}$ ($\{\ldots\}$ denoting equivalence class under $\equiv$). A line element may be pictured as:

a point (the base point) with a short line through it. The set of all line elements on M is called the line element bundle of M, denoted $P(M)$. It comes equipped with a map

$$\pi : P(M) \longrightarrow M,$$

the map which to each line element $\{t\}$ associates its base point $t(0)$. (Another good, and common,

name, for $P(M)$ is "the projectivized tangent bundle" of M.)

If $f : M \longrightarrow N$ is an <u>injective</u> map, we can construct a map $P(f) : P(M) \longrightarrow P(N)$ by putting

$$P(f)(\{t\}) = \{f \circ t\}.$$

Note that if f is not assumed injective, we cannot assert that $f \circ t$ is injective, whence it may not define a line element at all. So P is only functorial for <u>injective</u> mappings.

If we have a law Φ which to each point of M associates a line element through it, we have what we may call a <u>line element field</u>, or <u>direction field</u> on M. Example: Let M = set of points in the plane $A \times A$ away from $\underline{0}$; to each $(x,y) \in M$ we associate the direction represented by the tangent vector $D \longrightarrow M$ given by

$$d \longmapsto (x - dy,\ y + d \cdot x),$$

i.e. the direction <u>perpendicular</u> to the vector (x,y) itself:

The integration problem posed by this direction field is "find curves having everywhere tangent directions belonging to the direction field," and the solution to this differential equation is: the family of circles with $(0,0)$ as center.

The direction field, and the solution, has in this case a lot of symmetries: rotations, similitudes. Symmetries (with respect to transformations, or infinitesimal transformations (= vector fields, cf. §4) of direction fields is something we shall exploit theoretically later, in analogy with § 6).

We now want to describe the notion of <u>unparametrized</u> <u>curve</u> S in an object M (S being a subset of M). This can be done provided M is a <u>manifold</u> in the sense of [3], but we shall temporarily only deal with the case $M = A^n$. In analogy with the $D(2) \subseteq A \times A$ described in § 1, we define

$$D(n) \subseteq A^n$$

as the set

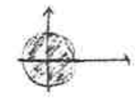
(11.2) $$D(n) = \{(x_1, \dots, x_n) \mid x_i \cdot x_j = 0 \quad \forall i,j = 1, \dots, n\}$$

If $\underline{v} \in A^n$, we call the set

$$\{\underline{v} + \underline{w} \mid \underline{w} \in D(n)\}$$

the <u>1-monad</u> <u>around</u> $\underline{v}$. Visualize it as an infinitesimal n-dimensional ball around $\underline{v}$, the parallel translate of $D(n)$ to $\underline{v}$. Does $D(n)$ look like a ball? On p. 4, we drew a shape

to visualize $D(2)$, but



is not bad either; for, it is stable under linear maps (in particular rotations):

<u>Exercise</u> Let $F: A^n \longrightarrow A^n$ be an A-linear map. Prove that $F(D(n)) \subseteq D(n)$. (Hint: F is given by an $n \times n$ matrix).

Now let $M = A^n$. A subset $S \subseteq A^n$ is said to be <u>locally</u> a <u>curve</u> <u>around</u> $s \in S$ if there is a pull-back diagram of form

(11.3)
$$\begin{array}{ccc} S & \hookrightarrow & M \\ \uparrow & \nearrow & \uparrow \\ D & \longrightarrow & m_1(s) \end{array}$$

where $m_1(s)$ is the 1-monad around s. S is called a <u>curve</u> if it is locally a curve around each of its points.

In more set-theoretic terms: S is locally a curve around $s \in S$ if $S \cap m_1(s)$ is $\cong D$

(via an isomorphism $D \longrightarrow S \cap m_1(s)$ taking 0 to s.) Such isomorphisms are not unique (alternatively, pull-back diagrams are only determined up to isomorphism) but the equivalence class of the composite

$$(11.4) \qquad D \cong S \cap m_1(s) \hookrightarrow M$$

(or equivalently, of the dotted arrow in (11.3)) under the equivalence relation $\equiv$ is well defined. So the equivalence class of (11.4) does define a line element in $s \in S$, called the tangent direction of S at s. A curve S thus has a tangent direction in each of its points.

12. Contact tangents to $P(M)$.

Some tangent vectors $t: D \longrightarrow P(M)$ are better than others. The good ones are those where the tangent vector $\pi \circ t : D \longrightarrow M$ belongs to the direction given by $t(0) \in P(M)$, i.e. if for some (and hence all) $s \in t(0)$,

$$\pi \circ t = b \cdot s$$

for some scalar $b \in A$. Note that this makes sense: $t(0)$ is an equivalence class of (proper) tangent vectors at $\pi(t(0)) \in M$; $\pi \circ t$ is a

tangent vector at $\pi(t(0))$, so it makes sense to ask whether the latter is a scalar multiple of the former (recall that $b \cdot s$ is given by $d \mapsto s(b \cdot d)$ - see § 3 (p. 10 bottom)).

Geometrically: $t: D \to P(M)$ is good if the direction in which the base points move by t (t being a small parametrized family of line elements, each one having a base point) agrees with the direction of the 0'th line element of the family. Thus, the family of line elements

is not good, because $t(0)$ has direction / whereas the base points move almost horizontally.

(One can even construct, for M "two-dimensional", [3], a 1-form ω on $P(M)$ (in the sense of § 10) which to each $t: D \to P(M)$ measures how much t differs from being "good"; ω is called the contact 1-form.)

A bijective $\varphi : P(M) \longrightarrow P(M)$ which preserves the good (or the 'contact') tangents (meaning

$$\boxed{D \xrightarrow{t} P(M) \text{ good}} \Rightarrow \boxed{D \xrightarrow{t} P(M) \xrightarrow{\varphi} P(M) \text{ good}}\,)$$

is called a contact transformation on M (not on $P(M)$).

We give two examples of contact transformations.

1. Let $f : M \to M$ be bijective. As we have seen, it induces a map

$$P(f) : P(M) \to P(M).$$

We claim that it is a contact transformations (contact transformations of this kind are called point-transformations. To see that $P(f)$ preserves contact tangents (= good tangents), let $t : D \to P(M)$ be good. This means that for some (or any) representative s of $t(0) \in P(M)$,

$$s : D \rightarrowtail M,$$

we have

$$\pi \circ t = b \cdot s \qquad \text{for some scalar } b \in A.$$

Now $f \circ s$ is a representative of $P(f)(t(0))$. And we have

$$\pi \circ P(f) \circ t = f \circ \pi \circ t = f \circ (b \cdot s) = b \cdot (f \circ s),$$

proving that $\pi \circ P(f) \circ t$ "belongs to" the direction (line element) represented by $f \circ s$. Thus $P(f) \circ t$ is good.

2. The following is a more specific example of a contact transformation $P(M) \to P(M)$, but more typical than 1., since it is not induced by a map $M \to M$. It is furthermore geometric: it requires that we in the plane $A \times A$ can drop the perpendicular from any point onto any line. This requires a property of A which we have to pose as an axiom:

A is a formally-real ring, meaning: for any integer n, and any n-tuple $x_1, \ldots, x_n \in A$, the element $1 + \sum_{i=1}^{n} x_i^2$ is invertible (actually, we need it only for $n = 1$)

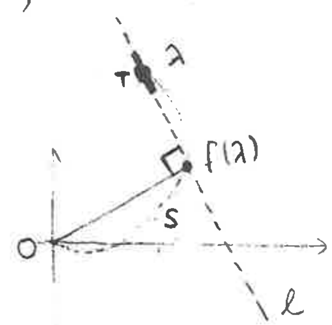
(Exercise The rings $\mathbb{Q}$ and $\mathbb{R}$ are formally real. The ring $\mathbb{C}$ of complex numbers is not formally real.)

We explain first the example in purely geometric terms. Let $M = A \times A$. Fix a point O in M; for simplicity, let $O = (0,0)$. Now declare a map

$$f : P(M) \longrightarrow M$$

by sending a line element λ to the point where the perpendicular from O to the line ℓ, determined by λ, hits ℓ:

(12.1)



From the theory of "angles in circles" ["synsvinkel-buer"] we know that $f(\lambda)$ must lie on the circle S having OT as diameter (T being the base point of λ) (see e.g. [7] Sætn. 77). We claim now that there is at most one contact transformation $\Phi : P(M) \to P(M)$ making the triangle

$$\begin{tikzcd} P(M) \arrow[r, dashed, "\Phi"] \arrow[dr, "f"'] & P(M) \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

commutative, and, if it exists at all, must send the line element λ (displayed in (12.1)) to the line element consisting of the point $f(\lambda)$ (displayed in (12.1)) and the direction given by the tangent to the circle S in $f(\lambda)$. For, let $t : D \to P(M)$ be a tangent vector to λ (i.e., $t(0) = \lambda$) which only varies the direction, but does not vary T:

(12.2)

Such a t is automatically good; for, $\pi \circ t$ being constant with value T belongs to any direction, in particular to $t(0) = \lambda$. If Φ exists and preserves good tangents, $\Phi \circ t$ is good, so that

$$f \circ t = \pi \circ \Phi \circ t \quad \text{belongs to} \quad \Phi(t(0)).$$

But since all line elements $t(d)$ $(d \in D)$ have base point T, the "synsvinkelbue"-argument above tells us that $f(t(d))$ belong to the circle S for all $d \in D$. Thus $f \circ t$ is a tangent to the circle S (at $f(\lambda)$). Thus this tangent must belong to $\Phi(t(0)) = \Phi(\lambda)$; therefore $\Phi(\lambda)$ is determined: it must be that circle tangent (at least provided we know that $f \circ t$ is proper.)

13. A^3 coordinatizes part of $P(A^2)$.

Some line elements in A^2 have representatives of form

(13.1) $$d \longmapsto (x + d,\ y + p \cdot d), \qquad d \in D$$

Namely, (13.1) represents the direction through (x, y) with slope p. (Heuristically, all line elements in the plane A^2 are of this form, except the vertical line elements.) The line element represented by (13.1) can be labelled by the triple $(x, y, p) \in A^3$. In this way, we represent the "set of non-vertical line elements" in A^2 by A^3; alternatively, we have constructed an injective map

(13.2) $$A^3 \overset{i}{\rightarrowtail} P(A^2)$$
$$(x, y, p) \longmapsto \big(\text{line element represented by } (13.1)\big).$$

When is a tangent at $P(A^2)$ of form

(13.3) $$D \overset{t}{\dashrightarrow} A^3 \xrightarrow{\ i\ } P(A^2)$$

a contact tangent? Let

(13.4) $$t(d) = (x + d \cdot \xi,\ y + d \cdot \eta,\ p + d \cdot \rho) \qquad \forall d \in D$$

Then (13.3) is a contact tangent if the tangent to A^2 given by

(13.5) $$d \mapsto (x + d \cdot \xi,\ y + d \cdot \eta) \qquad d \in D$$

belongs to the direction given by p, i.e. if the tangent (13.5) is a scalar multiple of (13.1). The scalar has to be ξ; thus, the condition is

$$d \cdot \eta = p \cdot d \cdot \xi \qquad \forall d \in D$$

or, cancelling the universally quantified d,

(13.6) $$\eta = p \cdot \xi,$$

which is thus a necessary and sufficient condition for (13.3) to be a contact tangent.

We have a 1-form (in the sense of § 10) on A^3, namely the one which sends a tangent vector given by (13.4) to $\eta - p \cdot \xi$ (standard notation: $dy - p \cdot dx$). This 1-form is called the contact 1-form, since its null-space consists precisely of the contact tangents.

Exercise. The map $\Phi : P(A^2) \longrightarrow P(A^2)$ considered on p. 56 can, in the coordinates given by (13.2), be described

$$(x,y,p) \longmapsto \left(-\frac{(y-p\cdot x)p}{1+p^2}, \frac{y-p\cdot x}{1+p^2}, \frac{xp^2-x-2yp}{yp^2-y+2xp} \right).$$

(it is called the pedal transformation with pole $(0,0)$).

A parametrized curve $A \xrightarrow{f=\langle f_1,f_2\rangle} A\times A$ in the plane gives, provided its speed vector $f'(a)$ $(a \in A)$ is always proper, rise to a curve in $P(A\times A)$, namely

(13.7) $$a \longmapsto (f(a), \text{direction given by } f'(a)) \qquad a \in A$$

Further, if $f'(a)$ is never vertical, we may represent (13.7) in the coordinate system (13.2), where it becomes

(13.8) $$a \longmapsto \left(f_1(a), f_2(a), \frac{f_2'(a)}{f_1'(a)}\right)$$

Then (13.8) (or more precisely, (13.8) composed with the i of (13.2)) is a contact curve (parametrized) in the sense that, for any fixed $a \in A$ the tangent t to $P(A^2)$ given by

$$d \longmapsto \left(f_1(a+d), f_2(a+d), \frac{f_2'(a+d)}{f_1'(a+d)}\right) \qquad d \in D$$

is a contact tangent. Now

$$f_1(a+d) = f_1(a) + d \cdot f_1'(a),$$

so if we use the notation of (13.4), $f_1(a) = x$, $f_1'(a) = \xi$, etc., $p = f_2'(a)/f_1'(a)$; and the condition (13.6) $\eta = p \cdot \xi$ becomes

$$f_2'(a) = \left(f_2'(a)/f_1'(a)\right) \cdot f_1'(a)$$

which evidently is satisfied. This proves the contact-condition.

It can also be formulated as follows. Let F denote the map (13.8), and let ω denote the contact 1-form. Then $F^*(\omega) = 0$. Note: if $f(x) = (x, g(x))$, then $F(x) = (x, g(x), g'(x))$

We would like not only to lift parametrized curves $f: A \to A^2$ into parametrized curves $F: A \longrightarrow P(A^2)$, but also to lift unparametrized curves $S \subseteq A^2$ into unparametrized curves $\tilde{S} \subseteq P(A^2)$, and without any assumption on "non-verticalness" of tangents. But to talk about unparametrized curves of $P(A^2)$, we need to define what its 1-monads are. So we insert a § with generalities:

14. C^1-manifolds, and 1-monads

An object N is said to be an *n-dimensional* C^1-*manifold around* $a \in N$, if there exists a subset $\mathfrak{M} \subseteq N$ with $a \in \mathfrak{M}$, and satisfying:

(i) any $D(k) \xrightarrow{\tau} N$ with $\tau(\underline{0}) = a$ factors through $\mathfrak{M}$

(ii) there exists a bijective $D(n) \xrightarrow{\cong} \mathfrak{M}$ taking $\underline{0}$ to a.

Clearly, such an $\mathfrak{M}$ is unique if it exists. We call it the *1-monad* around a. (Also the dimension number n is well defined, at least provided we assume that if $D(n) \cong D(n')$, then $n = n'$.)

We have k inclusions $\mathrm{incl}_j : D \longrightarrow D(k)$:

(14.1) $$\mathrm{incl}_j(d) = (0, 0, \ldots, \underset{\substack{\uparrow \\ j\text{'th position}}}{d}, 0, \ldots, 0)$$

Extending slightly the notion "infinitesimal linearity" of §3, we say that M is *infinitesimally linear* if for any $m \in M$ and any k, and for any k-tuple of maps $t_j : D \longrightarrow M$ with $t_j(0) = m$ $(j = 1, \ldots, k)$, there exists a unique $\tau : D(k) \longrightarrow M$ with

$$\tau \circ \mathrm{incl}_j = t_j \qquad j = 1, \ldots, k.$$

It is easy to see that if M_1 and M_2 are infinitesimally linear, then so is $M_1 \times M_2$. We shall assume that A is infinitesimally linear (hence A^n is also).

We proceed to show* that A^n is an n-dimensional C^1 manifold around each of its points. For simplicity, consider $\underline{0}$.

* The proof [illegible] also give [illegible] the monad notion here agrees with the one [illegible] § 11.

We prove that $D(n) \subseteq A^n$ will work as the monad around $\underline{0}$. Clearly (ii) holds. Let $\tau : D(k) \longrightarrow A^n$ have $\tau(\underline{0}) = \underline{0}$. Consider

$$D \xrightarrow{\mathrm{incl}_j} D(k) \xrightarrow{\tau} A^n \xrightarrow{\mathrm{proj}_i} A$$

It takes 0 to 0, and thus by axiom 1 is of form

$$d \longmapsto a_{ij} \cdot d$$

for some unique $a_{ij} \in A$. Consider the linear map $A^k \longrightarrow A^n$ with matrix the $n \times k$ matrix $\{a_{ij}\}$, and let τ' be the restriction of this map to $D(k) \subseteq A^k$. Clearly

$$\tau' \circ \mathrm{incl}_j = \tau \circ \mathrm{incl}_j \qquad j = 1, \ldots, k$$

whence by the uniqueness assertion in "infinitesimal linearity", for A^n, $\tau = \tau'$. Thus for $(d_1, \ldots, d_k) \in D(k)$

$$\tau(d_1, \ldots, d_k) = \Big(\sum_{j=1}^{k} a_{ij} d_j\Big)_{i=1,\ldots,n}$$

which belongs to $D(n)$ (multiply out!). This proves (i).

We can in a similar spirit prove:

Proposition 14.1. If M is an p dimensional C^1-manifold around $m \in M$, and N is a q-dimensional C^1-manifold around $n \in N$, then $M \times N$ is a $(p+q)$-dimensional manifold around (m, n).

Proof. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the monads, and let there be given isomorphisms

$$D(p) \xrightarrow[\cong]{\varphi} \mathfrak{M}, \qquad D(q) \xrightarrow[\cong]{\psi} \mathfrak{N}$$

with $\varphi(\underline{0}) = m$, $\psi(\underline{0}) = n$. We claim that the composite

$$D(p+q) \subseteq D(p)\times D(q) \xrightarrow[\cong]{\varphi\times\psi} \mathcal{M}\times\mathcal{N} \subseteq M\times N$$

will be the monad $\underbrace{\text{(Or "have the monad}}_{\text{around } (m,n)}$ as image."). Suppose $D(k) \xrightarrow{\tau=(\tau_1,\tau_2)} M\times N$ maps $\underline{0}$ to (m,n). Since φ and ψ are monads, $\tau_1 : D(k) \longrightarrow M$ factors across φ, and τ_2 across ψ. Thus we have a commutative

$$\begin{tikzcd} D(p)\times D(q) \arrow[r, tail, "\varphi\times\psi"] & M\times N \\ D(k) \arrow[u, "\nu"] \arrow[ur, "(\tau_1,\tau_2)"'] & \end{tikzcd}$$

The arrow ν takes $\underline{0}$ to $\underline{0}$, and since $D(p)\times D(q) \subseteq A^{p+q}$, where we know that the monad around $\underline{0}$ is $D(p+q)$, we conclude that ν factors through $D(p+q)$. This proves the Proposition.

Proposition 14.2 If N is an n-dimensional C^1-manifold around $a\in N$, then the tangent space $(N^D)_a$ carries a natural structure of A-module, which is in fact isomorphic (but not canonically) to A^n. If further N is an n-dimensional C^1-manifold around each of its points, then it is infinitesimally linear, and the A-module structure on $(N^D)_a$ deriving from infinitesimal linearity (§ 3) agrees with the one derived from the C^1-property.

Proof. Choose an isomorphism $\varphi : D(n) \xrightarrow[\cong]{} \mathcal{M}_1(a) \subseteq N$ taking $\underline{0}$ to a. Since $D(n)$ is infinitesimally linear by Corollary 3.5, $(D(n)^D)_{\underline{0}}$ carries an A-module structure. We have an invertible map

$$(D(n)^D)_{\underline{0}} \xrightarrow[\cong]{\varphi^D} (\mathcal{M}_1(a)^D)_a$$

which we use to transport the A-module structure on the left hand object onto the right hand object. But note that by property (i') for 1-monad:

$$(\mathcal{M}_1(a)^D)_a = (N^D)_a .$$

We have to verify that the A-module structure thus constructed on $(N^D)_a$ is independent of the choice of isomorphism $\varphi : D(n) \longrightarrow \mathcal{M}_1(a)$. If $\varphi' : D(n) \longrightarrow \mathcal{M}_1(a)$ is another isomorphism, let $\varphi^{-1}\circ\varphi' = \theta$ be their composite,

$$\theta : D(n) \xrightarrow{\cong} D(n) \qquad \theta(\underline{0}) = \underline{0}$$

It then suffices to see that the map which θ^D induces

$$(D(n)^D)_{\underline{0}} \longrightarrow (D(n)^D)_{\underline{0}}$$

is A-linear. This follows from the functoriality of the fibrewise A-module structure on infinitesimally linear objects (Remark p.12).

To prove that $(N^D)_a \cong A^n$ as an A-module (but not canonically), is achieved when we prove $(D(n)^D)_{\underline{0}} \cong A^n$, as an A-module. This latter isomorphism is canonical, in fact

$$(D(n)^D)_{\underline{0}} = ((A^n)^D)_{\underline{0}} \cong A^n ,$$

the latter isomorphism being given by taking principal port"; but that isomorphism *is* A-linear, by Proposition 3.2.

Finally, let N be a C^1-manifold around each of its points. To prove N infinitesimally linear, let t_1 and t_2 be tangent vectors at the same point $a \in N$. Let $\varphi : D(n) \xrightarrow{\cong} \mathcal{M}_1(a)$ be as above. By property (i) of 1-monads, both t_1 and t_2 factor through φ, and since $D(n)$ is infinitesimally linear (Corollary 3.5), we get a map $D(2) \longrightarrow D(n)$ and thus a map l: $D(2) \longrightarrow \mathcal{M}_1(a) \subseteq N$ restricting to t_1 and t_2 on the axes. Uniqueness of such map: if we have two maps $D(2) \longrightarrow N$ restricting to t_1 and t_2 on the axes, they both factor through $\mathcal{M}_1(a)$ and thus through φ, using again property (i). Since the two resulting maps $D(2) \to D(n)$ agree on the two axes of $D(2)$, they are equal, by infinitesimal linearity of $D(n)$. Thus the two given maps $D(2) \longrightarrow N$ are also equal.

Finally, the fact that the fibrewise linear structure on N^D derived from infinitesimal linearity agrees with the one carried over by

$$(D(n)^D)_0 \xrightarrow{(\varphi^D)_0} (N^D)_a$$

follows from $(\varphi^D)_0$ being linear for both kind of structures on $(N^D)_a$.

This proves the Proposition.

15. Some technicalities concerning *proper* vectors

Our whole approach prevents us from calling a vector *proper* just when it is "not zero." (Cf. the introduction).

We already called a *tangent* vector $t : D \longrightarrow M$ *proper* provided it is injective as a map $D \to M$.

On the other hand, we shall call a vector $\underline{v} \in V$ (V a "vector space", meaning an A-module) *proper* if the map $A \longrightarrow V$ defined by $a \longmapsto a \cdot \underline{v}$ is injective.

A tangent vector $t : D \longrightarrow M$ is (provided M is infinitesimally linear) a vector, namely in $(M^D)_m$ (where $m = t(0)$). Our two notions of 'proper' have to be compared.

Proposition 15.1 Let $t : D \longrightarrow M$ be a tangent vector at $m \in M$. If t is a proper tangent vector, then it is proper as a vector in $(M^D)_m$. The converse holds provided M is a C^1-manifold (of arbitrary dimension).

Proof Let $t : D \longrightarrow M$ be a proper tangent. To prove $t \in (M^D)_m$ to be a proper vector in the vector space $(M^D)_m$ means to prove that: if $b \in A$ has $b \cdot t = 0$, then $b = 0$. Now $b \cdot t$ is the tangent vector $D \longrightarrow M$ given by $d \longmapsto t(b \cdot d)$, and to say that it is the *zero* tangent vector means that it has value constant m. So

$$t(b \cdot d) = m \quad \forall d \in D$$

Since $t : D \to M$ is injective, this implies

$$b \cdot d = 0 \quad \forall d \in D,$$

but "cancelling the universally quantified d" gives $b = 0$.