This is a retyping of Chapter VIII in the Section "Comments on Parts I-1 and I-2" of "Charles Ehresmann oevres complètes et commentées", ed. Andrée Charles Ehresmann, Amiens 1984. We have attempted to follow the typography of the original as closely as possible. Updatings (2017) are indicated by [[...]]. A list of the specific articles by C. Ehresmann quoted in the chapter is added at the end, and are indicated in accordance with the complete bibliography of the "oevres complètes" by /.../.

EHRESMANN AND THE FUNDAMENTAL STRUCTURES OF DIFFERENTIAL GEOMETRY SEEN FROM A SYNTHETIC VIEWPOINT

by Anders KOCK

1. Jets as the synthetic foundation of differential geometry.

On the eve of the 1950's, C. Ehresmann made the discovery of the notion of *jet* as the fundamental element of differential geometry. Many of his papers from this decade deal with and exploit this foundation, which soon became a standard tool for differential geometers.

His work may be seen as an effort to understand differential geometry by concepts which synthesize rigour and intuition; one might say, by synthetic concepts. For instance, a *tangent vector* of a manifold M is, from such synthetic viewpoint, a 1-jet of a map $\mathbf{R} \to M$ to which it gives rise.

The advent in the mid 70's of synthetic differential geometry, in the specific sense (of [4], or [7], say, inspired by a lecture of Lawvere in 1967), which we shall here for short denote SDG, provides a setting in which jet-theoretic constructions become more transparent. This is not surprising since the basic assumption of SDG is that

the notion of jet should be representable

("Kock-Lawvere Axiom"); this means that an r-jet at $x \in M$ is not an equivalence class of maps but *is* a map defined on a certain infinitesimal subset ("r-monad") $\mathfrak{M}_r(x)$ around x. This assumption will hold if one enlarges the category of C^∞-manifolds to the category of "C^∞-schemes" [2], or certain toposes related hereto [1, 5, 7, 16]. For instance, the notion of 1-jet at $0 \in \mathbf{R}$ is represented by a scheme D whose only point is 0, but whose structure sheaf is the ring $\mathbf{R}[\epsilon]$, of dual numbers; a tangent vector at $x \in M$ then is a map $D \to M$ with $0 \mapsto x$.

The role of $\mathbf{R}[\epsilon]$ for representing the notion of tangent vector goes back to A. Weil's [20] 1953; he states that his theory of "Points proches" has a double source: Fermat, and "la théorie des jets développée dans ces dernieres années par C. Ehresmann".

Thus SDG is a natiral continuation and completion of Ehresmann's foundational work of the 50's, and it is not surprising that a revisiting of this work with ideas and notions of SDG leads to clarification and simplification.

One needs, however, one step beyond Weil's "local algebras" or the corresponding "infinitesimal manifolds" of SDG, to be able to make this revisiting, namely to represent the notion of r-jet at a point $x \in M$, where M is something more general than a coordinate space. To describe the necessary notion of r-*monad* around $x \in M$, one needs a relation of "r-neighbours", $x \sim_r y$ between the elements of M, or, equivalently [1] a subset $M_{(r)} \subseteq M \times M$, "the r'th neighbourhood of the diagonal". Then the r-monad $\mathfrak{M}_r(x)$ is the set of r-neighbours of x and represents r-jets at x. The neighbourhoods $M_{(r)}$ of the diagonal have a prehistory in algebraic geometry (Grothendieck et al), and analogous schemes $M_{(r)}$ were utilized later by Malgrange [15] and Kumpera-Spencer [13] for representing jet bundles in differential geometry. An axiomatic-synthetic approach to a neighbour relation for representing the jet notion was initiated in [6]; such an approach is necessary for the SDG formulations below.

The relationship between the scheme theoretic and the axiomatic approach to "first neighbourhood of the diagonal" has been studied in [3].

In what follows we shall attempt a synopsis of part of C. Ehresmann's work on jet theory and some subsequent developments, by paraphrasing it into the "neighbour"-language of SDG. Some such paraphrasing was carried out in [6], notably concerning Lie algebroids of Lie groupoids, G-structures, Lie equations, and prolongation of differentiable categories. This will not be repeated here.

To make sense to the paraphrasing, one needs only to know about the r-neighbour relation that it is reflexive symmetric, not transitive (but rather:

$$x \sim_r y \ , \ y \sim_s z \ \Rightarrow \ x \sim_{r+s} z \ )$$

and that it is preserved by any map.

If $f : \mathfrak{M}_r(x) \to N$ is an r-jet at x, with values in N, and $f(x) = y$, one calls /33/ x the *source* and y the *target*, of the jet f. Since f preserves the relation $\sim_r$, it follows immediately that f factors through

$$\mathfrak{M}_r(y) \subset N,$$

[1] Only from the viewpoint of SDG is it true that the two ideas mentined here are equivalent. From the scheme theoretic viewpoint, only the latter makes sense since M and M_r have the same points and only differ by their structure sheaves, so that one cannot talk about a *pair* of neighbour points. We have followed the tradition of SDG in omitting the word "differentiable" everywhere, and writing "set" instead of "manifold" or "scheme". This use, or abuse, of language stems from the fact that SDG embeds the category of manifolds into a topos (a well adapted model for SDG) about which we talk as if it were the category of sets. This nay be justified in terms of categoical logic (see e.g. [7], Ch. II), which in turn is an offspring of the idea of "functorial semmantics" [14], or of "realizing the sketch in a category" (Part IV of these works)).

and so can be composed with any r-jet at y. Thus r-jets form a category. The category of thise r-jets whose source and target belong to the same manifold M is denoted C_rM. This is the prime type of example of what C. Ehresmann calls a *differentiable category* in /50/ (or category internal to the category of manifolds). Thus important structures in differential geometry *are* categories. We discuss these in §2 below.

The "non-holonomous" jets introduced in /41/ play a role in the context of formal integrability considerations, which we shall discuss at the end of §2. The notion itself can be paraphrased as follows. Let $\sim$ denote the 1-neighbour relation. A *non-holonomous* r-jet at $x \in M$ (with values in N, say) is a law which to each "r-chain" in M

$$x \sim x_1 \sim x_2 \sim \ldots \sim x_r$$

associates an element of N. The set of such r-chains may be called the *non-holonomous r-monad* $\widetilde{\mathfrak{M}}_r(x)$ around x. If

$$f(x, x_1, \ldots, x_r)$$

depends only on x_r, f may be identified with an ordinary ("holonomous") r-jet at x, since the map

$$\partial_1 : \widetilde{\mathfrak{M}}_r(x) \to \mathfrak{M}_r(x)$$

associating to an r-chain its extremity x_r may be assumed (for nice N) to be surjective.

2. Categories in differential geometry.

This is the title of the last survey article /116/ by C. Ehresmann, who in categories recognized a decisive motive force in the foundations of differential geometry. Among the roots of this recognition are Lie's theory of continuous transformation groups, and the Lie-Klein Erlangen program. Lie etc. considered in this context structures more general than those we now call Lie groups - namely "pseudogroups" (possibly infinite-dimensional) and with multiplication only locally defined.

To understand differential geometry in the spirit of the Erlangen program means thus more than understanding the formal notions of *group* and *group actions*. The necessary comprehensive notion for this were brought to the day by E. Cartan and C. Ehresmann, in particular with the realization that pseudogroups are categories (in fact groupoids).

With the concept of (small) category in the center of the study of the foundations of differential geometry, C. Ehresmann was led to two deepenings of this concept in two directions:

1. ordered categories,
2. differentiable categories.

The deepening 1 comes about when refelcting on the word "local", and an example is the groupoid of all local diffeomorphisms of a manifold, with the notion of "restriction" providing the ordering. Ordered categories are commented on in these Oeuvres, II. - The deepening 2 comes about when reflecting on jets, and realizing that jets form categories which themselves are manifolds. We shall, in terms of SDG, comment in more detail on 2, in particular on aspects of the theory of connections.

Just as Lie groups G are considered in conjunction with their actions (i.e., as groups of transformations of a manifold), differentiable categories Φ, and in particular differentiable groupoids are considered in conjunction with their actions on maifolds E fibered over M ($M =$ the manifold of objects of Φ), /42/. This, in the language of present day category theory, is the notion of discrete fibration, or presheaf on Φ. In SDG (in fact in any topos), one may, equivalently, express this structure in terms of a functor over M, $\Phi \to \mathrm{Full}(E)$, where $\mathrm{Full}(E)$ is the category over M with maps from one E-fibre to another as arrows.

In case Φ is the groupoid $\Pi_r M$ of invertible r-jets with source and target in M, an action of it on some $E \to M$ is called an r'th order *prolongation* of M, and a cross section σ of $E \to M$ is an rth order *infinitesimal structure* on M /32/-/38/. Any tensor bundle over M is a 1st order prolongation. An example of a 2nd order structure on M is an affine connection on M. The relevant 2nd order prolongation here is an example of a bundle of connection elements, see below.

Any r'th order infinitesimal structure σ on M gives rise to a subgroupoid $\mathrm{Fix}(\sigma)$ of $\Pi_r M$, consisting of those r-jets that preserve σ. One of the benefits which C. Ehresmann derived from the systematic consideration of such differentiable categories was the "right" comprehensive notion of connection, namely connection with values in a category (usually a groupoid) /46/. The intuitive idea behind any connection notion is that it is a law for infinitesimal transport, and that this transport preserves some given infinitesimal structure σ. Thus it should be "something" that takes values in the groupoid $\mathrm{Fix}(\sigma)$.

The combinatorial aspects (the neighbour relation), which SDG introduces, allows us to describe such notion in a simple way [6]. Let Φ be a groupoid (or category) with M as its set of objects. A (k'th order) *connection element X at $x \in M$ with values in* Φ is a law which, to each y with $y \sim_k x$ associates an arrow $X(y) : x \to y$ *of* Φ (one requires $X(x)$ to be the identity arrow at x). These connection elements form a bundle $Q^k(M, \Phi)$ over M (which is in fact a k'th order prolongation of M), and a cross section ot this bundle is a Φ-valued k'th order *connection* on M.

One could even replace $\sim$ by an arbitrary (oriented) *graph* Γ over M; by this we understand a set Γ equippped with "face" maps

$$\partial_0\ ,\ \partial_1\ : \Gamma \to M$$

and a "degeneracy" map $i : M \to \Gamma$ splitting ∂_0 and ∂_1. A category Φ over M is evidently a graph, by forgetting the composition of arrows; and also a refelxive relation Φ is a graph. Thus, we may describe the notion of Φ-valued connection relative to Γ : it is a map $\Gamma \to \Phi$ of graphs over M. The groupoid valued connections were studied from this viewpoint in [10] (which is close to /28/ in content), and in [11], which takes departure in /41, 43/ : the relevant graph Γ to describe r'th order non-holonomous connections in combinatorial terms is the graph $\Gamma = \tilde{M}(r)$ of r-chains

$$\underline{x} = (x_0 \sim x_1 \sim \ldots \sim x_r) \quad \text{with}$$

$$\partial_0(\underline{x}) = x_0\ ,\ \partial_1(\underline{x}) = x_r, \ \text{ and } \ i(\underline{x}) = (x \sim x \sim \ldots \sim x).$$

If for instance ∇ is a Φ-valued 1st order connection, we get a 2nd order non-holonomous connection $\nabla_*\nabla$ by

$$\nabla_*\nabla(x_0, x_1, x_2) := \nabla(x_1, x_2) \circ \nabla(x_0, x_1),$$

cf. /46/, and [19, 12, 11]. (Then ∇ is curvature free iff $\nabla_*\nabla$ is holonomous, cf below.)

An alternative connection notion was described by Joyal (unpublished, but see [9]), who considered for an arbitrary "bundle" $E \to M$ an action of the graph $\sim$ on E (in analogy with an action of a categor over M on E). It becomes a special case of the groupoid-valued connection notion, by taking $\Phi = \text{Full}(E)$. Conversely, a Φ-valued connection for a general groupoid Φ over M can be seen as a "bundle"-connection in the bundle $\partial_1 : \Phi \to M$ with a certain equivariance property. In fact, both notions and the proof of their equivalence can be paraphrased from /28/ Section 4.

While groupoid-valued higher order connections always give rise to "absolute differentiation" /46/, essentially only 1st order connections give rise to(global) parallel transport. The relationship between the integration problem raised by the latter, and the notion of curvature of the connection, was stated in /46/, and in [19]. SDG allows us to paraphrase the notion of curvature and its role in integrability questions in an "unforgettable" way :

Any category Ψ over M gives rise to a graph $\Psi_{(1)}$ over M, consisting of the "near-identities" of Ψ, i.e. those arrows f with $f \sim \partial_0(f)$. In attempting to extend a graph map

$$\nabla : \Psi_{(1)} \to \Phi$$

(where Φ is a category over M) into a functor $\Psi \to \Phi$, which is what the integration problem amounts to, it is necessary that ∇ preserves those composites which happen to exist in $\Psi_{(1)}$. To say that ∇ is *curvature free* is to say that it does preserve such composites. The *Representation Theorem for Lie groupoids* [17, 18] gives conditions on Φ and Ψ in order that the curvature is the only obstruction to extending ∇ to a functor.

The integrability theorem for curvature free connections is the special case when

$$\Psi = M \times M, \quad \text{so} \quad \Psi_{(1)} = M_{(1)},$$

the 1-neigbour relation. A special case of that, again, is the integration theorem of [8], giving conditions on M when Lie group valued 1-forms on M are exact (take $\Phi = M \times G \times M$).

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