GENERALIZED FIBRE BUNDLES

Anders Kock

Math. Inst.
Aarhus University
Ny Munkegade
8000 Aarhus C, Denmark

As the title suggests, this is a rather formal paper: it generalizes some notions just in order to have a perfect symmetry of concepts. Thus we symmetrize the set up of our paper "Fibre bundles in general categories" |8|, and arrive at what may be called "Haefliger structures" in general categories; such structures occur in foliation theory, cf. |4| and |10|. We intend the present paper to be readable without knowledge of |6|,|7|, or |8|.

We shall work in a category $\underline{E}$ with finite inverse limits, and we shall talk about the objects of $\underline{E}$ as if they were sets. If $q: C \to B$ is a stable regular epimorphism, we may represent 'elements' of B by 'elements' of C, provided the outcome is independent of choice of representatives.

We shall in particular consider such stable regular epimorphisms which are descent maps, in a sense we shall recapitulate, and which is a crucial concept in any fibre bundle theory, allowing one to glue objects together out of compatible local data.

I acknowledge fruitful discussions with J. Pradines and J. Duskin; some of the formalisms of the present paper are closely related to ideas occuring, in some form, in their work, |9|, and |1|, §5. Also the debt to Ehresmann's groupoid theoretic foundations of differential geometry is obvious. Finally, I would like to thank S. Eilenberg for pushing me into adopting a notation that makes calculations obvious (rather than the one of |6|,|7|,|8|). The paper is in final form.

§1. Double equivalence relations, and pregroupoids.

A double equivalence relation is a special case of a double category, in the same way as an equivalence relation is a special case of a category. This in fact defines the notion, but let us make it more explicit:

A double equivalence relation on an object $X \in \underline{E}$ is a subobject $\Lambda \subset X^4$ (a 4-ary relation), such that

1) the two binary relations H and V on X given by

From "Categorical Algebra and its Applications" Proceedings, Louvain-la-Neuve 1987
Springer Lecture Notes in Math. 1348

$$H(x,y) \text{ iff } \Lambda(x,y,x,y)$$

and

$$V(x,z) \text{ iff } \Lambda(x,x,z,z),$$

respectively, are equivalence relations on X,

2) the binary relation $\sim_h$ on X^2 given by

$$(x,y) \sim_h (z,u) \text{ iff } \Lambda(x,y,z,u)$$

is an equivalence relation on H, and the binary relation $\sim_v$ on X^2 given by

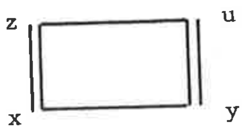
$$(x,z) \sim_v (y,u) \text{ iff } \Lambda(x,y,z,u)$$

is an equivalence relation on V.

(In particular, $\Lambda(x,y,z,u)$ implies $(x,y)\varepsilon H$, $(z,u)\varepsilon H$, $(x,z)\varepsilon V$, and $(y,u)\varepsilon V$.)

We depict the statement $\Lambda(x,y,z,u)$ by a diagram

(1.1)



We say that a double equivalence relation is a *pregroupoid*, or a *pregroupoid structure on* X, if for any x,y,z with $(x,y)\varepsilon H$ and $(x,z)\varepsilon V$, there exists a unique u such that $\Lambda(x,y,z,u)$. Then this u may be denoted $yx^{-1}z$.

This pregroupoid notion is more general than the one of |7| or |8|, since there we assume that $H = X\times X$.

Example 1.1. Let H be the set of arrows of a groupoid, with S as set of objects. Let $A\subset S$, $B\subset S$ be two subsets. We construct a pregroupoid structure on H by putting $\Lambda(x,y,z,u)$ whenever

$$d_0(x) = d_0(y) \quad \varepsilon \quad A$$

$$d_1(x) = d_1(z) \quad \varepsilon \quad B$$

and

$$u = y\circ x^{-1}\circ z$$

This example is a case of a slightly more structured pregroupoid notion: let X be given, together with maps $\alpha : X \to A$, $\beta : X \to B$, for some objects A and B. A pregroupoid structure on X with respect to $A \xleftarrow{\alpha} X \xrightarrow{\beta} B$ is just a pregroupoid structure on X such that H is the kernel pair of α and V is the kernel pair of β.

In this case, the pregroupoid structure Λ on X may be given in terms of the ternary operation $yx^{-1}z$, defined whenever $\beta(z) = \beta(x)$ and $\alpha(y) = \alpha(x)$. (This is the approach of |6| and |7|, where A = 1, and $yx^{-1}z$ is denoted $\lambda(x,y,z)$.)

§2. Descent; and the groupoids associated to a regular pregroupoid.

We assume some familiarity with internal category theory in $\underline{E}$, as in |5| 2.1. When we say that $\mathbf{H} = (H \rightrightarrows A)$ is a category object in $\underline{E}$, we mean that A is the 'set' of objects, H is the 'set' of arrows, and the two displayed maps are d_0 and d_1 (domain and codomain formation), respectively. We call $\mathbf{H}$ a groupoid if there is an inversion map $(\ )^{-1}: H \to H$ with the expected properties.

An internal diagram, or a left action for $\mathbf{H} = (H \rightrightarrows A)$ consists of data $\varphi : F \to A$, together with a left action by $\mathbf{H}$, so $h \cdot f$ makes sense whenever $\varphi(f) = d_0(h)$, and then $\varphi(h \cdot f) = d_1(h)$; an associative and unitary law is assumed. The information of such internal diagram may be encoded by a certain category object $\mathbf{F} = (F_1 \rightrightarrows F_0)$ (with $F_0 = F$), together with a functor $\mathbf{F} \to \mathbf{H}$ (with certain properties: a discrete opfibration). We call $\mathbf{F}$ the total category and $\mathbf{F} \to \mathbf{H}$ the discrete opfibration of the internal diagram, or of the left action. Note that the total category for an action by a groupoid is itself a groupoid.

The category of internal diagrams for $\mathbf{H}$ is denoted $\underline{E}^{\mathbf{H}}$. There is a dual notion of right actions for $\mathbf{H}$, they form a category $\underline{E}^{\mathbf{H}^{op}}$.

We call a left action by a groupoid free if the associated total category $\mathbf{F}$ is an equivalence relation, meaning that $F_1 \to F_0 \times F_0$ is mono (and symmetric - this is automatic here). And we call the action principal homogeneous if the associated total category $\mathbf{F}$ is codiscrete, meaning that $F_1 \to F_0 \times F_0$ is iso.

An internal functor $q : \mathbf{G} \to \mathbf{H}$ gives rise to a functor $q^* : \underline{E}^{\mathbf{H}} \to \underline{E}^{\mathbf{G}}$, essentially by pulling back along q. If $\mathbf{H} = (H \rightrightarrows B)$ is a category, and $q : C \to B$ is a map, we may form a category $\mathbf{G} = (G \rightrightarrows C)$ by pulling $H \to B \times B$ back along $q \times q : C \times C \to B \times B$. This is the "full image" of $\mathbf{H}$ along q; it comes with a functor back to $\mathbf{G}$. In particular, if $\mathbf{H} = B$ is the discrete category on B (meaning $\mathbf{H} = (B \rightrightarrows B)$), the full image of

$q: C \to B$ is just the kernel pair $\mathbb{R}_q = (R \rightrightarrows C)$ of q; it comes with a functor to B.

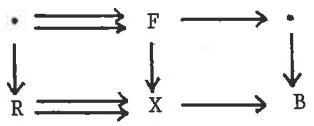
It is easy to see that if q is a stable regular epi, then the induced functor

$$\underline{E}/B \simeq \underline{E}^B \xrightarrow{q^*} E^{\mathbb{R}_q} \tag{2.1}$$

is full and faithful. We say that q is a descent map if q is stable regular epi, and if the functor (2.1) is an equivalence of categories. An equivalent, less sophisticated version of the notion is given in |8|. It is anyway quite standard (perhaps rather under the name effective descent map). Yet another simple, but non-elementary, way of describing the notion is in terms of the full and faithful left exact embedding $i: \underline{E} \to \tilde{\underline{E}}$, where $\tilde{\underline{E}}$ is the topos of sheaves for the canonical topology on $\underline{E}$; it takes stable regular epis to epis, and a stable regular epi q is a descent map if $i(q)$ has the property that pulling back along it reflects the property of being representable (= in the image of i).

Using this description, one may see that descent maps are stable under composition and pull-back, and that $q \circ p$ descent $\Rightarrow q$ descent.

By a descent situation in E we mean a diagram



in which the right hand square is a pull back, each of the two rows exact (kernel pair/coequalizer), and the left hand square is the discrete opfibration associated to an action of the category $R \rightrightarrows X$ on $F \to X$.

We say that an equivalence relation $\mathbb{R} = (R \rightrightarrows C)$ is a descent equivalence (-relation) if it has a coequalizer q, of which it is a kernel pair, and which is a descent map. In a topos, any equivalence is a descent equivalence, and any epi is a descent map.

Let X, Λ be a pregroupoid. We call it regular if the equivalence relations H and V on X, derived out of Λ as in §1, are descent equivalences. Thus there are descent maps

$$\alpha : X \to A, \quad \beta : X \to B,$$

such that $H = X \times_A X$, $V = X \times_B X$. The equivalence relation $\sim_h$ on $H = X \times_A X$ now comes about as total category of a left action of $X \times_B X \rightrightarrows X$ on $p_0: X \times_A X \to X$, namely the one given by

$$(x,z) \cdot (x,y) := (z, yx^{-1}z),$$

and since $X \to B$ was assumed a descent map, it follows that we have an object $X^* \to B$ in $\underline{E}/B$ participating in a descent situation

(2.2)
$$\begin{array}{ccccc} \sim_h & \rightrightarrows & X\times_A X & \xrightarrow{\beta'} & X^* \\ \downarrow & & p_0 \downarrow & & \downarrow d_0 \\ X\times_B X & \rightrightarrows & X & \xrightarrow[\beta]{} & B \end{array}$$

in particular

(2.3)
$$X\times_B X^* \;=\; X\times_A X.$$

We shall denote $\beta'(x,y)$ by yx^{-1}. Note that

$$yx^{-1} = uz^{-1} \quad \text{iff} \quad u = yx^{-1}z \, .$$

We have a groupoid structure $X^* \rightrightarrows B$ on X^* given by

$$d_0(yx^{-1}) := \beta(x) \qquad d_1(yx^{-1}) := \beta(y)$$

$$(tu^{-1})\circ(yx^{-1}) := (tu^{-1}y)x^{-1}$$

(here we use that $\beta': X\times_A X \to X^*$ is stable regular epi, which means that we may represent elements in X^* by elements $(x,y) \in X\times_A X$). The checking of independence of choice of representatives is easy from the definitions. We note that d_0 is the map $X^* \to B$ occurring in (2.2), and by the stability properties of descent maps, d_0 is a descent map.

The groupoid X^* acts on the left on $\beta : X \to B$, the action being given by

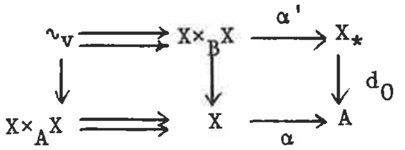
$$(yx^{-1}).z := yx^{-1}z.$$

For each $a \in A$, $X_a := \alpha^{-1}(a)$ is stable under the X^*-action, and the action of X^* on X_a is principal homogeneous: to any $x,y \in X_a$, there exists a unique $g \in X^*$ with $g \cdot x = y$, namely yx^{-1}.

By completely symmetric arguments, the equivalence $\sim_v$ on $V = X\times_B X$ comes about by a left action of $X\times_A X \to X$ on $p_0 : X\times_B X \to X$, namely the one given by

$$(x,y)\cdot(x,z) := (y, yx^{-1}z),$$

and since $X \to A$ was assumed a descent map, it follows that we have an object $X_* \to A$ in $\underline{E}/A$ participating in a descent situation



We denote $\alpha'(x,z)$ by $x^{-1}z$, and

$$x^{-1}z = y^{-1}u \quad \text{iff} \quad u = yx^{-1}z.$$

X_* carries the structure of groupoid $X_* \rightrightarrows A$, namely

$$d_0(x^{-1}z) := \alpha(z) \qquad d_1(x^{-1}z) := \alpha(x).$$

$$(x^{-1}z)\circ(u^{-1}t) := x^{-1}(zu^{-1}t).$$

As for the X^*-case, $d_0 : X_* \to A$ is a descent map.

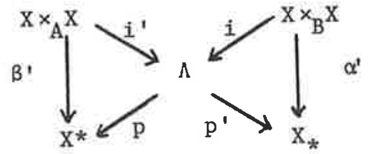
The groupoid X_* acts on the right on $\alpha : X \to A$, the action being given by

$$y\cdot(x^{-1}z) := yx^{-1}z.$$

For each $b \in B$, $X_b := \beta^{-1}(b)$ is stable under the X_*-action, and the action of X_* on X_b is principal homogeneous.

Finally, the X^*- and X_*-actions commute with each other.

<u>Note</u>. Part of the information contained in a regular pregroupoid and its associated groupoids may be recorded in the following "butterfly" diagram, considered by Pradines |9| (for reasons related to ours):



where $i(x,z) = (x,x,z,z)$, $p(x,y,z,u) = yx^{-1}$ etc., and where the diagonal sequences in a certain sense are exact sequences of groupoids (for a "diagonal" groupoid structure on Λ).

<u>Example.</u> Assume that the groupoid $G \rightrightarrows A$ acts in a free way on the right of some $\alpha : X \to A$. Then we can equip X with structure of pregroupoid, namely by letting $\Lambda \subset X^4$ consist of the 'set' of all $(x,y,x\cdot g,y\cdot g)$ where $\alpha(x) = \alpha(y) = d_1(g)$. If α is a descent map, and the equivalence relation on X given by the action of G is a descent equivalence, with quotient $\beta : X \to B$, say, the pregroupoid is a pregroupoid with respect to $A \leftarrow X \to B$, and is regular. By the

constructions above, we thus get groupoids $X_* \rightrightarrows A$ and $X^* \rightrightarrows B$ acting on X from the right and the left, respectively. It is easy to see that $X_* \rightrightarrows A$ is canonically isomorphic to $G \rightrightarrows A$ (in a way compatible with the action), the isomorphisms $X_* \to G$ and $G \to X_*$ being given by

$$x^{-1}(x \cdot g) \mapsto g$$

and

$$g \mapsto x^{-1}(x \cdot g) \quad \text{(for some/any } x \in X_{d_1(g)}\text{)},$$

respectively. (Note that $X_{d_1(g)}$ is inhabited since α is stable regular epi.)

On the other hand, the groupoid $X^* \rightrightarrows B$ provides a new way of encoding some of the information of the G-action on X. For the case where $A = 1$, this is essentially Ehresmann's construction |2| of a groupoid XX^{-1} out of a free action of a group on a space, i.e. out of a principal fibre bundle.

§3. Generalized fibre bundles.

Assume that $A \overset{\alpha}{\leftarrow} X \overset{\beta}{\rightarrow} B$ is a regular pregroupoid. To make $\varepsilon: E \to B$ into a <u>generalized fibre bundle for</u> X <u>with fibre</u> $\varphi: F \to A$ means, by definition, to give an invertible map

$$X \times_A F \xrightarrow{\sigma} X \times_B E, \tag{3.1}$$

commuting with the projections to X, and satisfying (3.2) below; to state (3.2), let us write $x(f)$ for the unique element in E satisfying

$$\sigma(x,f) = (x, x(f))$$

(where $x \in X$, $f \in F$, and $\alpha(x) = \varphi(f)$), and similarly, we write $x^{-1}(e)$ for the unique element in F satisfying

$$\sigma^{-1}(x,e) = (x, x^{-1}(e)),$$

(where $x \in X$, $e \in E$, and $\beta(x) = \varepsilon(e)$). We then have $\varepsilon(x(f)) = \beta(x)$ and $\varphi(x^{-1}(e)) = \alpha(x)$.

The condition we require on σ is, in this notation, expressed

$$(yx^{-1}z)(f) = y(x^{-1}(z(f))) \tag{3.2}$$

whenever it makes sense, i.e. whenever

$$\varphi(f) = \alpha(z), \quad \beta(x) = \beta(z), \quad \alpha(x) = \alpha(y);$$

or, equivalently

(3.2') $$(yx^{-1}z)^{-1}(e) = z^{-1}(x(y^{-1}(e)))$$

whenever it makes sense, i.e. whenever

$$\varepsilon(e) = \beta(y),\ \beta(x) = \beta(z),\ \alpha(x) = \alpha(y).$$

Note that the data σ of (3.1) may (in the category of sets, say) be thought of as providing for each $x \varepsilon X$ a bijection from the $\alpha(x)$-fibre of F to the $\beta(x)$-fibre of E, in a way which is compatible with the pregroupoid structure, by (3.2). The pregroupoid of bijective maps from fibres of F to fibres of E is a special case of Example 1.1, by taking $H \rightrightarrows S$ there to be the groupoid of all (bijections between) sets.

A <u>generalized fibre bundle for</u> X is a triple $\langle \varphi : F \to A, \varepsilon : E \to B, \sigma \rangle$, with σ as in (3.1) and satisfying (3.2). We denote it short $\langle F,E,\sigma \rangle$. They form a category Fib(X), a morphism $\langle F,E,\sigma\rangle \to \langle F',E',\sigma'\rangle$ being a pair of maps $F \to F'$, $E \to E'$, compatible with the φ's, ε's, and σ's in the evident way. There are evident forgetful functors $\mathrm{Fib}(X) \to \underline{E}/A$ and $\mathrm{Fib}(X) \to \underline{E}/B$.

If $\langle F,E,\sigma \rangle$ is a fibre bundle fo X, there is a natural left action of the groupoid $X_* \rightrightarrows A$ on $F \to A$, and a natural left action of the groupoid $X^* \rightrightarrows B$ on $E \to B$, decribed by

$$(x^{-1}z)\cdot f := x^{-1}(z(f))$$

where $\varphi(f) = \alpha(z)$, $\beta(x) = \beta(z)$, and

$$(yx^{-1})\cdot e := y(x^{-1}(e))$$

where $\varepsilon(e) = \beta(x)$, $\alpha(x) = \alpha(y)$. These descriptions are well defined, due to (3.2) and (3.2'), respectively, and provide in fact functors

(3.3) $$\mathrm{Fib}(X) \to \underline{E}^{X_*}\ ,\quad \mathrm{Fib}(X) \to \underline{E}^{X^*},$$

respectively.

When $A = 1$, one talks about <u>fibre bundles</u> instead of <u>generalized</u> fibre bundles. This is the situation considered in |8|, where we prove (Theorem 4.1) that the functors of (3.3) in this case are equivalences of categories. The proof does not depend on the assumption $A = 1$, so that we have

<u>Theorem 3.1.</u> The functors (3.3) are equivalences of categories.

In Remark 4.4 below, we shall indicate the inverse functors; their construction of course depends on constructing objects by descent.

§4. A general descent construction.

Let $\mathbf{G} = (G \rightrightarrows A)$ be a groupoid in $\underline{E}$, and B an object of $\underline{E}$. A $\mathbf{G}$-<u>valued cocycle on</u> B consists of a descent map $q: C \to B$ and a functor $\gamma = (\gamma_1, \gamma_0)$ from the kernel pair $\mathbf{R} = (R \rightrightarrows C)$ of q to $G \rightrightarrows A$. (Sometimes we also say that γ is a $\mathbf{G}$-valued cocycle <u>on the covering</u> C <u>of</u> B). Such $\gamma: \mathbf{R} \to \mathbf{G}$ gives rise to a functor $\underline{E}^{\mathbf{G}} \to \underline{E}^{\mathbf{R}}$, and thus, since q is a descent map, i.e. $\underline{E}^{\mathbf{R}} \simeq \underline{E}/B$, we get by composition a functor

$$\gamma^! : \underline{E}^{\mathbf{G}} \to \underline{E}/B.$$

Now $d_1: G \to A$ carries a canonical left action by $\mathbf{G}$, given by composition, so it is an object of $\underline{E}^{\mathbf{G}}$. With γ a cocycle as above, we thus get an object $\gamma^!(d_1)$; we denote it also $\gamma^!(\mathbf{G})$. We have

<u>Theorem 4.1.</u> Assume that $d_1: G \to A$ (or equivalently $d_0: G \to A$), and $\gamma_0: C \to A$ are descent maps. Then $\gamma^!(\mathbf{G})$ carries canonically a structure of regular pregroupoid, with respect to maps to A and B; and $(\gamma^!(\mathbf{G}))_* \cong \mathbf{G}$ canonically.

<u>Proof/construction.</u> By construction of $\gamma^!$, $\gamma^!(\mathbf{G})$ sits in a pull back diagram

(4.1)
$$\begin{array}{ccc} G \times_A C & \xrightarrow{q'} & \gamma^!(\mathbf{G}) \\ \downarrow & & \downarrow \beta \\ C & \xrightarrow[q]{} & B \end{array}$$

Since q is descent, q' is descent, and we define a map $\alpha: \gamma^! \mathbf{G} \to A$ by defining it on representatives from $G \times_A C$: $\alpha(g,c) := d_0(g)$.

Since the left hand vertical map in (4.1) comes about by pulling $d_1: G \to A$ back, and d_1 was assumed a descent map, it is itself a descent map, and from the stability properties of such, it follows that β is descent. Also, α sits by construction in a commutative square

$$\begin{array}{ccc} G \times_A C & \xrightarrow{q'} & \gamma^!(\mathbf{G}) \\ \downarrow & & \downarrow \alpha \\ G & \xrightarrow[d_0]{} & A \end{array}$$

whose left hand vertical map comes about by pulling back $\gamma_0 : C \to A$, and since d_0 is descent, it follows from the stability properties of descent maps that α is descent. Now a (regular) pregroupoid structure can be defined on $A \leftarrow \gamma^! \mathbf{G} \to B$, namely by the ternary operation defined (on representatives from $G \times_A C$) by

$$(4.2) \qquad (g_2,c_2)(g_1,c_1)^{-1}(g_3,c_3) := (g_2 \circ g_1^{-1} \circ \gamma_1(c_1,c_3)^{-1} \circ g_3, c_2),$$

(note that $(c_1,c_3) \in R \subset C \times C$, since (g_1,c_1) and (g_3,c_3) are supposed to represent elements in the same B-fibre; also $d_0(g_1) = d_0(g_2)$, since (g_1,c_1) and (g_2,c_2) are supposed to represent elements in the same A-fibre). We leave the further details to the reader. The isomorphism $(\gamma^!(\mathbf{G}))_* \cong \mathbf{G}$ is given by

$$(g_1,c_1)^{-1}(g_3,c_3) \mapsto g_1^{-1} \circ \gamma(c_1,c_3)^{-1} \circ g_3$$

in one direction, and by

$$g \mapsto (1_a,c)^{-1} (g,c)$$

in the other (for some/any c with $\gamma(c) = d_1(g) = a$).

This theorem has as a special case the construction of a principal fibre bundle out of a system of coordinate transformations. The next theorem has similarly as a special case the construction of fibre bundles associated with it.

Let $\mathbf{G} = (G \rightrightarrows A)$ and γ be as above; then

Theorem 4.2. Assume that $\gamma_0 : C \to A$ and $d_1 : G \to A$ are descent maps. Then the functor $\gamma^! : \underline{E}^{\mathbf{G}} \to \underline{E}/B$ lifts to a functor $\underline{E}^{\mathbf{G}} \to \mathrm{Fib}(\gamma^!(\mathbf{G}))$. This functor is an equivalence.

Proof/construction. Let $\varphi : F \to A$ be equipped with a left $\mathbf{G}$-action. We construct a map

$$\gamma^! \mathbf{G} \times_A F \xrightarrow{\sigma} \gamma^! \mathbf{G} \times_B \gamma^! F$$

as follows:

$$\sigma((g,c),f) := ((g,c),(g \cdot f,c))$$

where $d_1(g) = c$, $\varphi(f) = d_0(g)$. It has an inverse, given by

$$\sigma^{-1}((g,c),(f',c')) := ((g,c), g^{-1} \circ \gamma(c',c) \cdot f')).$$

We omit the details in checking the well-definedness of σ and σ^{-1}, and that the σ does indeed provide $\gamma^! F$ with structure of fibre bundle for $\gamma^! G$ with fibre F.

The fact that the functor is an equivalence follows from $Fib(\gamma^! G) \cong \underline{E}^{(\gamma^!(G))_*}$ (Theorem 3.1), together with $(\gamma^!(G))_* \cong G$ (Theorem 4.1).

In the standard applications, the 'covering' $q: C \to B$ considered in the present § will typically be derived (with $\underline{E}$ the category of topological spaces, say) from an open covering $\{U_i \subset B \mid i \in I\}$, with $C = \coprod U_i$, the disjoint union of the U_i's; then the γ_1 will be a family $\{\gamma_{ij} \mid (i,j) \in I \times I\}$ of transition maps.

However, coverings may be taken quite more general than that, (and is perhaps a novelty in our presentation), and this generality comes to work now:

Let $A \leftarrow X \to B$ be a regular pregroupoid; the we have a canonical X_*-valued cocycle on B, defined on the covering $\beta: X \to B$, (which is usually not of the form $\coprod U_i \to B$). This canonical cocycle is described as follows; γ_0 is just $\alpha: X \to A$; and $\gamma_1: X \times_B X \to X_*$ is given by

$$\gamma_1(x,z) = z^{-1}x.$$

Since $d_1: X_* \to A$ and γ_0 ($= \alpha$) are descent maps, the construction in Theorem 4.1 applies, so that $\gamma^!(X_*)$ is a regular pregroupoid w.r.to A and B. We have

Proposition 4.3. Let $A \leftarrow X \to B$ be a regular pregroupoid, and consider the canonical X_*-valued cocycle γ on the covering $X \to B$. Then $\gamma^!(X_*) \cong X$, canonically.

Proof/construction. Let (g,x) represent an element of $\gamma^!(X_*)$, so $g \in X_*$ and $d_1(g) = \alpha(x)$. Associate to it the element $x \cdot g \in X$. Conversely, to $x \in X$, associate the element in $\gamma^!(X_*)$ represented by $(1_{\alpha(x)}, x)$.

Remark 4.4. We may remark that the $\gamma^!$-construction, when applied to the canonical X_*-valued cocycle γ for a regular pregroupoid $A \leftarrow X \to B$, yields the inverse for the equivalence described in Theorem 3.1. We have in fact functors

$$\underline{E}^{X_*} \xrightarrow{\gamma^!} Fib(\gamma^!(X_*)) \to Fib(X),$$

the first one by Theorem 4.2, and the second (isomorphism) by Proposition 4.3.

§5. Foliations.

A groupoid $H \rightrightarrows B$ is usually called <u>transitive</u> if the map (d_0,d_1): $H \to B\times B$ is epic in some strong sense, say a descent map. If $A \leftarrow X \rightarrow B$ is a regular pregroupoid in $\underline{E}$, the groupoids $X^* \rightrightarrows B$ and $X_* \rightrightarrows A$ need not be transitive: in the category of sets, for example, take R to be an equivalence relation on an inhabited set X, let $A = X/R$, $B = X$, with obvious α and β, and define Λ by

$$\Lambda(x,y,z,u) \text{ iff } x = z, \quad y = u, \quad xRy.$$

Then X_* is the discrete groupoid on A, whereas $X^* \rightrightarrows B$ is $R \rightrightarrows B$. So X^* is transitive iff R is the codiscrete equivalence relation on X, iff $A = 1$, iff X_* is transitive.

In general, if $A \leftarrow X \rightarrow B$ is a regular pregroupoid, we have a commutative

$$\begin{array}{ccc} X\times_A X & \hookrightarrow & X\times X \\ \downarrow & & \downarrow{\scriptstyle \beta\times\beta} \\ X^* & \xrightarrow[(d_0,d_1)]{} & B\times B \end{array}$$

If $A = 1$, the top map is an isomorphism, and then the fact that $\beta\times\beta$ is descent implies that (d_0,d_1) is descent, so that

<u>Proposition 5.1.</u> If $A \leftarrow X \rightarrow B$ is a regular pregroupoid, and $A = 1$, then $X^* \rightrightarrows B$ is transitive.

This is the situation which occurs for fibre bundles in contrast to the present <u>generalized</u> fibre bundles, which rather come up in foliation theory, as we shall now sketch (essentially following |4|).

A smooth foliation $\underline{F}$ of codimension q on a manifold B may be presented by giving an open cover $\{U_i \mid i\in I\}$ of B, and for each i, a smooth surjective submersion f_i: $U_i \to A_i \subset \mathbb{R}^q$, with connected fibres. The A_i's may be assumed disjoint. The leaves of the foliation are the equivalence classes for the equivalence relation $\equiv_{\underline{F}}$ on B generated by the relation $\sim$, where

$$b \sim b' \text{ iff } (\exists i \text{ with } b,b' \in U_i \text{ and } f_i(b) = f_i(b')).$$

If $b \in U_i \cap U_j$, there is a unique germ $t = t_{b,i,j}$ of a diffeomorphism from $f_i(b)$ to $f_j(b)$, with $t\circ f_i$ having same germ at b as f_j. Let $A = \bigcup A_i$, let $G \rightrightarrows A$ be the groupoid of germs generated algebraically by germs of form $t_{b,i,j}$.

The data of the f_i and $t_{b,i,j}$ then provide a cocycle γ on the covering $\coprod U_i \to B$ of B with values in the groupoid $G \rightrightarrows A$. (Such a cocycle, or rather, an equivalence class of such, is a <u>Haefliger structure</u> on B with values in the groupoid $G \rightrightarrows A$.)

The corresponding groupoid $(\gamma^! \mathbf{G})^* \rightrightarrows B$ deserves the name <u>holonomy groupoid</u> of the foliation. An example of a generalized fibre bundle for the pregroupoid $A \leftarrow \gamma^! \mathbf{G} \to B$ is the normal bundle of the foliation, which one gets by applying the $\gamma^!$-construction to the tangent bundle $\mathbb{R}^q \times A \to A$ of A.

The holonomy groupoid is a glorified version of the equivalence relation $\equiv_F$, in the sense that two points in B are $\equiv_F$-related iff they can be connected by an arrow of the holonomy groupoid $(\gamma^! \mathbf{G})^*$. To prove this in a general context, we need that we out of a groupoid $H \rightrightarrows B$ can induce an equivalence relation R on B, namely by taking the 'image' of $(d_0, d_1) : H \to B \times B$,

$$H \twoheadrightarrow R \rightarrowtail B \times B;$$

so let us assume for simplicity that $\underline{E}$ is the category of sets (or any topos).

We consider, as in §4, a groupoid $\mathbf{G} = (G \rightrightarrows A)$ with d_0 and d_1 surjective, and a $\mathbf{G}$-valued cocycle γ defined on a covering $q: C \to B$; we assume as in §4 that $\gamma_0 : C \to A$ is surjective, so that the construction in §4 provides us with a regular pregroupoid $\gamma^! \mathbf{G}$ with respect to A and B, and with $(\gamma^! \mathbf{G})_* \cong \mathbf{G}$.

<u>Proposition 5.2.</u> The equivalence relation R induced on B by the groupoid $(\gamma^! \mathbf{G})^* \rightrightarrows B$ equals the relation given by: $b \sim b'$ iff

$\exists c, c' \in C$ with $q(c) = b$, $q(c') = b'$, and $\exists$ an arrow $\gamma(c) \to \gamma(c')$ in $\mathbf{G}$.

In particular, the relation thus described is an equivalence relation.

<u>Proof.</u> Let $b \sim b'$ in virtue of $g : \gamma(c) \to \gamma(c')$ in $\mathbf{G}$. Consider the elements x and y in $\gamma^! \mathbf{G}$ given by

$$x = (1_{\gamma(c)}, c) \qquad y = (g, c').$$

Then $\beta(x) = b$, $\beta(y) = b'$, and $\alpha(x) = \alpha(y) = \gamma(c)$. So yx^{-1} makes sense and is an arrow in $(\gamma^! \mathbf{G})^*$ from b to b', so bRb'.

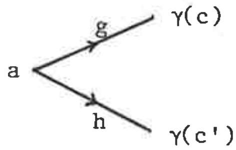
To prove the converse, we first prove

<u>Lemma 5.3.</u> Let $q(c) = b$, $q(c') = b'$. Any arrow $b \to b'$ in $(\gamma^! G)^*$ may be written in form uz^{-1}, for elements $u, z \in \gamma^! \mathbf{G}$, with

$$z = (1_{\gamma(c)}, c), \qquad u = (k, c'), \tag{5.1}$$

where $k: \gamma(c) \to \gamma(c')$ is an arrow of $\mathbf{G}$.

<u>Proof.</u> Let the given arrow be yx^{-1}; x and y may be represented (in fact uniquely) in form $x = (g,c)$, $y = (h,c')$, with g and h arrows of $\mathbf{G}$



Let $k := h \circ g^{-1}$, and let z and u then be given by (5.1). To prove $yx^{-1} = uz^{-1}$ in $(\gamma^! \mathbf{G})^*$ means to prove $u = yx^{-1}z$ in $\gamma^! \mathbf{G}$, but the recipe (4.2) for this ternary operation of $\gamma^! \mathbf{G}$ yields $(h \circ g^{-1} \bullet \gamma(c,c)^{-1} \circ 1, c')$, which is u.

To finish the proof of the proposition, we may by the Lemma assume that bRb' holds in virtue of an arrow uz^{-1}, with u and z as in (5.1). Then $k: \gamma(c) \to \gamma(c')$ witnesses $b \equiv b'$.

<u>Bibliography.</u>

1. J.Duskin, Free groupoids, trees, and free groups, Journ. Pure Appl. Alg. (to appear).
2. C.Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, Coll. de Topo., Bruxelles C.B.R.M. (1950), 29-55. (Oeuvres Vol I, Amiens 1984, 237-250).
3. C.Ehresmann, Catégories topologiques et catégories différentiables, Coll. Géom. Diff. Globale, Bruxelles C.B.R.M. (1959), 137-150. (Oeuvres Vol I, Amiens 1984, 237-250).
4. A.Haefliger, Gropoïdes d'holonomie et classifiant, Astérisque 116 (1984), 70-97.
5. P.T.Johnstone, Topos theory, Academic press 1977.
6. A.Kock, The algebraic theory of moving frames, Cahiers de Top. et Géom. Diff. 23 (1982), 347-362.
7. A.Kock, Combinatorial notions relating to principal fibre bundles, Journ. Pure Appl. Alg. 39 (1986), 141-151.
8. A.Kock, Fibre bundles in general categories, Aarhus Univ. Preprint Series 1986/87 No.27. To appear in Journ. Pure Appl. Alg.
9. J.Pradines, Au coeur de l'oeuvre de Charles Ehresmann et de la géometrie differentielle: Les groupoïdes différentiables. In C.Ehresmann, Oeuvres Vol I, Amiens 1984, 526-539.
10. B.L.Reinhart, Differential geometry of foliations, Springer Verlag 1983.