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FORMS AND INTEGRATION IN SYNTHETIC DIFFERENTIAL GEOMETRY

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FORMS AND INTEGRATION IN SYNTHETIC DIFFERENTIAL GEOMETRY

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The context of the present article is synthetical differential geometry. The main theorem proved (or reproved) is Stokes' Theorem; it turns out that the infinitesimal version of this theorem is true by the very definition of the coboundary operator for differential forms, namely as the dual of the classical boundary operator for singular cubical chains (slightly generalized). The integration axiom introduced in [9] allows us to pass to the finite form of Stokes' theorem via a theory of "additive rectangle functions", in §4.

It should be said that our "differential form" notion looks quite different from the classical one, since an n-form on M, with us, is something that assigns scalars to "infinitesimal n-cubes" $D^n \to M$, rather than to n-tuples of tangent vectors attached at the same point. However, for suitably smooth objects, (in particular, for all manifolds in a well-adapted model in the sense of [1],[5]), the two notions can be proved to be equivalent.

§1. Infinitesimal prelude: infinitesimal cubes, chains and cochains

We let R be a ring object of line type (see e.g. [3]), fixed throughout. As usual, D denotes the "set" $\{d \in R \mid d^2 = 0\}$. We assume that 2 is invertible in R.

Given an object M, we define a singular infinitesimal n-cube ($n \geq 0$) on M to be a pair

$$\gamma = \langle c, \underline{d} \rangle \in M^{D^n} \times D^n.$$

(We think of γ as the restriction of the "open" cube $c: D^n \to M$ to the infinitesimal rectangle $[0,d_1] \times \ldots \times [0,d_n]$; since we did not define any notion of interval such as $[0,d_1]$ yet, this remark is purely heuristical. The point is that we need "faces" of D^n in order to define the boundary operator).

A singular infinitesimal cubical n-chain on M is then an R-linear combination of singular infinitesimal n-cubes, i.e. we define $C_n(M)$ as the free R-module generated by $M^{D^n} \times D^n$.

The definition of the boundary operator

$$\partial_n: C_n(M) \longrightarrow C_{n-1}(M)$$

(for $n \geq 1$) is almost immediate from the heuristic remark above. Explicitely for γ as above, let $\delta \in D$ and $i = 1,\ldots,n$, and denote by $c_i(\delta)$ the map $D^{n-1} \to M$ given by

$$(d_1,\ldots,d_{n-1}) \longmapsto c(d_1,\ldots,d_{i-1},\delta,d_i,\ldots,d_{n-1}).$$

(So c_i itself is a map $D \to M^{D^{n-1}}$). Put for $i = 1,\ldots,n$ and

$\alpha = 0,1$

$$F_{i\alpha}(\gamma) = \langle c_i(\alpha \cdot d_i), \underline{d}(\hat{i}) \rangle$$

where $\underline{d}(\hat{i})$ is the $(n-1)$-tuple obtained from $(d_1,\ldots,d_n)$ by omitting the i'th component. Finally, define a map

$$\partial = \partial_n: M^{D^n} \times D^n \longrightarrow C_{n-1}(M)$$

by putting

$$\partial_n(\gamma) = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} F_{i\alpha}(\gamma).$$

Extending this by R-linearity gives an R-linear map $\partial_n: C_n(M) \to C_{n-1}(M)$ as desired. The proof that $\partial_n \circ \partial_{n+1} = 0$ is now so close to the classical one for singular cubical homology (c.f. e.g. [2] pp.321-322) that we omit it.

Given an R-module V, a map

$$\omega: M^{D^n} \to V \qquad (n \geq 0)$$

is called a singular infinitesimal cubical n-cochain on M. They form an R-module $C^n(M,V)$. By definition, n-cochains act on "open" n-chains, so are not dual to chains. However, there is a bilinear pairing

$$C_n(M) \times C^n(M,V) \xrightarrow{\int} V$$

defined on generators of $C_n(M)$ as follows. Given $\gamma = \langle c, \underline{d} \rangle \in M^{D^n} \times D^n$ and $\omega: M^{D^n} \to V$, we define

$$(1.1) \qquad \int_\gamma \omega := d_1 \cdot \ldots \cdot d_n \cdot \omega(c),$$

and interpret this as an infinitesimal integration.

We shall make $C^*(M,V) = \oplus_{n\geq o} C^n(M,V)$ into a cochain complex provided V is Euclidean (cf. [3] or [11]), i.e., for any $t: D \to V$ there is a unique $v \in V$ (called the "principal part" of the "tangent vector t") such that

$$t(d) = t(0) + d \cdot v \qquad \forall d \in D$$

We also write $D_o(t)$ for v ("derivative" of t at 0). The guiding line for defining the coboundary $d\omega$ of a cochain is that we would like

$$(1.2) \qquad \int_{\partial\gamma} \omega = \int_\gamma d\omega$$

to hold, for $\gamma \in C_n(M), \omega \in C^{n-1}(M,V)$. So let us start by calculating the left hand side of (1.2): for $\gamma = \langle c, \underline{d} \rangle \in M^{D^n} \times D^n$, we have

$$\int_{\partial\gamma} \omega = \sum_{i,\alpha} (-1)^{i+\alpha} \int_{F_{i\alpha}(\gamma)} \omega$$

$$= \sum_{i=1}^{n} \left[\prod_{k \neq i} d_k \cdot (-1)^{i+1} \{\omega(c_i(d_i) - \omega(c_i(0))\} \right]$$

Now for each $i = 1, \ldots, n$,

$$D \xrightarrow{c_i} M^{D^{n-1}} \xrightarrow{\omega} V$$

is a map from D to V, and since V is Euclidean, the terms

in the curly brackets reduce to

$$d_i \cdot D_o(\omega o c_i);$$

we thus obtain

$$\int_{\partial\gamma} = d_1 \cdot \ldots \cdot d_n \cdot \sum_{i=1}^{n} (-1)^{i+1} \cdot D_o(\omega o c_i).$$

The following definition is now evident

$$(1.3) \qquad d\omega(c) := \sum_{i=1}^{n} (-1)^{i+1} D_o(\omega o c_i)$$

so that we end up with

$$\int_{\partial\gamma} \omega = d_1 \cdot \ldots \cdot d_n \cdot d\omega(c);$$

or, using (1.1)

<u>Theorem 1.1</u> ("Infinitesimal Stokes"):

$$(1.2) \qquad \int_{\partial\gamma} \omega = \int_\gamma d\omega.$$

We shall see in a moment (Corollary 1.3 below) that, for Euclidean V, (1.3) is the only possible way to define $d\omega$ if we want (1.2) to hold. This means of course that Stokes' Theorem is built into our construction of the coboundary operator, i.e. is true by definition. It seems to be a frequent phenomenon that things become "evident in the infinitesimally small".

We first note the following quite general

<u>Proposition 1.2</u>. Let V be an R-module. Then the following are equivalent:

(i) V satisfies the following condition W_n for all $n \geq 1$

(W_n): if $t: D^n \to V$ is such that $t(d_1,\dots,d_n) = 0$ as soon as one of the d_i's is 0, then there is a unique $v \in V$ for which

$$t(d_1,\dots,d_n) = d_1 \cdot \ldots \cdot d_n \cdot v \quad \forall (d_1,\dots,d_n) \in D^n$$

(ii) V satisfies condition W_1

(iii) V is Euclidean.

<u>Proof</u>. The equivalence of (ii) and (iii) is straightforward, and (i) ⇒ (ii) is obvious. To prove (ii) ⇒ (i), let us assume W_1 and W_{n-1} hold for some $n \geq 2$, and consider $t: D^n \to V$ with $t(d_1,\dots,d_n) = 0$ if $d_i = 0$ for some i. For each $d \in D$, let $t_d: D^{n-1} \to V$ send $(d_1,\dots,d_{n-1})$ to $t(d_1,\dots,d_{n-1},d)$. By W_{n-1}, there is a unique $v = v(d) \in V$ for which

$$t_d(d_1,\dots,d_{n-1}) = d_1 \cdot \ldots \cdot d_{n-1} \cdot v(d). \tag{1.4}$$

We claim that $v(0) = 0$. In fact

$$d_1 \cdot \ldots \cdot d_{n-1} \cdot v(0) = t(d_1,\dots,d_{n-1},0) = 0$$

for all $(d_1,\dots,d_{n-1}) \in D^{n-1}$; apply the uniqueness assertion in

W_{n-1} to conclude $v(0) = 0$. Now, apply W_1: there is a unique $v \in V$ such that $v(d) = d \cdot v$ $\forall d \in D$. This $v \in V$ witnesses validity of W_n.

<u>Corollary 1.3</u>. For $\omega_1,\omega_2: M^{D^n} \to V$ with V Euclidean, if

$$\int_\gamma \omega_1 = \int_\gamma \omega_2 \qquad \text{for all } \gamma \in M^{D^n} \times D^n$$

then $\omega_1 = \omega_2$.

<u>Proof</u>. Consider, for fixed $c \in M^{D^n}$, the maps $D^n \to V$ given by

$$\underline{d} \longmapsto \int_{\langle c,\underline{d}\rangle} \omega_i = d_1 \cdot \ldots \cdot d_n \cdot \omega_i(c), \qquad (i = 1,2).$$

They agree by assumption for any $\underline{d}$, hence by the uniqueness assertion in W_n, $\omega_1(c) = \omega_2(c)$.

From the Corollary and the corresponding properties for ∂, it is now immediate to infer that $d \circ d = 0$ and that d is linear. So $C^*(M,V)$ is made into a cochain complex.

We are going to define <u>differential</u> n-<u>forms</u> on M (with values in a Euclidean module V) as singular infinitesimal cubical n-cochains with the properties: alternating, and n-linear (in a sense to be explained). The set of these will be proved to form a subcomplex $\Lambda^*(M,V) \subseteq C^*(M,V)$, (Thus, the "deRham complex" Λ^* is more closely related to a certain singular complex C^* than is usually the case).

We say that a cochain $\omega: M^{D^n} \to V$ is <u>alternating</u> if for all permutations $\pi: [n] \to [n]$, and all $c: D^n \to M$

$$\omega(c \circ D^\pi) = \mathrm{sgn}(\pi)\cdot\omega(c).$$

As to n-linearity, even if we may not have any fibrewise linear structures present in M^{D^n}, we have in any case n different actions of the multiplicative monoid $(R,\cdot)$ on M^{D^n}: The k'th of these, denoted $\cdot_k$, is given, for $c: D^n \to M$ and $a \in R$, by

(1.5) $$(a\cdot_k c)(d_1,\dots,d_n) := c(d_1,\dots,a\cdot d_k,\dots,d_n).$$

We say ω is <u>n-linear</u> if for any $c: D^n \to M$, any $k = 1,\dots,n$, and any $a \in R$

(1.6) $$\omega(a\cdot_k c) = a\cdot\omega(c).$$

In general, one needs some extra properties on M (like "infinitesimal linearity", [11],[7]) in order to have an addition in the fibres of $M^D \to M$ and thus also in the fibres of $M^{D^n} \to M^{D^{n-1}}$. But anyway, our terminology is justified, since we can prove that whenever an R-module structure is present, the multilinearity can in fact be deduced from the homogeneity-requirements like (1.6). Explicitely, we have the following Proposition (expected by Lawvere [10], and proved in [6]):

<u>Proposition 1.4</u>. If V is a Euclidean R-module, then for any R-module U and any map $f: U \to V$, if f is R-homogeneous, i.e.

(1.7) $$f(a\cdot u) = a\cdot f(u) \qquad \forall a \in R \quad \forall u \in U,$$

then f is R-linear.

<u>Proof</u>. For any $g: U \to V$, $x \in U$, $y \in U$ and $d \in D$, we have by Taylor's formula,

$$g(x+d\cdot y) = g(x) + d\cdot D_y g(x),$$

where $D_y g$ is the directional derivative of f in direction y (compare Theorem 2.2 and Proposition 3.1 in [3]). In particular, for any $d \in D$,

$$\begin{aligned} d\cdot f(x+y) &= f(d\cdot x + d\cdot y) \\ &= f(d\cdot x) + d\cdot D_y f(d\cdot x) \\ &= f(d\cdot x) + d\cdot(D_y f(0) + d\cdot D_x D_y f(0)) \\ &= f(d\cdot x) + d\cdot D_y f(0)) \end{aligned}$$

(since $d^2 = 0$)

$$\begin{aligned} &= f(d\cdot x) + f(d\cdot y) \\ &= d\cdot f(x) + d\cdot f(y), \end{aligned}$$

the first and last equality sign utilizing the assumed homogeneity condition on f. Since this holds for all $d \in D$, we get by the uniqueness assertion in the Euclideanness condition that $f(x+y) = f(x) + f(y)$, as desired.

Remark. We actually only used the condition (1.7) for $a \in D$, in other words, it suffices, for additivity that f is "infinitesimally homogeneous".

Proposition 1.5. The differential forms form a subcomplex $\Lambda^*(M,V)$ of the cochain complex $C^*(M,V)$. More generally, if $\omega \in C^{n-1}(M,V)$, then

(i) ω alternating $\Rightarrow d\omega$ alternating

(ii) ω $(n-1)$-linear $\Rightarrow d\omega$ n-linear.

Proof. (i) is a straightforward calculation which we omit. To prove (ii), let $c: D^n \to M$, $a \in R$, $k \in \{1,\ldots,n\}$, and put $\bar{c} = a \cdot_k c$. For any $i = 1,\ldots,n$, we compare c_i and $\bar{c}_i: D \to M^{D^{n-1}}$. It is easy to see that

$$\bar{c}_i(d) = \begin{cases} c_i(a\cdot d) & \text{if } k = i \\ a\cdot_k c_i(d) & \text{if } k < i \\ a\cdot_{k-1} c_i(d) & \text{if } k > i \end{cases}$$

Hence, if $i = k$, one has for all $d \in D$

$$\begin{aligned} d\cdot D_o(\omega o \bar{c}_i) &= \omega(c_i(a\cdot d)) - \omega(c_i(0)) \\ &= a\cdot d\cdot D_o(\omega o c_i), \end{aligned}$$

and this gives $D_o(\omega o \bar{c}_i) = a\cdot D_o(\omega o c_i)$ since the d's are universally quantified. On the other hand, if $i \neq k$, there is $k' \leq n-1$ such that

$$\omega(\bar{c}_i(d)) = \omega(a\cdot_{k'} c_i(d)) = a\cdot\omega(c_i(d)) \qquad \forall d \in D,$$

i.e.

$$\omega o \bar{c}_i = a\cdot(\omega o c_i).$$

This implies $D_o(\omega o \bar{c}_i) = a\cdot D_o(\omega o c_i)$ (since D_o is R-homogeneous). Hence $d\omega(a\cdot_k c) = a\cdot d\omega(c)$.

We conclude this section by observing that the complexes C_*, C^*, and Λ^* introduced for a given object M, in an obvious way depend functorially on M (the two last-mentioned contravariantly). Specifically, given $f: N \to M$, an arbitrary map. Define

$$f_*: C_n(N) \to C_n(M)$$

by letting

$$f_*(\langle c,\underline{d}\rangle) = \langle f o c,\underline{d}\rangle$$

on generators of $C_n(N)$, and extending linearly. Similarly, f induces for all n a map

$$f^*: C^n(M,V) \to C^n(N,V)$$

given by

$$(f^*(\omega))(c) = \omega(f o c).$$

It is then immediate to verify that f_* and f^* are chain maps, such that

$$\int_\gamma f^*\omega = \int_{f_*\gamma} \omega. \tag{1.8}$$

Also f^* restricts to a chain map on Λ^*.

§2. Classical and synthetic differential forms

The differential n-forms ω we consider take values on "open infinitesimal n-cubes" $c\colon D^n \to M$, whereas, classically, an n-form takes value on any n tangent vectors (at the same point); such n vectors can be obtained from c by restricting c to the n "axes" through $\underline{0} \in D^n$. We shall prove in this section that for a large class of objects M, the two "form"-notions are nevertheless equivalent.

We shall assume for this §, that $M^D \to M$ (the tangent bundle) is actually a standard vector bundle, i.e. an R-module with multiplication by scalars as in (1.5) for $k = 1$. This will be the case if M is infinitesimally linear, or if M is a Euclidean module. Then M^{D^2} has two vector bundle structures over M^D with projections

$$c \xrightarrow{h_1} c(0,-) \qquad c \xrightarrow{h_2} c(-,0) \tag{2.1}$$

respectively $(c\colon D \times D \to M)$. We denote by $+$ and $\oplus$ the additions in the fibres of h_1 (respectively the fibres of h_2). Thus $c+c'$ is defined if $c(0,d) = c'(0,d)$ $\forall d \in D$, and $c \oplus c'$ is defined if $c(d,0) = c'(d,0)$ $\forall d \in D$.

By induction, one finds n different vector bundle structures of M^{D^n} over $M^{D^{n-1}}$ with multiplication by scalars as in (1.5). Thus n-forms on M are multilinear in the ordinary sense, by Proposition 1.4.

For any n and $1 \leq i \leq n$, we let h_i denote the map $D^{n-1} \to D^n$ which inserts a 0 in the i'th position. If we assume that M is equipped with a connection in the sense of [8], i.e. a splitting ∇ of the map

$$M^{D^2} \xrightarrow{\langle h_1,h_2\rangle} M^D \times_M M^D$$

then it is easy to prove (cf. [13]) that for each n and each $i<j\leq n$, the map $\langle h_i,h_j\rangle\colon M^{D^n} \to M^{D^{n-1}} \times M^{D^{n-1}}$ factors through the pull-back object M_{ij} in the pull-back diagram

$$\begin{array}{ccc} M_{ij} & \xrightarrow{h_j} & M^{D^{n-1}} \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle h_i} \\ M^{D^{n-1}} & \xrightarrow[h_{j-1}]{} & M^{D^{n-2}} \end{array}$$

and that the map $K_{ij}\colon M^{D^n} \to M_{ij}$ thus obtained can be canonically split, using that data ∇. Call the splitting ∇_{ij}. In [13], Proposition 4, it is proved that a splitting ∇_n of "restrictions to the axes"

$$M^{D^n} \xrightarrow{K_n} M^D \times_M M^D \times_M \cdots \times_M M^D$$

can be constructed from the ∇_{ij}'s, and that, in fact, the composite $\nabla_n \circ K_n$ can be written as a composite of $\nabla_{ij} \circ K_{ij}$'s.

Recall that "Axiom 2" of [11] states that a map $c\colon D^2 \to M$ which has constantly the same value on the two axes, i.e.

$$c(0,d) = c(d,0) = c(0,0) \qquad \forall d \in D$$

is of form $c(d_1,d_2) = t(d_1 \cdot d_2)$ for a unique $t\colon D \to M$.

We can now prove

Theorem 2.1. Assume M satisfies "Axiom 2" of [11], and that M can be equipped with a connection ∇, and also that

$M^D \to M$ is a standard vector bundle. Then, for any differential form $\omega\colon M^{D^n} \to V$ (with V a Euclidean R-module), and for any n-cube $c\colon D^n \to M$, we have that $\omega(c)$ depends only on $K_n(c)$, the restriction of c to the n axes.

<u>Proof</u>. Let ∇ be a connection on M, and let the ∇_{ij} and ∇_n be constructed from ∇ as above. It suffices to prove

$$\omega(\nabla_n(K_n(c))) = \omega(c).$$

Since $\nabla_n \circ K_n$ can be constructed by iterating $\nabla_{ij} \circ K_{ij}$'s it suffices to prove, for all $i < j$,

$$\omega(\nabla_{ij}(K_{ij}(c))) = \omega(c).$$

Now c and $\nabla_{ij} K_{ij} c = \bar{c}$ agree on the i'th coordinate hyperplane, as well as on the j'th coordinate hyperplane. They can thus be viewed as maps

$$D^2 \longrightarrow M^{D^{n-2}}$$

that agree on the two axes of D^2, and the n-form ω on M gets reinterpreted as a 2-form on $M^{D^{n-2}}$. So it suffices to prove the following lemma (whose hypothesis is easily seen to be true for $N = M^{D^{n-2}}$):

<u>Lemma 2.2</u>. Let ω be a 2-form on an object N that satisfies "Axiom 2" and whose tangent bundle is a standard vector bundle. Then if $c, \bar{c}\colon D^2 \to M$ agree on the two axes of D^2,

we have $\omega(c) = \omega(\bar{c})$.

The proof of this depends on another lemma, which may be useful in other situations also. Recall the two fibrewise additive structures $+$ and $\oplus$ of N^{D^2} over N^D, introduced at the beginning of this §. The corresponding minus signs will be denoted $-$ and $\ominus$ respectively.

<u>Lemma 2.3</u>. If $N^D \to N$ has vector bundle structure, and $c, \bar{c}\colon D^2 \to N$ are such that

$$h_1(c) = h_1(\bar{c}) \quad \text{and} \quad h_2(c) = h_2(\bar{c})$$

(i.e. c and $\bar{c}$ agree on the two axes), then

$$\tau := (c - \bar{c}) \ominus (c - c)$$

can be performed and is constant on the axes.

<u>Proof</u>. For $v, w \in N^{D^2}$, we have that $v - w$ is defined, iff $h_1(v) = h_1(w)$, and $v \ominus w$, is defined iff $h_2(v) = h_2(w)$; in the former case $h_1(v-w) = h_1(v) = h_1(w)$, and dually for the latter. Also, the processes $v \mapsto h_1(v)$ and $v \mapsto h_2(v)$ are homomorphisms of vector bundles. Now $c - \bar{c}$ can be performed since $h_1(c) = h_1(\bar{c})$ by assumption. Also $c - c$ can be performed. Further

$$h_2(c-\bar{c}) = h_2(c) - h_2(\bar{c}) = h_2(c) - h_2(c) = h_2(c-c).$$

(the middle equality sign by assumption). But this means that

τ can be performed. Finally

$$h_2(\tau) = h_2((c-\bar{c}) \ominus (c-c)) = h_2(c-c) = h_2(c) - h_2(c) = 0$$

(meaning the zero vector in the fibre of $N^D \to N$ over $c(0,0)$, so means a constant map). Likewise

$$h_1(\tau) = h_1((c-\bar{c}) \ominus (c-c)) = h_1(c-\bar{c}) - h_1(c-c) = h_1(c) - h_1(c)$$
$$= 0.$$

This proves Lemma 2.3.

Proof of Lemma 2.2. Let $\tau: D \times D \to N$ be constructed from c and $\bar{c}$ as in Lemma 2.3. By "Axiom 2", there is a $t: D \to N$ such that

$$\tau(d_1,d_2) = t(d_1 \cdot d_2).$$

Therefore τ is invariant under the twisting $D\times D \to D\times D$, so since ω is alternating and $1+1$ is assumed invertible, we conclude $\omega(\tau) = 0$. But also $\omega(c-c) = 0$ since $c-c$ is zero for one of the additive structures. Hence

$$0 = \omega(\tau) = \omega((c-\bar{c}) \ominus (c-c)) = \omega((c-\bar{c}) - \omega(c-c)$$
$$= \omega(c-\bar{c}) = \omega(c) - \omega(\bar{c})$$

using that ω preserves each of the two additive structures, by Proposition 1.4 . This proves the Lemma 2.2 and thus the theorem.

We can use the theorem to describe the structure of k-forms on $M = R^n$ in terms of standard multilinear algebra. We shall only do it for $k = n$, which is the only case we shall use here. Let ω be an n-form on R^n. Let $c: D^n \to R^n$ be an n-cube. It is of form

$$(d_1,\dots,d_n) \longmapsto \underline{a} + \sum_{i=1}^{n} d_i \cdot \underline{b}_i + \sum_{i \neq j} d_i \cdot d_j \cdot f(\underline{d})$$

for some $f: D^n \to R^n$, by Taylors formula. But on the coordinate axes, this c agrees with the $\bar{c}$ given by

$$(d_1,\dots,d_n) \longmapsto \underline{a} + \sum d_i \cdot \underline{b}_i \ .$$

Now, the i'th additive structure $(i = 1,\dots,n)$ for such elements in M^{D^n} is simply adding the $\underline{b}_i$'s (this follows from Proposition 2 in [4]), and is defined provided the $\underline{a}$ and the $\underline{b}_j$'s for $j \neq i$ are pairwise equal), so that the value of ω on c is a function of $\underline{a},\underline{b}_1,\dots,\underline{b}_n$ which is multilinear and alternating in its dependence on the $\underline{b}_i$'s. Then it is of the form

(2.1) $$\omega(c) = f(\underline{a}) \cdot \det(\underline{b}_1,\dots,\underline{b}_n)$$

(det denotes the determinant of the matrix formed by the coordinates of the $\underline{b}_i$'s). The n-form ω given by (2.1), we shall denote $f \cdot dx_1 \wedge \dots \wedge dx_n$.

§3. Cubical chains

As in [9] we assume that the basic ring object R is equipped with a preorder relation $\leq$ compatible with the ring structure and such that, for any nilpotent element $d \in R$ we have $0 \leq d \wedge d \leq 0$.

For $a \leq b$, we let $|[a,b]|$ denote $\{x | a \leq x \leq b\}$. Notice that a and b are not determined by $|[a,b]|$. We shall want $[a,b]$ to denote not just the set $|[a,b]|$ but, formally the triple $\langle a,b,\{x | a \leq x \leq b\}\rangle$. In particular, we may construct a map: $I \to R$ (also denoted $[a,b]$) from the data $[a,b]$, namely that unique affine map which sends 0 to a and 1 to b.

We note that sets of form $|[a_1,b_1]| \times \ldots \times |[a_n,b_n]|$ are subeuclidean; stable under addition of nilpotents in any of the n directions, so that functions defined on this set may be partially differentiated in any of the n directions any number of times.

We let I denote $[0,1]$.

Let M be an arbitrary object. A finite singular n-cube on M is a map $I^n \to M$.

A (finite) singular n-chain on M is a formal linear combination of such. For the developments here, it makes no difference whether we take coefficients from $\mathbb{Z}$ or from R, so let us take the latter, to be specific. The set of singular n-chains on M is denoted $CI_n(M)$, to distinguish it from the set $C_n(M)$ of infinitesimal singular chains of §1.

The definition of a boundary operator $\partial: CI_n(M) \to CI_{n-1}(M)$ is standard, (cf. e.g. [2] 8.3), and analogous to the one of §1. This analogy will be exploited later (§5) where we will compare them explicitely, to obtain Stokes' theorem for finite singular

chains, out of the infinitesimal form of §1.

We need, for technical reasons, to consider some specific finite singular n-cubes on R^n, which we call n-rectangles. An n-rectangle is a map $I^n \to R^n$ which is of form

$$\underline{x} \to \underline{a} + (b_1 \cdot x_1, \ldots, b_n \cdot x_n) \tag{3.2}$$

($\underline{x} = (x_1, \ldots, x_n)$ etc.). The map (3.2) is completely determined by $\underline{a} \in R^n$ and $\underline{b} \in R^n$, and will be denoted $[[\underline{a},\underline{b}]]$. Note that for $n = 1$, $[[a,b]] = [a,a+b]$. If the values of (3.2) lie in I^n for all $\underline{x} \in I^n$, it may be considered an element of $CI_n(I^n)$, (as well as of $CI_n(R^n)$). In particular, if $\underline{a} \in I^n$ and $\underline{b} \in D^n$, the rectangle $[[\underline{a},\underline{b}]]$ will be such, since $I^n \subseteq R^n$ is subeuclidean Such rectangles we call D-small. Note that a D-small rectangle is a finite singular cube. But it determines an infinitesimal singular cube on I^n, namely $\langle t_{\underline{a}}, \underline{b}\rangle$, where $t_{\underline{a}}: D^n \to I^n$ is given by

$$t_{\underline{a}}(\underline{\delta}) = \underline{a} + \underline{\delta} \qquad \forall \underline{\delta} \in D^n$$

We shal in §5 introduce integration of n-forms against (finite) singular n-chains, $\int_\rho \omega$. It will be linear in ρ and ω and satisfy

$$\int_{f_*(\rho)} \theta = \int_\rho f^*\theta \tag{3.3}$$

whenever $f: M \to N$, θ is an n-form on N and ρ an n-chain on M. We shall also prove (Proposition 5.2) that under the correspondence above between D-small n-rectangles on I^n and (some of) the infinitesimal n-cubes on I^n, the (finite) integration of the former agrees with the infinitesimal integration of the latter (infinitesimal integration being introduced in §1); explicitely, for $(\underline{a},\underline{d}) \in I^n \times D^n$,

$$(3.4)_n \qquad \int_{[[\underline{a},\underline{d}]]} \omega = \int_{<t_{\underline{a}},\underline{d}>} \omega$$

for any n-form on I^n. Using this information ($\forall n$), we shall now derive, purely combinatorially, that

$$(3.5) \qquad \int_{\partial[[\underline{a},\underline{d}]]} \theta = \int_{\partial<t_{\underline{a}},\underline{d}>} \theta$$

for any (n-1)-form θ on I^n.

To prove this (and at the same time describe the boundary operator for finite chains explicitely), we first introduce some slightly heavy notation. For $X = I$ or D, for $i = 1,\dots,n$, and for $x \in X$, we denote by $\eta(i,x)$ the map

$$X^{n-1} \to X^n$$

given by

$$(x_1,\dots,x_{n-1}) \mapsto (x_1,\dots,x_{i-1},x,x_i,\dots,x_{n-1}).$$

The boundary operator for $CI_*(M)$ can be described by

$$\partial\rho = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \rho\circ\eta(i,\alpha)$$

for $\rho: I^n \to M$; the boundary operator given in §1 for C_*M can be described (for $c: D^n \to M, \underline{d} \in D^n$), by

$$\partial<c,\underline{d}> = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} < c\circ\eta(i,\alpha\cdot d_i), \underline{d}(\hat{i})>$$

($\underline{d}(\hat{i})$ obtained from $\underline{d}$ by omitting the i'th entry).

Now, each side of the equality sign in (3.5) is a sum of 2n terms. We prove equality of the (i,α)'th term on the left with the (i,α)'th term on the right. The left (i,α)-term is modulo sign the integral of θ over the (n-1)-cube

$$[[\underline{a},\underline{d}]]\circ\eta(i,\alpha) = \eta(i,a_i+\alpha d_i)\circ[[\underline{a}(\hat{i}),\underline{d}(\hat{i})]]$$

so that the integral, by (3.3), equals

$$(3.6) \qquad \int_{[[\underline{a}(\hat{i}),\underline{d}(\hat{i})]]} (\eta(i,a_i+\alpha\cdot d_i))^*\theta.$$

Similarly, the right hand side has for its (i,α) term $\pm$ the (infinitesimal) integral of θ over

$$<t_{\underline{a}}\circ\eta(i,\alpha\cdot d_i),\underline{d}(\hat{i})> = <\eta(i,a_i+\alpha\cdot d_i)\ t_{\underline{a}(\hat{i})},\underline{d}(\hat{i})>$$

so that the integral is, by (1.8),

$$(3.7) \qquad \int_{<t_{\underline{a}(\hat{i})},\underline{d}(\hat{i})>} \eta(i,a_i+\alpha\cdot d_i)^*\theta.$$

But (3.6) and (3.7) agree, by $(3.4)_{n-1}$. Since the sign is also the same, namely $(-1)^{\alpha+i}$, the two terms to be compared agree. This proves (3.5).

Given a singular n_1-cube c_1 on M_1 and a singular n_2-cube c_2 on M_2, we get a singular n_1+n_2-cube on $M_1 \times M_2$:

$$I^{n_1+n_2} = I^{n_1} \times I^{n_2} \xrightarrow{c_1 \times c_2} M_1 \times M_2$$

which in turn induces a bilinear map (also denoted $\times$):

$$CI_{n_1}(M_1) \times CI_{n_2}(M_2) \to CI_{n_1+n_2}(M_1 \times M_2)$$

It is a standard fact that

(3.8) $$\partial(c_1 \times c_2) = \partial c_1 \times c_2 + (-1)^{n_1} c_1 \times \partial c_2$$

Note that a rectangle $[\![\underline{a},\underline{b}]\!]$ on I^n or R^n can be written as

$$[\![a_1,b_1]\!] \times \ldots \times [\![a_n,b_n]\!] \quad \text{or} \quad [a_1,a_1+b_1] \times \ldots \times [a_n,a_n+b_n]$$

On the set of linear combinations of rectangles on the I^n's, we introduce the smallest additive equivalence relation $\sim$ such that

(i) $[a,b] + [b,c] \sim [a,c]$

(ii) $c \sim c' \Rightarrow c'' \times c \sim c'' \times c'$

(iii) $c \sim c' \Rightarrow c \times c'' \sim c' \times c''$

We can then prove

(3.9) $$c \sim c' \Rightarrow \partial c = \partial c'$$

For, R-linearity of ∂ takes care of R-linearity of $\sim$; formula (3.8) takes care of the clauses (ii) and (iii), and

$$\partial([a,b] + [b,c]) = \partial([a,c])$$

is obvious, so that (i) is taken care of.

§4. Integration and additive rectangle functions

We now state the main axiom (taken from [9]) needed to integrate "over finite rectangles".

<u>Axiom 4.1</u>. $\forall a \leq b \;\; \forall f:[a,b] \to R \;\; \exists! g:[a,b] \to R$:

$$g(a) = 0 \wedge \forall x \in [a,b] \, g'(x) = f(x).$$

Recall [3] that $g'(x) = f(x)$ means $g(x+d) = g(x) + d \cdot f(x) \;\; \forall d \in D$. Therefore, the axiom may be restated in terms of "1-dimensional additive rectangle functions", in the sense given below. We refer to [12] for this formulation).

As usual we put $\int_a^b f(t)\,dt = g(b)$. The following properties are easily checked (cf. [9])

(4.1) $\int_a^b f(t)\,dt$ is R-linear in f

(4.2) $\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt$, for $a \leq b \leq c$

(4.3) Let $h:[a,b] \to R$ be defined by $h(s) = \int_a^s f(t)\,dt$. Then $h' = f$.

(4.4) Let $g:[a,b] \to R$. Then $\int_a^b g'(t)\,dt = g(b) - g(a)$.

(4.5) Let $f:[a_1,b_1] \times [a,b] \to R$. Let $h:[a_1,b_1] \to R$ be defined by $h(s) = \int_a^b f(s,t)\,dt$. Then

$$h'(s) = \int_a^b \frac{\partial f}{\partial s}(s,t)\,dt.$$

Furthermore, a consequence of Fubini's Theorem can also be proved from this context:

(4.6) Let $f:[a_1,b_1] \times [a_2,b_2] \to R$. Then

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)dx = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)dy.$$

<u>Proof</u>. Let

$$F(x) = \int_{a_2}^{b_2} f(x,y)\,dy, \quad G(x) = \int_{a_2}^{b_2}\int_{a_1}^{x} f(t,y)\,dt\,dy.$$

Then $G'(x) = F(x)$, by (4.5). Furthermore, $G(a_1) = 0$ by (4.1) and (4.2). Hence, using (4.4),

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)dx = \int_{a_1}^{b_1} F(x)\,dx = G(b_1)$$
$$= \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)dy.$$

On the basis of Axiom 4.1, we shall prove that certain R-valued functions defined on the set of D-small rectangles may be extended uniquely to "additive rectangle functions" defined on the set of all rectangles (we interpret this as "integrating infinitely many small rectangles").

We shall call a rectangle $[[\underline{a},\underline{\underline{b}}]]$ on R^n (or I^n) <u>degenerate</u> if at least one of the coordinates of $\underline{b}$ is 0.

Equivalently

$$[a_1,b_1]\times\dots\times[a_n,b_n]:\ I^n \to R^n$$

is degenerate if $a_i = b_i$ for some $1 \leq i \leq n$.

We can now state the main result of this section.

<u>Proposition 4.2</u>. Assume that φ is an R-valued function defined on the set of D-small rectangles on R^m (or I^m) and taking the value 0 on degenerate ones. Then there is a unique extension $\tilde{\varphi}$ to arbitrary m-rectangles on R^m (or I^m) which is an <u>additive rectangle function</u> in the sense that, for rectangles c, c_1, c_2,

$$c \sim c_1 + c_2 \quad \text{implies} \quad \tilde{\varphi}(c) = \tilde{\varphi}(c_1) + \tilde{\varphi}(c_2).$$

<u>Proof</u>. We first notice that any additive rectangle function is 0 on degenerate rectangles. Let $[a_1,b_1]\times\dots\times[a_n,b_n]$ (with $a_1 \leq b_1,\dots,a_m \leq b_m$) be an arbitrary n-rectangle on R^n. Since R is of line type, it satisfies the conditions of Proposition 1.2. So since φ is 0 on degenerate D-small rectangles, there is a unique function

$$f:|[a_1,b_1]|\times\dots\times|[a_n,b_n]| \to R$$

such that

$$d_1\dots d_n\cdot f(x_1,\dots,x_n) = \varphi([x_1,x_1+d_1]\times\dots\times[x_n,x_n+d_n])$$

for all $(d_1,\dots,d_n) \in D^n$ and $a_1 \leq x_1 \leq b_1,\dots,a_n \leq x_n \leq b_n$.

We now define $\tilde{\varphi}$ by

$$\tilde{\varphi}([a_1,b_1]\times\dots\times[a_n,b_n]) = \int_{a_1}^{b_1}\left(\dots\left(\int_{a_n}^{b_n} f(x_1,\dots,x_n)\,dx_n\right)\dots\right)dx_1.$$

We notice that $\tilde{\varphi}$ may be extended uniquely, by R-linearity, to the free R-module $C_n'(R^n) \subseteq CI_n(R^n)$ generated by the rectangles (rather than by the arbitrary singular n-cubes).

The additivity of $\tilde{\varphi}$ is an obvious consequence of

<u>Proposition 4.3</u>. Assume that $c \sim c'$ with $c, c' \in C'_n(R^n)$. Then $\tilde{\varphi}(c) = \tilde{\varphi}(c')$.

<u>Proof</u>. Immediate from the definition of $\sim$ and the properties of $\int$: for the clause (i) of the definition of $\sim$, this is just (4.2), whereas for (ii) we use (4.1); for (iii), we use (4.6).

The uniqueness of the extension follows from the following Lemma, easily proved by using "the fundamental theorem of calculus" (4.4).

<u>Lemma 4.4</u>. Let $g : |[a_1,b_1]| \times \ldots \times |[a_n,b_n]| \to R$ be such that $g(x_1,\ldots,a_i,\ldots,x_n) = 0$ for all $1 \le i \le n$ and all $a_1 \le x_1 \le b_1, \ldots, a_n \le x_n \le b_n$. Then

$$g(x_1,\ldots,x_n) = \int_{a_1}^{x_1}(\ldots(\int_{a_n}^{x_n} \frac{\partial^n g}{\partial t_1 \ldots \partial t_n} dt_n)\ldots)dt_1$$

Now suppose ψ_1 and ψ_2 are two additive rectangle functions extending φ. The difference $\psi = \psi_1 - \psi_2$ is still additive, and is 0 on D-small rectangles. We claim that ψ is 0.

The function

$$g(x_1,\ldots,x_n) = \psi([a_1,x_1] \times \ldots \times [a_n,x_n])$$

satisfies the hypothesis of Lemma 4.4 (since additive functions are 0 on degenerate rectangles). By developing g in Taylor series, and using additivity of ψ, we obtain for any $d_1,\ldots,d_n \in D$:

$$d_1 \cdot \ldots \cdot d_n \cdot \frac{\partial^n g}{\partial x_1 \ldots \partial x_n} = \psi([x_1,x_1+d_1] \times \ldots \times [x_n,x_n+d_n]) = 0$$

the rectangle occurring here being D-small. Since this holds for all $(d_1,\ldots,d_n) \in D^n$, we conclude

$$\frac{\partial^n g}{\partial x_1 \ldots \partial x_n} \equiv 0.$$

So by Lemma 4.4, $g \equiv 0$, and hence also $\psi \equiv 0$. This proves the uniqueness.

We finally remark that the content of the present §4 as well as the next immediately generalizes to functions with values in an arbitrary Euclidean module V instead of R provided V satisfies an Axiom analogous to Axiom 4.1, for maps $[a,b] \to V$.

§5. Integration of forms against finite chains

Assume that ω is an n-form on M, and $\tau : I^n \to M$ a singular n-cube on M. Then $\tau^*\omega$ is an n-form on I^n, and therefore by §2 of form

$$\tau^*\omega = f \cdot dx_1 \wedge \ldots \wedge dx_n$$

for some unique $f : I^n \to R$. We define

$$\int_\tau \omega := \int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

and extend by linearity the definition of $\int_\tau \omega$ to arbitrary chains $\tau \in CI_n(M)$. Clearly, $\int_\tau \omega$ depends also linearly on ω. Also, (3.3) holds, as is seen immediately from the definition.

For the particular case where $M = R^n$ or I^n, we have

Proposition 5.1. For ω an n-form on R^n or I^n, $\int_{-} \omega$ defines, by restriction to the set of rectangles, an additive rectangle function in the sense of §4.

Proof. Write $\omega = f \cdot dx_1 \wedge \ldots \wedge dx_n$. For the rectangle

$$\rho = [a_1, b_1] \times \ldots \times [a_n, b_n] : I^n \to R^n$$

we then have

$$\rho^*\omega = \prod_{i=1}^{n} (b_i - a_i) \cdot (f \circ \rho) \cdot dx_1 \wedge \ldots \wedge dx_n \tag{5.1}$$

so that

$$\int_\rho \omega = \int_0^1 \ldots \int_0^1 \Pi (b_i - a_i) \cdot f(\rho(x_1, \ldots, x_n))\, dx_1 \ldots dx_n$$
$$= \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n,$$

by n-fold application of the substitution rule ((1.5) in [9]. Now the additivity of $\int_{-} \omega$ follows from (4.1), (4.2) and (4.6) just as in the proof of Proposition 4.3.

Let ω be an n-form on I^n, and consider the D-small rectangle

$$\rho = [a_1, a_1 + d_1] \times \ldots \times [a_n, a_n + d_n]$$

given by $(\underline{a}, \underline{d}) \in I^n \times D^n$. Consider also the infinitesimal singular n-cube $\langle t_{\underline{a}}, \underline{d} \rangle$ on I^n (where $t_{\underline{a}}(\delta) = \underline{a} + \underline{\delta}$ for any $\underline{\delta} \in D^n$). We can now state

Proposition 5.2. Infinitesimal integration of ω as introduced in §1 agrees with the integration of ω against D-small rectangles, i.e., with the notation just introduced

$$\int_\rho \omega = \int_{\langle t_{\underline{a}}, \underline{d} \rangle} \omega .$$

Proof. Let $\omega = f\, dx_1 \wedge \ldots \wedge dx_n$. Then $\rho^*\omega$ is, in analogy with (5.1), given by (5.1) with $b_i - a_i = d_i$, and as in the calculation above, we get

$$\int_\rho \omega = \int_{a_1}^{a_1 + d_1} \ldots \int_{a_n}^{a_n + d_n} f(t_1, \ldots, t_n)\, dt_1 \cdot \ldots \cdot dt_n .$$

But generally $\int_a^{a+d} f(t)\,dt = d\cdot f(a)$, as can be seen immediately from the definition of integration (§4) (or from (4.3), say). Applying this principle n times to our expression yields $\Pi d_i \cdot f(g)$.

On the other hand

$$\int_{<t_{\underline{a}},\underline{d}>} \omega = \Pi d_i \cdot \omega(t_{\underline{a}}).$$

But to say $\omega = f\cdot dx_1\wedge\ldots\wedge dx_n$ is equivalent to saying $\omega(t_{\underline{a}}) = f(\underline{a})\ \forall \underline{a}$. This proves Proposition 5.2 (= formula $(3.4)_n$).

We can now prove Stokes' theorem for arbitrary singular chains. Let M be an arbitrary object.

<u>Theorem 5.3</u>. (Stokes). If σ is a singular n-chain on M, and ω is an $(n-1)$-form on M, we have

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega.$$

<u>Proof</u>. It suffices to prove this for the case where σ is a singular <u>cube</u> $I^n \overset{\sigma}{\to} M$. It suffices to prove that for an arbitrary rectangle $\rho: I^n \to I^n$

$$(5.2)\qquad \int_{\partial(\sigma\circ\rho)} \omega = \int_{\sigma\circ\rho} d\omega$$

since the identity map on I^n appears as a particular rectangle. Now consider both sides of (5.2) as rectangle functions, i.e. in their dependence of ρ. By the uniqueness assertion in Proposition 4.2, it suffices to see that they are both <u>additive</u> rectangle functions, and that they agree on <u>D-small</u> rectangles.

Now if $\rho \sim \rho_1 + \rho_2$, then

$$\int_{\partial(\sigma\circ\rho)} \omega = \int_{\sigma_*(\partial\rho)} \omega = \int_{\partial\rho} \sigma^*\omega$$

and

$$\int_{\partial(\sigma\circ\rho_1)} \omega + \int_{\partial(\sigma\circ\rho_2)} \omega = \int_{\partial\rho_1} \sigma^*\omega + \int_{\partial\rho_2} \sigma^*\omega$$

but these two expressions agree since $\partial\rho = \partial\rho_1 + \partial\rho_2$ by (3.9).

To see the right hand side of (5.2) an additive rectangle function is immediate from Proposition (5.1) applied to $\rho^*(d\omega)$.

So all that remains is to see that the two sides in Stokes' Theorem agree on D-small rectangles σ,

$$\sigma = [a_1, a_1+d_1]\times\ldots\times[a_n, a_n+d_n].$$

But

$$(5.3)\qquad \int_{\partial(\rho\circ\sigma)} \omega = \int_{\partial\sigma} \rho^*\omega = \int_{\partial<t_{\underline{a}},\underline{d}>} \rho^*\omega$$

by (3.5). On the other hand

$$(5.4)\qquad \int_{\rho\circ\sigma} d\omega = \int_\sigma d(\rho^*\omega) = \int_{<t_{\underline{a}},\underline{d}>} d(\rho^*\omega)$$

by $(3.4)_n$ = Proposition 5.2, and functorality of exterior derivation of forms. But the right hand sides of (5.3) and (5.4) and equal by Theorem 1.1, the infinitesimal form of Stokes' Theorem.

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