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PRINCIPAL BUNDLES, GROUPOIDS, AND CONNECTIONS

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Abstract. We clarify in which precise sense the theory of principal bundles and the theory of groupoids are equivalent; and how this equivalence of theories, in the differentiable case, reflects itself in the theory of connections. The method used is that of synthetic differential geometry.

Introduction. In this note, we make explicit a sense in which the theory of principal fibre bundles is equivalent to the theory of groupoids; and in particular, how the differential geometric notion of connection appears in this equivalence. For the latter, we shall utilize the method of Synthetic Differential Geometry, which has for its base the notion of "first neighbourhood of the diagonal" of a manifold.

Basically, the notions of "principal bundle" and "groupoid", their essential equivalence, and the notion of connection in this context, were described by Ehresmann, [4], [5] etc.

It is classical to formulate the notion of electromagnetic field in terms of a connection in a principal G-bundle, with G the (abelian) group $U(1)$. Recent particle physics is extending this "gauge theory" viewpoint to non-commutative G, and also to higher "connective structures", see e.g. [3] and references there. To cope with the mathematical complications arising in this extension process, Breen and Messing [2], [3] found it helpful to utilize some of the formal or "synthetic" method elaborated in the last decades (as in e.g. [7]). The present note hopefully also provides a contribution to such "synthetic gauge theory", by combining it more firmly with the theory of groupoids.

The main vehicle for the relationship between principal bundles and groupoids is a functor which to a principal bundle P associates a transitive groupoid PP^{-1}; this "Ehresmann functor" (terminology of Pradines [20]) is however, as pointed out in loc. cit., not an equivalence of categories in the sense of category theory, since it is not

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even faithful. We provide in Section 3 a faithful extension E of the Ehresmann functor. This E is *the enveloping groupoid of a pregroupoid*; the notion of pregroupoid is a slight (and purely algebraic) extension of the notion of principal bundle, and we begin with an exposition of it. (It is also known under other names, e.g. *affinoid*, cf. [23] (Weinstein).)

Sections 5–7 deal with the theory of connections, their connection forms, and curvature; unlike a classical treatment, like [16], say, the method of synthetic differential geometry bypasses the need for linearization (Lie groupoid into a Lie algebroid, principal bundle into an Atiyah sequence, ...). In this sense, the notions related to connections become truly "combinatorial" or "synthetic".

The text is written in a "synthetic" style; this essentially means that one speaks about the spaces (manifolds), (including quotient spaces for equivalence relations, say) *as if* they were just sets; and leaves the construction of the relevant smooth structure to general categorical principles – principles that may be axiomatized for instance by singling out in the category D of "spaces", a class S of good epimorphisms, typically the surjective submersions, and a class I if good monos, together satisfying certain exactness properties (like a Godement Theorem). Such exactness properties have been axiomatized by the Pradines notion of "Diptych", cf. [20], [21]. Other ways of encoding the relevant exactness go via Dubuc's notion of *well adapted topos model* for differential geometry, cf. e.g. [7]. Pradines [20] formulates the method as follows:

> "decrire les constructions algébrique au moyen de diagrammes de $\underline{Ens}$, mettre en évidence les injections et surjections, puis réécrire ces diagrammes dans $(D; I, S)$".

Thus, the words "set" and "manifold" become almost synonyms! For Sections 5–7, we need an extended version of this principle: even a somewhat abstract object (scheme) as the first neighbourhood of the diagonal $M_{(1)}$ of a manifold M is treated *as if* it were just a *set*, even though its *set* of points is the same as the *set* of points of M itself. For this extended use of the synthetic method, the topos approach provides a complete justification, cf. e.g. [7]. The point, however, is: the set theoretic descriptions are good enough to convey the idea about the essence of the constructions and the geometry.

Some of the notions, paraphrasings and results presented here, I have presented, in scattered form, in some previous prepublications and publications. Thus the notion of pregroupoid was presented in [10] (and the special case relevant for principal fibre bundles already in [9]). The construction of the enveloping groupoid of a pregroupoid was presented in [15], and in a rudimentary form in [13] (an expanded version of this is [14]). The preprint [13] (and also [14]) contains the material presented in Sections 4–7. On the other hand, [13], [15], and [14] (except the last section) are subsumed in the present note.

I would like to thank the organizers of the "Geometry and Topology of Manifolds" conferences for inviting me to present this and related material to an audience of differential geometers.

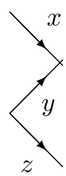
1. Equational theory of pregroupoids. We consider a groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$ (so G_1 is the set of arrows, G_0 the set of objects of $\mathbf{G}$). For any two subsets $A \subseteq G_0$ and $B \subseteq G_0$, we let $\mathbf{G}(A, B)$ denote the set of arrows $\in G_1$ whose domain is in A and whose codomain is in B. If $A = B$, this carries structure of groupoid, the full subgroupoid on A, which thus here is denoted $\mathbf{G}(A, A)$.

There are evident book-keeping maps $d_0 : \mathbf{G}(A, B) \to A$ ("domain") and $d_1 : \mathbf{G}(A, B) \to B$ ("codomain").

In $\mathbf{G}$, we compose from left to right, denoting composition by $\circ$. Then composition in $\mathbf{G}$, together with the book-keeping maps, provide $X = \mathbf{G}(A, B)$ with a certain partially defined algebraic structure: a ternary operation denoted $xy^{-1}z$, defined whenever $\beta(x) = \beta(y)$ and $\alpha(y) = \alpha(z)$, (and then $\alpha(xy^{-1}z) = \alpha(x)$ and $\beta(xy^{-1}z) = \beta(z)$). Namely

$$xy^{-1}z := x \circ y^{-1} \circ z.$$

The reader may find the following display useful. The vertices (objects) are elements of A and B, respectively (with A-objects in the left hand column, B-objects in the right hand column).



The following equations trivially hold for this ternary operation (whenever the expressions are defined):

$$xx^{-1}z = z, \tag{1}$$

$$xy^{-1}y = x, \tag{2}$$

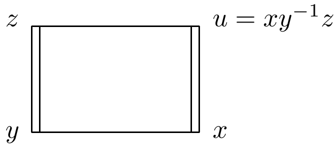
("unit laws"),

$$vx^{-1}(xy^{-1}z) = vy^{-1}z, \tag{3}$$

$$(xy^{-1}z)z^{-1}w = xy^{-1}w, \tag{4}$$

("concatenation laws"). The reason for the latter name is motivated by the following diagrammatic device (also used in [10]):

We indicate the assertion that $u = xy^{-1}z$ by a diagram



(5)

Here, the single lines connect elements in X which have same codomain, double lines connect elements with same domain. Quadrangles that arise in this way, $u = xy^{-1}z$, we shall call *parallelograms*, and (3) (resp. (4)) then expresses that parallelograms may be concatenated horizontally (resp. vertically). The display of (3) in terms of quadrangles in fact is

$$
\begin{tikzcd}
z \arrow[r, dash] \arrow[d, equal] & u = xy^{-1}z \arrow[r, dash] \arrow[d, equal] & vy^{-1}z \arrow[d, equal] \\
y \arrow[r, dash] & x \arrow[r, dash] & v
\end{tikzcd}
\tag{6}
$$

Definition 1.1. A *pregroupoid* ("on A, B") is set X equipped with surjections $\alpha : X \to A$, $\beta : X \to B$ and with a partially defined ternary operation, denoted $xy^{-1}z$, defined whenever $\beta(x) = \beta(y)$ and $\alpha(y) = \alpha(z)$; and then $\alpha(xy^{-1}z) = \alpha(x)$ and $\beta(xy^{-1}z) = \beta(z)$; and the equations (1), (2), (3), (4) are supposed to hold.

(In [10], essentially the same notion was considered, from a less equational viewpoint. It has later been considered by Johnstone [6] under the name *herdoid* and by Weinstein under the name *affinoid*; it is closely related to Pradines' notion of *butterfly diagram*, [19], [20], [21].)

Example. If $\pi : P \to A$ is a principal G bundle (i.e. the group G acts on the right on P, the action is free, and the orbit set is A), then P carries a natural structure of pregroupoid on the pair of sets A, 1 (with $\alpha : P \to A$ the given map π, and with the unique map $P \to 1$ as β). Namely, given x, y, z as in (5), with $\pi(y) = \pi(z)$, i.e. with y and z in the same orbit, there exists a unique $g \in G$ with $y \cdot g = z$; now take u in (5) to be $x \cdot g$. Then (1) and (3) are trivial, whereas (2) and (4) follow from the unit- and associative law for the G-action. Let us as a sample prove (4). The expressions in (4) are defined when y, z, and w are in the same orbit. Let $y \cdot g = z$, and let $z \cdot h = w$, with g and h in G. Since $y \cdot (gh) = w$, by the associative law for the G-action, the right hand side of (4) is $x \cdot (gh)$. We have $xy^{-1}z = x \cdot g$, by definition, say $= u$, and so the left hand side of (4) is $u \cdot h$. But

$$u \cdot h = (x \cdot g) \cdot h = x \cdot (gh),$$

again by the associative law for the G-action.

Of course, this calculation becomes entirely trivial, once the theorem about enveloping groupoids is proved, since then it is a calculation in a groupoid, so that the occurrences of z and z^{-1} in the middle of the left hand side of (2) cancel.

Equational deductions from the defining equations are somewhat cumbersome; luckily, once they have been carried to the point of proving the existence of an enveloping groupoid for a pregroupoid (in the form of Theorem 3.1, say), further calculations become superfluous, since they then reduce to calculations in a groupoid, meaning calculations with a *binary* operation (namely composition of arrows), which is furthermore associative and with units and inverses. We shall not present the calculations leading to the existence of enveloping groupoid here; the full details may be found in [15]. Suffice it to say that the defining equations (1)–(4) imply that the set of parallelograms (i.e. diagrams coming about like (5)) are stable under horizontal concatenation (like in (6)), and also under vertical concatenation. Also horizontal and vertical reflection of a parallelogram is a parallelogram. (Reflection of a parallelogram in its diagonal will in general not be a parallelogram; that is a commutativity condition. This is the reason for using different display for the horizontal and vertical edges in (5).)

Since the notion of pregroupoid is purely algebraic (except for the surjectivity requirement for α and β), it is clear how to organize pregroupoids into a category: if

$$A \xleftarrow{\alpha} X \xrightarrow{\beta} B$$

and

$$A' \xleftarrow{\alpha'} X' \xrightarrow{\beta'} B'$$

are pregroupoids, a morphism ξ from the first to the second consists of maps $\xi_0 : A \to A'$, $\xi_1 : B \to B'$ and $\xi : X \to X'$ commuting with the structural maps α and β and preserving the ternary operation. It is usually harmless to omit the subscripts and just write ξ for all three maps in question.

There is an evident "forgetful" or "underlying" functor

$$\text{groupoids} \;\to\; \text{pregroupoids}$$

taking the groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$ to the pregroupoid $\mathbf{G}(G_0, G_0)$, (so the ternary operation $xy^{-1}z$ is given by $x \circ y^{-1} \circ z$).

2. Groupoids from pregroupoids. To every pregroupoid P, there is associated two groupoids PP^{-1} and $P^{-1}P$; for the case of a principal G-bundle P, PP^{-1} is the "Ehresmann groupoid" of P, as alluded to in the introduction, (and $P^{-1}P$ is the group G). We shall recall these constructions in pregroupoid terms (cf. also [10]). We shall also describe a natural third groupoid $E(P)$, the *"enveloping groupoid* of P", [15]; it solves a universal problem, and it contains PP^{-1} and $P^{-1}P$ as full subgroupoids. Specifically, let P be a pregroupoid on A, B. The groupoid PP^{-1} is a groupoid with A as its set of objects; an arrow from $a \in A$ to $a' \in A$ is represented by a pair (x, y) of elements in P over a and a' (i.e. $\alpha(x) = a, \alpha(y) = a'$) respectively, and with $\beta(x) = \beta(y)$; and the pair (x, y) represents the same arrow as (u, z) if y, x, z, u form a parallelogram, as in (5). The arrow represented by the pair (x, y) is denoted xy^{-1}. So with y, x, z, u as in (5), $xy^{-1} = uz^{-1} \in PP^{-1}$. Arrows compose by $(x, y) \circ (y, v) := (x, v)$, more generally $(x, y) \circ (z, v) := (xy^{-1}z, v)$, and this is well defined because parallelograms concatenate. The notation makes the law for composition look almost tautological:

$$xy^{-1} \circ zv^{-1} := (xy^{-1}z)v^{-1}.$$

Identities are represented by trivial pairs (x, x) (so an identity at $a \in A$ exists because we assumed that $\alpha : P \to A$ was surjective); and inversion takes (x, y) to (y, x) and is well defined because parallelograms may be reflected in a vertical axis.

Note that this construction is a generalization of the familiar construction of a vector space out of an affine space: a vector is an equivalence class of pairs of points, the equivalence relation on the set of pairs being given by the (affine) notion of parallelogram. (The formula describing the composition structure of this groupoid is repeated in the table (7) below.)

Similarly, the groupoid $P^{-1}P$ is a groupoid with B as its set of objects; an arrow from $b \in B$ to $b' \in B$ is represented by a pair (y, z) of elements in P over b and b', (i.e. $\beta(y) = b, \beta(z) = b'$) respectively, and with $\alpha(y) = \alpha(z)$; and the pair (y, z) represents the same arrow as (x, u) if y, x, z, u form a parallelogram, as in (5). The arrow represented by the pair (y, z) is denoted $y^{-1}z$. Composition is then given by

$$y^{-1}z \circ v^{-1}w := y^{-1}(zv^{-1}w).$$

Identities are given by pairs (y, y), and they exist because β was assumed surjective. Inversion takes (y, z) to (z, y).

For the case where P is a principal G-bundle on A, PP^{-1} is the Ehresmann groupoid (a groupoid with A as set of objects), whereas $P^{-1}P$ is (canonically isomorphic to) the group G (a groupoid with 1 as set of objects): $y^{-1}z$ being identified with the unique $g \in G$ with $y \cdot g = z$. – I believe these "conjugate" roles of PP^{-1} and $P^{-1}P$ was first observed by Pradines in [19]; see also [10]. These two groupoids (for a general pregroupoid P) are sometimes called the *edge* groupoids of P, rather than the Ehresmann groupoids, because they appear as the two edges of a certain bisimplicial set.

The edge groupoids of P *act* on P, from the left and the right, respectively, and these actions are principal, and commute with each other, cf. [10]. Their description will be given in the next Section (table (7)), as part of the description of the enveloping groupoid.

3. The enveloping groupoid of a pregroupoid. We first describe the enveloping groupoid, then we describe the universal problem which it solves. Consider a pregroupoid P on A, B, so there are structural maps $\alpha : P \to A$, $\beta : P \to B$, and a ternary operation ("rule of three") $xy^{-1}z$, as in Section 1, with the book-keeping conditions as stated there. The enveloping groupoid $E = E(P)$ will have the disjoint union $A + B$ of the sets A and B for its set of objects. Its set of arrows must therefore be a disjoint union of four sets,

$$E(A, A), E(B, B), E(A, B), E(B, A),$$

where $E(A, B)$ denotes the set of those arrows in $E(P)$ whose domain is an element in A and whose codomain is an element in B, – and similarly for the three other cases. The three first of these sets we already have:

$$\begin{aligned} E(A, A) &:= PP^{-1}, \\ E(B, B) &:= P^{-1}P, \\ E(A, B) &:= P \end{aligned}$$

and we get the remaining fourth by a "rule of three on a higher level", namely by taking a new copy of P itself; this new copy we denote P^{-1}, so

$$E(B, A) := P^{-1}.$$

(Here, $P^{-1}P$ denotes just the set of *arrows* of the groupoid $P^{-1}P$, and similarly for PP^{-1}.) In so far as the book-keeping maps d_0 and d_1 (domain and codomain) are concerned, we already have them for $P^{-1}P$ and PP^{-1}, as part of the description of these groupoids; for P, they are the α and β given as part of the pregroupoid structure on P. Finally, for P^{-1}, we interchange the α and β; more precisely, if $x \in P$ is considered as an element in P^{-1}, we denote it x^{-1}, and we put $d_0(x^{-1}) := \beta(x) \in B$, $d_1(x^{-1}) := \alpha(x) \in A$. For the compositions, we have already given the compositions in PP^{-1} and $P^{-1}P$ (which appear as full subgroupoids of E), but for completeness, we collect all the possible compositions in a table (7) below. First, we give a description of the d_0 and d_1 of $E(P)$ in terms of α and β.

$$\begin{array}{lll}
E(A,A) := PP^{-1}; & d_0(xy^{-1}) := \alpha(x), & d_1(xy^{-1}) := \alpha(y), \\
E(B,B) := P^{-1}P; & d_0(y^{-1}z) := \beta(y), & d_1(y^{-1}z) := \beta(z), \\
E(A,B) := P; & d_0(x) := \alpha(x), & d_1(x) := \beta(x), \\
E(B,A) := P^{-1}; & d_0(x^{-1}) := \beta(x), & d_1(x^{-1}) := \alpha(x).
\end{array}$$

Here is the description of the composition in the form of a multiplication table. We compose from left to right; the "type" of the left hand factor is listed in the column on the left, the type of the right hand factor in the row on the top. (The reader may keep track on the book-keeping conditions by making a display of the five elements x, y, z, v, w in the form of a "zig-zag", by extending the zig-zag figure in Section 1, adjoining v and w at the bottom.)

	$A \to A$	$A \to B$	$B \to A$	$B \to B$
$A \to A$	$xy^{-1} \circ zv^{-1}$ $:= (xy^{-1}z)v^{-1}$	$xy^{-1} \circ z$ $:= xy^{-1}z$		
$A \to B$			$x \circ y^{-1}$ $:= xy^{-1}$	$x \circ y^{-1}z$ $:= xy^{-1}z$
$B \to A$	$z^{-1} \circ yx^{-1}$ $:= (xy^{-1}z)^{-1}$	$y^{-1} \circ z$ $:= y^{-1}z$		
$B \to B$			$z^{-1}y \circ x^{-1}$ $:= (xy^{-1}z)^{-1}$	$y^{-1}z \circ v^{-1}w$ $:= y^{-1}(zv^{-1}w)$

(7)

If we label the entries in this matrix in the standard way known from linear algebra, the entry (1,1) describes the composition in the Ehresmann groupoid PP^{-1}, entry (1,2) describes the (left) action of PP^{-1} on P, and similarly (2,4) describes the right action of $P^{-1}P$ on P. Note that if we combine (2,3) with (1,2), we get

$$(x \circ y^{-1}) \circ z = xy^{-1}z. \tag{8}$$

The proof that the composition thus defined in $E(P)$ is in fact associative is a purely equational exercise (not quite trivial); the sceptical reader may find these calculations in [15]. Existence of units is easy, and existence of inverses is almost tautological (see [15]). Thus we have a groupoid $E = E(P)$ whose object set is $A + B$, and where the full subgroupoid given by $A \subseteq A + B$ is PP^{-1}, and the full subgroupoid given by $B \subseteq A + B$ is $P^{-1}P$; P itself appears as the set of arrows from elements in A to elements in B. Thus every pregroupoid P comes about from a groupoid, as in the example given in the introduction to Section 1. Let us summarize

THEOREM 3.1. *To every pregroupoid P on A, B, there exists a groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$ with composition $\circ$, with A and B subsets of the set G_0, such that P is the set of arrows* *from elements in A to elements in B, and such that $xy^{-1}z = x \circ y^{-1} \circ z$, whenever this makes sense.*

Now a principal bundle $P \to A$ (with group G, say) may be seen as a pregroupoid on $A, 1$. We therefore we get as a Corollary the following result, which will be the main vehicle for the connection theory in Sections 5 and 7. (As before, 1 denotes the one-point set; the only point of it we denote $*$.)

THEOREM 3.2. *For every principal G-bundle $P \to A$, there exists a groupoid E with $A+1$ as object set, and with P itself the set of arrows from points a of A to $*$; G appears as $\underline{Hom}_E(*, *)$, acting on P by post-composition.*

(Recall that we compose from left to right.) An arrow $a \to *$ may be thought of as a *(co-)frame* at a (an isomorphism of a with the "standard" object $*$). Thus, the Theorem may also be stated succinctly: *every principal bundle is a frame bundle*.

With the category of pregroupoids as described in Section 1, it is clear that each of the constructions $P \mapsto PP^{-1}$, $P \mapsto P^{-1}P$, and $P \mapsto E(P)$ are in fact functors from the category of pregroupoids to the category of groupoids. A rationale for considering $E(P)$ is that it solves a universal problem:

THEOREM 3.3. *The functor $E(-)$ from pregroupoids to groupoids is left adjoint to the "forgetful" functor from groupoids to pregroupoids described in Section 1; E is a faithful functor, and the unit (front adjunction) for the adjointness is injective.*

Proof. Consider a pregroupoid on A, B. The set of objects of $E(P)$ is $A + B$, and the set of arrows with domain in A and codomain in B is, by construction, just the set P, so we have an injective mapping η from P to (the set of arrows of) $E(P)$. This mapping is a morphism of pregroupoids into the underlying pregroupoid of $E(P)$; this follows from $x \circ y^{-1} \circ z = xy^{-1}z$ (cf. (8)) (the book-keeping compatibility trivially holds, by the construction). So it just remains to check the universal property of η. A pregroupoid homomorphism from $A \leftarrow P \to B$ into the underlying pregroupoid of a groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$ consists of $\phi_0 : A \to G_0$, $\phi_1 : B \to G_0$, and a map $\phi : X \to G_1$. The maps ϕ_0 and ϕ_1 together define a map $A + B \to G_0$, which is the object part of the desired functor $\overline{\phi} : E(P) \to \mathbf{G}$. The value of $\overline{\phi}$ on arrows in P is forced to be $\overline{\phi}(x) = x$ (since we require $\overline{\phi}$ composed with η to give ϕ), and then the remaining three cases are also forced if we want $\overline{\phi}$ to be a functor:

$$\overline{\phi}(x^{-1}) := (\phi(x))^{-1}, \quad \overline{\phi}(yz^{-1}) := \phi(y) \circ \phi(z)^{-1}, \quad \overline{\phi}(xy^{-1}) := \phi(x) \circ \phi(y)^{-1},$$

and it is clear from the defining formulas (from the table) that $\overline{\phi}$ preserves composition, and also clearly identities. This proves the Theorem. Faithfulness of E follows because η is injective.

(Note that P likewise embeds in PP^{-1}, but not *naturally*: such embedding requires *choice* of a "base point" x_0 in P ; then $y \in P$ gives rise to an arrow x_0y^{-1} in PP^{-1}.)

Because P embeds *naturally* into the groupoid $E(P)$, we have proposed the name *enveloping groupoid of P* for it. (In [13], we called it the *comprehensive groupoid.*) It is analogous to the enveloping associative algebra of a Lie algebra in the sense that all equations concerning the ternary operation $xy^{-1}z$ can be checked under the assumption that $xy^{-1}z$ is actually $x \circ y^{-1} \circ z$ for the (associative) composition $\circ$ of a groupoid.

4. Gauge group bundle. For any groupoid $\Psi \rightrightarrows M$ with object set M, there is a group bundle on M, namely

$$\Sigma_{a \in M} \Psi(a, a) \to M,$$

sometimes ([16]) called the *gauge group bundle* of Ψ, gauge(Ψ). It carries a right conjugation action by Ψ: if $\psi : a \to b$ in Ψ, and $h \in \Psi(a, a)$, then $h^{\psi} := \psi^{-1} \circ h \circ \psi \in \Psi(b, b)$. In particular, if P is a principal G-bundle on M, the group bundle on M, defined in this way from the groupoid $PP^{-1} \rightrightarrows M$, is sometimes called the *adjoint bundle* of P, ad(P) or gauge(P).

In the case where $P^{-1}P = G$ is commutative, gauge(P) $\to M$ may canonically be identified with the constant group bundle $M \times G \to M$ (with trivial PP^{-1}-action): an element of gauge(P) over $a \in M$ is given by a fraction

$$xy^{-1} \in PP^{-1}(a, a)$$

with x and y both in P_a. But then also $y^{-1}x \in P^{-1}P$ makes sense, and the process $xy^{-1} \mapsto y^{-1}x$ is well defined if G is commutative; the identification of gauge(P) with $M \times G$ is thus

$$xy^{-1} \mapsto (a, y^{-1}x) \in M \times G,$$

for $x, y \in P_a$.

5. Connections versus connection forms. Consider a principal bundle $\pi : P \to M$, with group G, as above. We shall assume that M and P are equipped with reflexive symmetric relations $\sim$, called the *neighbour* relation. The set of pairs $(x, y) \in M \times M$ with $x \sim y$ is a subset $M_{(1)} \subseteq M \times M$, called the *first neighbourhood of the diagonal*, and similarly for $P_{(1)} \subseteq P \times P$. We assume that $\pi : P \to M$ preserves the relation $\sim$, and also that π is a "submersion" in the sense that if $a \sim b$ in M, and $\pi(x) = a$, then there exists a $y \sim x$ in P with $\pi(y) = b$. In fact, we assume that for any "infinitesimal k-simplex" $a_0, \ldots, a_k$ in M (meaning a $k+1$-tuple of mutual neighbours), and for any $x_0 \in P$ above a_0, there exists an infinitesimal k-simplex $x_0, \ldots, x_k$ in P (with the given first vertex x_0) which by π maps to $a_0, \ldots, a_k$. Finally. the action of any $g \in G$ on P is assumed to preserve the relation $\sim$ on P.

This is motivated by Synthetic Differential Geometry (SDG), cf. [7], and more recently [12] and [3], where the notion of connection (infinitesimal parallel transport) and differential form is elaborated in these terms. In particular, let us remind the reader how the notion of "differential form on a manifold M with values in the Lie algebra of a Lie group G" is paraphrased synthetically. So let M be a set equipped with a (reflexive symmetric) neighbour relation $\sim$, as above, and G is a group. Then a k-form on M with values in G is a law ω which to any infinitesimal k-simplex $a_0, \ldots, a_k$ in M associates an element $\omega(a_0, \ldots, a_k)$ in G, and associates the unit $e \in G$ to any degenerate simplex (meaning one in which two entries are equal). In the context of SDG, it *follows* that such ω is alternating, meaning that if the a_i's are permuted, $\omega(a_0, \ldots, a_k) \in G$ is unchanged if the permutation is even, and changed into its (multiplicative) inverse, if the permutation is odd. In particular, for G-valued 1-forms, $\omega(y,x) = \omega(x,y)^{-1}$. (In Section 6, we shall consider forms with values in a group *bundle*.) Since π preserves the relation $\sim$, any form ω on M gives rise to a form $\pi^*(\omega)$ on P, $\pi^*(\omega)(x_0,\dots,x_k) := \omega(\pi(x_0),\dots,\pi(x_k))$.

The groupoid viewpoint for connections is in essence due to Ehresmann. In SDG, this connection notion becomes paraphrased (see [11] or [12], Section 8): for a groupoid $\Phi \rightrightarrows M$, a connection in it is just a map $\nabla : M_{(1)} \to \Phi$ of reflexive symmetric graphs over M, so for $a \sim b \in M$, $\nabla(a,b)$ is an arrow $a \to b$ in Φ. (This geometric/groupoid theoretic connection notion is related to the notion of "direct (or quasi-)connection" of Teleman, cf. [22].)

If $P \to M$ is a principal G-bundle, a connection in the groupoid $PP^{-1} \rightrightarrows M$ is sometimes called a *principal* connection in P. Arrows in $PP^{-1}(a,b)$ may be identified with right G-equivariant maps $P_b \to P_a$: to an arrow $n : a \to b$, associate its (left) action $n \cdot -$, as described in Section 3 (entry (1,2) in the matrix (7)).

Let $\pi : P \to M$ be a principal fibre bundle. To any connection ∇ in the groupoid PP^{-1} (i.e. to any principal connection), one may associate a 1-form ω on P with values in the group $G = P^{-1}P$, as follows. For u and v neighbours in P, with $\pi(u) = a$, $\pi(v) = b$, put

$$\omega(u,v) := u^{-1}(\nabla(a,b) \cdot v). \tag{9}$$

Note that both u and $\nabla(a,b)\cdot v$ are in the π-fibre over a, so that the "fraction" $u^{-1}(\nabla(a,b)\cdot v)$ makes sense as an element of $P^{-1}P$.

The defining equation is equivalent to

$$u \cdot \underbrace{\omega(u,v)}_{\in P^{-1}P} = \underbrace{\nabla(\pi(u),\pi(v))}_{\in PP^{-1}} \cdot v. \tag{10}$$

Let us agree that (for u, v in P a pair of neighbours in P) $\nabla(u,v)$ denotes $\nabla(\pi(u),\pi(v))$. Then the equation (10) may be written

$$u \cdot \omega(u,v) = \nabla(u,v) \cdot v. \tag{11}$$

It is therefore possible to represent the relationship between ∇ and the associated ω by means of a simple figure:

$$\begin{tikzcd} \bullet & v \arrow[l, "\nabla(u,v)\cdot"'] \\ u \arrow[u, "\cdot\,\omega(u,v)"] & \end{tikzcd}$$

The figure reflects something geometric, namely that $\omega(u,v)$ acts inside the fibre ("vertically"), whereas the action of ∇ defines a notion of *horizontality*.

We have the following two equations for ω. First, let $x \sim y$ in P, and assume that g

has the property that also $x \cdot g \sim y$. Then

$$\omega(x \cdot g, y) = g^{-1}\omega(x, y). \tag{12}$$

Also, for $x \sim y$ and *any* $g \in G$

$$\omega(x \cdot g, y \cdot g) = g^{-1}\omega(x, y)g. \tag{13}$$

To prove (12), let us denote $\pi(x) = \pi(x \cdot g)$ by a and $\pi(y)$ by b. Then we have, using the defining equation (10) for ω twice,

$$x \cdot g \cdot \omega(x \cdot g, y) = \nabla(a, b) \cdot y = x \cdot \omega(x, y),$$

and now we may calculate in the enveloping groupoid of P: first cancel the x on the left, then multiply the equation by g^{-1} on the left. To prove (13), we have, with a and b as above,

$$x \cdot g \cdot \omega(x \cdot g, y \cdot g) = \nabla(a, b) \cdot y \cdot g = x \cdot \omega(x, y) \cdot g,$$

by the defining equation (10) for $\omega(x \cdot g, y \cdot g)$, and by (10) for $\omega(x, y)$, respectively. From this, we get the result by calculating in the enveloping groupoid: first cancelling x and then multiplying the equation by g^{-1} on the left.

It is easy to conclude from (12) that for $g \sim e$ in G (whence $x \sim x \cdot g$ in P),

$$\omega(x, x \cdot g) = g;$$

more generally, using also (13), if $h \sim g$ in G, we may prove

$$\omega(x \cdot h, x \cdot g) = h^{-1}g. \tag{14}$$

The following Proposition is now the rendering, in our context, of the relationship between a connection ∇ and its connection 1-form ω. (In Ehresmann's formulation, [4], ("Deuxième Définition") he seems to be able to replace (13) by the weaker (14) which is a synthetic paraphrasing of his (c') equation 2 from loc. cit.: $\omega(h(s + ds)) = s^{-1}(s + ds)$ (where s and $s + ds$ are neighbours in G; I haven't been able to make this replacement in the present context.)

Theorem 5.1. *The process $\nabla \mapsto \omega$ just described establishes a bijective corresondence between 1-forms ω on P, with values in the group $P^{-1}P$ and satisfying (12) and (13), and connections ∇ in the groupoid PP^{-1}.*

Proof. Given a 1-form ω satisfying (12) and (13), we construct a connection ∇ as follows. Let $a \sim b$ in M. To define the arrow $\nabla(a, b) : a \to b$ in PP^{-1}, pick $u \sim v$ above $a \sim b$, and put

$$\nabla(a, b) := u(v \cdot \omega(v, u))^{-1}.$$

We first argue that this is independent of the choice of v, once u is chosen. Replacing v by $v \cdot g \sim u$, we are in the situation where (12) may be applied; we get

$$u(v \cdot g \cdot \omega(v \cdot g, u))^{-1} = u(v \cdot g \cdot g^{-1}\omega(v, u))^{-1} = u(v \cdot \omega(v, u))^{-1};$$

the left hand side is $\nabla(a, b)$ defined using $u, v \cdot g$, the right hand side is using u, v.

To prove independence of choice of u: any other choice is of form $u \cdot g$ for some $g \in G$. For our new v, we now chose $v \cdot g$ (the result will not depend on the choice, by the argument just given). Again we calculate; we calculate first $\nabla(a, b)$ in terms of $u \cdot g, v \cdot g$.

This is the first entry in the following string of equations; the first equality sign is by (13):

$$(u \cdot g)(v \cdot g \cdot \omega(v \cdot g, u \cdot g))^{-1} = (u \cdot g)(v \cdot g \cdot g^{-1} \cdot \omega(v, u) \cdot g)^{-1}$$
$$= (u \cdot g)(v \cdot \omega(v, u) \cdot g)^{-1} = u(v \cdot \omega(v, u))^{-1},$$

and the last expressions here is $\nabla(a, b)$, defined using u, v.

The calculation that the two processes are inverse to each other is trivial (using $\omega(u, v) = \omega(v, u)^{-1}$ and $\nabla(a, b) = \nabla(b, a)^{-1}$).

6. Gauge forms versus horizontal equivariant forms. We consider a principal fibre bundle $\pi : P \to M$ as in the previous section. The *horizontal* k-forms that we now consider, are k-forms on P with values in the group $G = P^{-1}P$. *Horizontality* for a k-form θ means that

$$\theta(u_0, u_1, \ldots, u_k) = \theta(u_0, u_1 \cdot g_1, \ldots, u_k \cdot g_k) \tag{15}$$

for any infinitesimal k-simplex $(u_0, u_1, \ldots, u_k)$ in P, and any $g_1, \ldots g_k \in P^{-1}P$ with the property that $(u_0, u_1 \cdot g_1, \ldots, u_k \cdot, g_k)$ is still an infinitesimal simplex (which is a strong "smallness" requirement on the g_i's).

Note that the connection form ω for a connection ∇ is *not* a horizontal 1-form, since $\omega(x, y \cdot g) = \omega(x, y)g$, not $= \omega(x, y)$.

We say that a k-form θ, as above, is *equivariant* if for any infinitesimal k-simplex $(u_0, \ldots, u_k)$, and *any* $g \in P^{-1}P$, we have

$$\theta(u_0 \cdot g, u_1 \cdot g, \ldots, u_k \cdot g) = g^{-1}\theta(u_0, u_1, \ldots, u_k)g. \tag{16}$$

Note that connection forms are equivariant in this sense, by (13).

Recall ([12], say) that a k-form on M with values in a group *bundle* $E \to M$ is a law which associates to an infinitesimal k-simplex $a_0, ..., a_k$ in M an element in the fibre E_{a_0} (and associates the neutral element e in this fibre if the simplex is degenerate, i.e. if two of the a_i's are equal). We are interested in the case where E is the gauge group bundle of a groupoid; such forms we call *gauge valued forms*, for brevity.

Example: If ∇ and ∇_1 are two connections in a groupoid $\Phi \rightrightarrows M$, we may form a 1-form $\nabla_1\nabla^{-1}$ with values in the gauge group bundle of Φ; it is given, for $a \sim b$ in M, by

$$(\nabla_1\nabla^{-1}(a, b)) := \nabla_1(a, b) \cdot \nabla(b, a).$$

PROPOSITION 6.1. *There is a natural bijective correspondence between horizontal equivariant k-forms on P with values in $G = P^{-1}P$, and k-forms on M with values in the gauge group bundle gauge(PP^{-1}).*

Proof/Construction. Given a horizontal equivariant k-form θ on P as above, we construct a gauge valued k-form $\check{\theta}$ on M by the formula

$$\check{\theta}(a_0, \ldots, a_k) := (u_0 \cdot \theta(u_0, \ldots, u_k))u_0^{-1}, \tag{17}$$

or equivalently

$$\check{\theta}(a_0, \ldots, a_k) \cdot u_0 = u_0 \cdot \theta(u_0, \ldots, u_k), \tag{18}$$

where $(u_0, \dots, u_k)$ is an arbitrary infinitesimal k-simplex mapping to the infinitesimal k-simplex $(a_0, \dots, a_k)$ by π (such exist, since π is a surjective submersion). Note that the enumerator and the denominator in the fraction defining the value of $\check{\theta}$ are both in the fibre over a_0, so that the value is an endo-map at a_0 in the groupoid PP^{-1}, thus does belong to the gauge group bundle. We need to argue that this value does not depend on the choice of the infinitesimal simplex $(u_0, \dots, u_k)$. We first argue that, once u_0 is chosen, the choice of the remaining u_i's in their respective fibres does not change the value. This follows from horizontality (15). To see that the value does not depend on the choice of u_0: choosing another one amounts to choosing some $u_0 \cdot g$, for some g. But then we just change $u_1, \dots, u_k$ by the same g; this will give the arrow in PP^{-1}

$$(u_0 \cdot g \cdot \theta(u_0 \cdot g, \dots, u_k \cdot g))(u_0 \cdot g)^{-1}.$$

Now we calculate in the enveloping groupoid Φ; so we drop partentheses and multiplication dots; using the assumed equivariance (16), this expression then yields

$$u_0 g g^{-1} \theta(u_0, \dots, u_k) g g^{-1} u_0^{-1},$$

which clearly equals the expression in (17).

Conversely, given a gauge valued k-form α on M, we construct a $P^{-1}P$-valued k-form $\hat{\alpha}$ on P by putting

$$\hat{\alpha}(u_0, u_1, \dots, u_k) := u_0^{-1}(\alpha(a_0, a_1, \dots, a_k) \cdot u_0) \tag{19}$$

where a_i denotes $\pi(u_i)$. Since, for $i \geq 1$, this expression depends on u_i only through $\pi(u_i) = a_i$, it is clear that (15) holds, so the form $\hat{\alpha}$ is horizontal. Also,

$$\hat{\alpha}(u_0 \cdot g, \dots, u_k \cdot g) = (u_0 \cdot g)^{-1}(\alpha(a_0, \dots, a_k) \cdot (u_0 \cdot g));$$

by calculation in the enveloping groupoid of P, this immediately calculates to $g^{-1} \cdot \hat{\alpha}(u_0, \dots, u_k) \cdot g$, proving equivariance.

Finally, a calculation with the the enveloping groupoid again (cancelling u_0^{-1} with u_0) immediately gives that the two processes $\theta \mapsto \check{\theta}$ and $\alpha \mapsto \hat{\alpha}$ are inverse to each other.

We may summarize the bijection $\alpha \mapsto \hat{\alpha}$ from gauge(PP^{-1})-valued forms on M to horizontal equivariant $P^{-1}P$-valued forms on P by the formula

$$u_0 \cdot \hat{\alpha}(u_0, , , , , u_k) = (\pi^* \alpha)(u_0, ..., u_k) \cdot u_0 \tag{20}$$

(which is essentially just a rewriting of (18)).

Consider now the case where the group $G = P^{-1}P$ is commutative. Then by the remarks in Section 4, we may identify the gauge group bundle with the constant group bundle $M \times G \to M$, and under this identification, we may (by commutativity) cancel the "external" u_0's, and get

$$\hat{\alpha}(u_0, ..., u_k) = (\pi^* \alpha)(u_0, ..., u_k),$$

for all infinitesimal k-simplices $u_0, ..., u_k$ in P. So under the identification of gauge valued forms with G-valued forms implied by Section 4, we have

$$\hat{\alpha} = \pi^* \alpha. \tag{21}$$

7. Curvature versus coboundary. Recall [12] that the *curvature* of a connection in a groupoid $\Phi \rightrightarrows M$ is the gauge(Φ)-valued 2-form $R = R_\nabla$ on M given by

$$R(a_0, a_1, a_2) = \nabla(a_0, a_1) \cdot \nabla(a_1, a_2) \cdot \nabla(a_2, a_0);$$

and recall [8], [12] that if ω is a 1-form with values in a group G, then $d\omega$ is the G-valued 2-form given by

$$d\omega(x_0, x_1, x_2) = \omega(x_0, x_1) \cdot \omega(x_1, x_2) \cdot \omega(x_2, x_0).$$

We apply this to the case where $\Phi = PP^{-1}$ and $G = P^{-1}P$, for a principal fibre bundle $\pi : P \to M$. Then the curvature R, which is a gauge(PP^{-1})-valued 2-form on M, gives, by Proposition 6.1, rise to a (horizontal and equivariant) $P^{-1}P$-valued 2-form $\hat{R}$ on P.

We then have the following:

Theorem 7.1. *Let $\pi : P \to M$ be a principal fibre bundle with group G, and let ∇ be a principal connection in it, i.e. a connection in the groupoid PP^{-1}. Let ω be the connection form of ∇ (a G-valued 1-form on P), and let R be the curvature of ∇ (a gauge(P)-valued 2-form on M). Then*

$$\hat{R} = d\omega, \tag{22}$$

(as G-valued 2-forms on P), called the curvature form of ∇, or equivalently

$$R = (d\omega)\check{}. \tag{23}$$

(as gauge(P)-valued 2-forms on M). In particular, the curvature form $d\omega$ is horizontal and equivariant.

For the case where G is commutative, we may identify gauge(P)-valued forms with G-valued forms; in particular, the curvature R may be seen as a G-valued 2-form on M; and (22) then reads, by (21),

$$\pi^*(R) = d\omega. \tag{24}$$

Proof. Let x, y, z be an infinitesimal 2-simplex in P, and let $a = \pi(x)$, $b = \pi(y)$, and $c = \pi(z)$. We calculate the effect of the (left) action of the arrow $R(a, b, c)$ on x (note that $R(a, b, c)$ is an endo-arrow at a in the groupoid):

$$\begin{aligned} R(a, b, c) \cdot x &= \nabla(a, b) \cdot \nabla(b, c) \cdot \nabla(c, a) \cdot x \\ &= \nabla(a, b) \cdot \nabla(b, c) \cdot z \cdot \omega(z, x) \\ &= \nabla(a, b) \cdot y \cdot \omega(y, z) \cdot \omega(z, x) \\ &= x \cdot \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x) \\ &= x \cdot d\omega(x, y, z), \end{aligned}$$

using the defining equations for R and for dw for the two outer equality signs, and using (10) three times for the middle three ones. Then (23) follows by formula (18).

Remark. By [7] I.18, or in more detail, [8], there is a bijective correspondence between G-valued k-forms θ on a manifold P (where G is a Lie group, say $P^{-1}P$), and differential k-forms $\overline{\theta}$, in the classical sense, with values in the Lie algebra $\underline{g}$ of G (i.e. multilinear alternating maps $TP \times_P ... \times_P TP \to \underline{g}$). Under this correspondence, the horizontal equivariant 2-form $d\omega$ on P considered in the Theorem corresponds to the classically considered "curvature 2-form" Ω on P, as in [18] II.4 or [1] 5.3, (perhaps modulo a factor ± 2, depending on the conventions chosen). This is not completely obvious, since Ω differs from the exterior derivative $d\overline{\omega}$ of the classical connection form $\overline{\omega}$ by a "correction term" $1/2\ [\overline{\omega}, \overline{\omega}]$ involving the Lie Bracket of $\underline{g}$; or, alternatively, the curvature form comes about by modifying $d\overline{\omega}$ by a "horizontalization operator" (this "modification" also occurs in the treatment in [17]). The fact that this "correction term" (or "modification") does not come up in our context can be explained by Theorem 5.4 in [8] (or see [7] Theorem 18.5); here it is proved that the formula $d\omega(x, y, z) = \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x)$ already contains this correction term, when translated into "classical" Lie algebra valued forms.

The Theorem has the following Corollary, which is essentially what [17] call the infinitesimal version of the Gauss-Bonnet Theorem (for the case where $G = SO(2)$):

COROLLARY 7.2. *Assume $P^{-1}P$ is commutative, and let the connection ∇ in PP^{-1} have connection form ω. Then the unique G-valued 2-form Ω on M with $\pi^*\Omega = d\omega$ is R (the curvature of ∇).*

Let us remark that [17] also gives a version of the Corollary for the non-commutative case, their Proposition 6.4.1; this, however, seems not correct. In this sense, our Theorem 7.1 is partly meant as a correction to this Proposition, partly a "translation" of it into the pure multiplicative principal bundle calculus, which is our main concern.

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