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ON THE SYNTHETIC THEORY OF VECTOR FIELDS

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The present note is an exposition of some of the general "synthetic differential geometry". The style of exposition is that it expresses maps, subobjects, and statemetns in set theoretic language. As long as one stays inside what Lawvere calls "cartesian logic", which is essentially negative free (but higher order) logic, then the maps, subobjects etc. described can be interpreted in any cartesian closed category with equalizers. So when we for instance say "ring", we mean "ring object in such a category".

Let A be a commutative ring with 1. Let $D \subseteq A$ be the set of elements of square zero. We say that A is of <u>line type</u> if every map $t: D \to A$ is of form

$$(0) \qquad t(d) = b + d \cdot c \qquad \forall d \in D$$

for some unique b and $c \in A$. Clearly $b = t(0)$. We denote the c occurring here by $t'(0)$. Similarly, if $f: A \to A$ is arbitrary, and $a \in A$, we define $f'(a)$ to be that unique dement in A such that

$$(1) \qquad f(a+d) = f(a) + d \cdot f'(a) \qquad \forall d \in D$$

(this element exists uniquely in virtue of A being of line type). We call (1) the Taylor expansion of f at a.

To a map $f: A \to A$ we have thus associated a new map, $f': A \to A$, its derivative. It is easy from (1) to prove

$$(f+g)' = f' + g' \qquad (f\cdot g)' = f'\cdot g + f\cdot g'$$
$$(f\circ g)' = (f'\circ g)\cdot g' \qquad (\text{identity})' \equiv 1$$
$$(\text{constant})' \equiv 0;$$

see [5]. In fact proofs of these laws explicitly using elements with vanishing square were used very early in the history of calculus (Fermat), but were later abandoned, perhaps due to

Proposition 1. No non-trivial rings in the category of sets are of lin type.

Proof. If A is non-trivial, then D must contain more than just $0 \in D$ (for, otherwise the c occurring in (0) could not be uniquely determined). So take some $\delta \in D$ with $\delta \neq 0$. Define a function $t: D \to A$ by

$$* \qquad \begin{cases} t(\delta) = 1 \\ t(d) = 0 \quad \text{for} \quad d \neq \delta. \end{cases}$$

By the line type axiom, t is of form $t(d) = b + d\cdot c$. Obviously $b = 0$, so $t(d) = d\ c\ \ \forall d \in D$. In particular

$$1 = t(\delta) = \delta\cdot c.$$

Multiplying this equation by δ, we obtain $\delta = \delta^2\cdot c = 9$, (since $\delta \in D$), contradicting the assumption $\delta \neq 0$.

The proof hinges on the construction principle $*$, which has no place in cartesian logic.

For the rest of this note, A is a fixed ring, assumed to be of line type.

We note that the uniqueness assertion about c in the line type notion can be formulated: for any $c \in A$

$$(c \cdot d = 0 \quad \forall d \in D) \Rightarrow (c = 0).$$

This principle, we refer to as "cancelling universally quantified d's".

Geometrically, D is the intersection of the unit circle around $(0,1) \in A \times A$ and the x-axis $A \times \{0\} \subseteq A \times A$, and is thus a unity of the opposites: "curved" and "straight". In fact, for any object M, a map $t: D \to M$ should be thought of as a <u>tangent vector</u> on M at the point $t(0) \in M$ (Lawvere, [3]). Likewise (ibid.), a vector field X on M is a law which to each m M associates a tangent vector $X(m,-): D \to M$. Thus, a vector field on M is a map

$$X: M \times D \to M$$

satisfying

$$X(m,0) = m \qquad \forall m \in M$$

Keeping a $d \in D$ fixed, we get a map

(2) $$X(-,d): M \to M$$

called an <u>infinitesimal transformation</u> belonging to X.

The classical work of Lie on differential equations (see e.g. [2]) makes wide use of these endomaps of M, which have no place in modern rigourons treatments.

It is natural to ask whether $X(-,d)$ is a bijective map, with inverse

$$X(-,-d): M \to M.$$

A condition on M that will guarantee this, and also will allow us to add tangent vectors at the same point, is the condition that M is <u>infinitesimally linear</u> in the following sense. For each natural number n, we let $D(n) \subseteq A^n$ be the subset

$$\{(d_1,\ldots,d_n) \in A^n \mid d_i \cdot d_j = 0 \quad \forall i,j\}$$

(in particular $d_i^2 = 0 \quad \forall i$). For $i = 1,\ldots,n$, we have the "i'th inclusion"

$$incl_i: D \to D(n)$$

given by

$$incl_i(d) = (0,0,\ldots,d,\ldots,0)$$

(the d in the i'th place).

We say that M is <u>infitesimally linear</u> [6], [8], if for each n and each n-tuple $t_i: D \to M$ $(i=1,\ldots,n)$ of tangent vectors at the same point $m \in M$, there exists a <u>unique</u> $l: D(n) \to M$ with

(3) $$l \circ incl_i = t_i \qquad i = 1,\ldots,n.$$

In particular, if M is infinitesimally linear, and t_1, t_2 are two tangent vectors at $m \in M$, there is a unique $l: D(2) \to M$ with (3) holding $(n=2)$, and we define $(t_1+t_2): D \to M$ to be the map given by

$$(t_1 + t_2)(d) = l(d,d)$$

(note that $d \in D$ implies $(d,d) \in D(2)$).

Likewise, if $t: D \to M$ is a tangent vector and $a \in A$ is a scalar, we define $a \cdot t$ to be the map $D \to M$ given by

$$(a \cdot t)(d) = t(a \cdot d)$$

(note that $d \in D$ and $a \in A$ implies $a \cdot d \in D$).

It is then easy to prove ([6],[8],[9]) that the set of tangent vectors at any given point m of M becomes an A-module, with the structures thus defined (one uses $D(3)$ to prove associativity; the higher $D(n)$'s are not used).

To prove

(4) $$X(X(m,d),-d) = m,$$

we shall more generally prove, for $(d_1,d_2) \in D(2)$

(5) $$X(X(m,d_1),d_2) = X(m,d_1 + d_2)$$

(note that $(d_1,d_2) \in D(2) \Rightarrow d_1 + d_2 \in D$, because when squaring $d_1 + d_2$, the double product vanishes by assumption) to prove (5), note that both sides define maps

$$l: D(2) \to M$$

with $l \circ incl_i = X(m,-)$ $(i=1,2)$, and thus are equal, by the uniqueness assertion in the infinitesimal linearity assumption.

We can add to vector fields X and Y on an infinitesimally linear object M, by letting $(X+Y)(m,-)$ be the sum (as already defined) of the two tangent vectors at m, $X(m,-)$ and $Y(m,-)$. We can also multiply a vectorfield X with a scalar valued function $\varphi: M \to A$, namely by putting

$$(\varphi \cdot X)(m,d) = X(m,\varphi(m)\cdot d).$$

In this way, the set of vector fields on M becomes a module over the ring of functions $M \to A$.

Recall [6] [7] that an A-module M is called Enclidean if each $t: D \to M$ is of form

$$t(d) = t(0) + d\cdot \underline{v}$$

for some unique $\underline{v} \in M$, called the principal part of t.

Proposition 2. If M is an Euclidian A-module which is also infinitesimally linear, then addition of tangent vectors at a given $\underline{m} \in M$ using infinitesimal linearity agrees with the obvious addition "adding principal parts". Similarly for multiplication by scalars.

Proof. Let

$$t_i(d) = \underline{m} + d\cdot \underline{v}_i \qquad i = 1,2$$

be two vectors at $\underline{m} \in M$. Their sum, using infinitesemal linearity is found from $l\colon D(2) \to M$ given by

$$l(d_1,d_2) = \underline{m} + d_1\ \underline{v}_1 + d_2 \cdot \underline{v}_2$$

since $l \circ incl_i = t_i$. So we have, for all $d \in D$

$$\begin{aligned}(t_1+t_2)(d) = l(d,d) &= \underline{m} + d \cdot \underline{v}_1 + d \cdot \underline{v}_2 \\ &= \underline{m} + d \cdot (\underline{v}_1 + \underline{v}_2),\end{aligned}$$

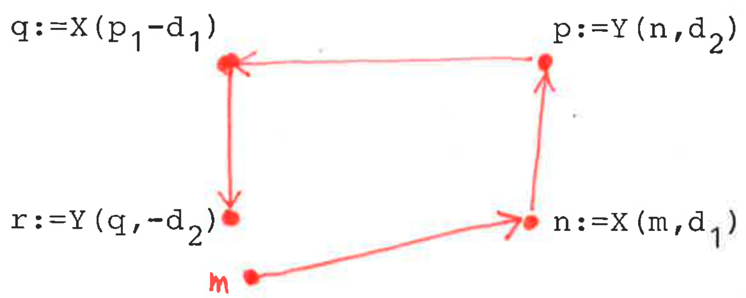
proving that $t_1 + t_2$ has principal part $\underline{v}_1 + \underline{v}_2$.

The last assertion of the Proposition is trivial.

We henceforth assume that A is of line type (hence Euclidean as an A-module), and infinitesimally linear; and M is assumed to be an arbitrary infinitesimally linear object.

We proceed to consider Poisson bracket of two vector fields X and Y on M. For fixed $d_1 \in D$ and $d_2 \in D$, we may consider the <u>commutator</u> of the two bijective endomaps $X(-,d_1)$ and $Y(-,d_2)$ of M. In other words, for fixed m, we consider the "circuit"

(6)



(recall from (4) that $X(-,d_1)^{-1} = X(-,-d_1)$, and similarly for Y).
For fixed m, the r obtained depends on $(d_1,d_2) \in D \times D$, so that we have a map

(7)
$$\begin{aligned} D \times D &\to M \\ (d_1,d_2) &\mapsto r \end{aligned}$$

If $d_1 = 0$, we have $n = m$ and $q = p$, so that

$$r = Y(q-d_2) = Y(p,-d_2) = n = m$$

the third equality sign by (4) and $Y(n,d_2) = p$. Similarly if $d_2 = 0$, we get likewise $r = m$. So the map (7) satisfies the condition for τ in the following requirement on M [6].

Requirement. For any map $\tau: D \times D \to M$ with

$$\tau(d,0) = \tau(0,d) = \tau(0,0) \quad \forall d \in M$$

there is a unique map $t: D \to M$ with

$$t(d_1 \cdot d_2) = \tau(d_1,d_2) \quad \forall (d_1,d_2) \in D \times D.$$

(evidently, then, $t(0) = \tau(0,0)$).

We assume henceforth that M satisfies this. Thus the map described in (7) is of form $(d_1,d_2) \to t(d_1 \cdot d_2)$ for some unique $t: D \to M$ with $t(0) = m$. We denote this t $[X,Y](m,-)$. Letting m vary, we obtain in this way a vector field $[X,Y]$ on M. It is characterized by

$$[X,Y](m,d_1 \cdot d_2) = r \quad \forall (d_1,d_2) \in D \times D,$$

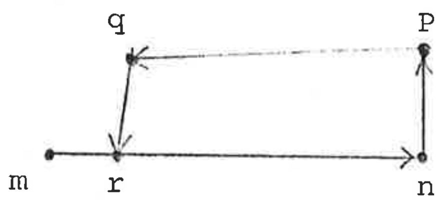
r obtained as in (6).

It is easy to prove that $[X,Y] = 0$ and $[X,Y] = -[Y,X]$. I belive that bilinearity and Jacobi identity for the bracket operation described here can be obtained by reinterpretation of the proofs for similar facts about the Lie algebra object of a monoid in [6].

Easier proofs exist (using Proposition 2) for the case where M is a Euclidean module, essentially by using the notion of "directional derivation anlong a vector field" which we shall discuss in a moment. However, we do not want to perform "a double-dualization" by identifying a vector field with a differential operator on a ring of functions. Thus, the following Theorem, which is essential in Lie's theory of differential equations, is stated and proved entirely in geometric terms (no differential operators!)

We shall call a vector field X proper if each $X(m,-): D \to M$ is an injective map (thus we make a positive assumption on X instead of the classical negative "$X(m,-)$ is always non-zero".) The theorem deals with two vector fields X,Y (with X proper) such that all circuits are X- trapezia, i.e. have shape

(8)



which to be precise, we take to mean that for each $m \in M$ and $(d_1,d_2) \in D$, the r constructed in (6) is of form $X(m,\delta)$ for some $\delta \in D$ (necessarily unique since X is proper).

We shall finally assume that A also satisfies the Requirement above. Then

Theorem 3. Let X,Y be vector-fields on M, with X proper. Then the following two conditions are equivalent:

i) all circuits of form (6) an X-trapezia, (8).

ii) $[X,Y] = \rho \cdot X$ for some scalar valued function $\rho: M \to A$.

<u>Proof</u>. Assume (i). Let m be fixed, and consider for $(d_1,d_2) \in D$ that unique $\delta = (d_1,d_2)$ such that

(9) $$r = X(m,\delta).$$

Arguing as for the map described in (7), we see that $\delta(d,0) = \delta(0,d) = 0$. Therefore ~~since~~ by the Requirement for A, $\delta(d_1,d_2) = t(d_1 \cdot d_2)$ for some unique $t: D \to A$. Since $t(0) = 0$, we get, since A is of line type, a unique $b \in A$ such that $t(d) = b \cdot d$ for all $d \in A$, so that

$$\delta(d_1,d_2) = b \cdot d_1 \cdot d_2 \qquad \forall\, (d_1,d_2) \in D \times D.$$

Now let m vary, and record the dependence of b on m by writing $b = \rho(m)$. Thus we have, for all $(d_1,d_2) \in D \times D$

$$[X,Y](m,d_1 \cdot d_2) = r = X(m,b \cdot d_1 \cdot d_2)$$

$$= X(m,\rho(m) \cdot d_1 \cdot d_2) = (\rho \cdot X)(m,d_1 \cdot d_2).$$

From the uniqueness in the Requirement. Then follows

$$[X,Y](m,d) = (\rho \cdot X)(m,d) \quad \text{for all} \quad d \in D$$

(and all m). This proves (ii).

The converse implication is trivial; if $r = (\rho \cdot X)(m,d_1 \cdot d_2)$, we have $r = X(m,\rho(m) \cdot d_1 \cdot d_2)$ witnessing that r is of form $X(m,\delta)$.

If we call two elements m_1 and m_2 of M X-neighbours provided there exists a $d \in D$ with

$$X(m_1,d) = m_2,$$

then it is easy to see that the conditions of the theorem in turn are equivalent to: for any $d \in D$, the permutation $Y(-,d)$ preserves the relation "being X-neighbours". Lie uses the phrase: "X admits Y"". The phrase "Y permutes X" makes a certain sense too in this connection, since by integration (which has no place in the present set up) the X-neighbour-relation passes into the relation being on the same streamline for the flow generated by X, wo that $Y(-,d)$ permutes the streamlines of X (possibly reparametrizing them).

We now discuss directional derivatives. Let X be a vector field on M, and $f: M \to V$ a function with values in a Euclidean module V (in particular, V might be A itself). Consider for fixed $m \in M$ the map $D \to V$ given by

$$d \mapsto f(X(m,d)).$$

By Euclidean-ness of V, this map is of form

$$d \mapsto f(m) + d \cdot \underline{v}$$

for some unique $\underline{v} \in V$, which we denote $X(f)(m)$. Thus $X(f): M \to V$ is the function characterized by

$$f(x(m,d)) = f(m) + d \cdot X(f)(m) \qquad \forall d \in D, \forall m \in M \tag{10}$$

("generalized Taylor formula").

The construction $f \mapsto f'$ previously mentioned is a special case, namely for X the vector field $\Uparrow$ on A given by

$$\Uparrow(a,d) = a+d.$$

It is proved in [7], Prop. 1.2 that $f \to X(f)$ is A-linear, and satisfies appropriate evident generalizations of Leibniz-rule:

$$X(\varphi\cdot f) \;''\; X(\varphi)\cdot f + \varphi\cdot X(f)$$

whenever $f\colon M \to V$ and $\varphi\colon M \to A$. (The proofs are easy from (10)). We proceed to investigate how $X(f)$ depends on X. We shal prove

<u>Proposition 4</u>. For any vactor fields X_1, X_2, Y on M, and any $\varphi : M \to A$, we have

(i) $$(X_1+X_2)(f) = X_1(f) + X_2(f)$$

(ii) $$(\varphi\cdot X)(f) = \varphi\cdot(X(f))$$

(iii) $$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

for any $f\colon M \to V$ (V a Euclidean infinitesimally linear module).

<u>Proof</u> (i): Let $L\colon M \times D(2) \;\; M$ be defined so that for any $m \in M$, $l = L(m,-,-)\colon D(2) \to M$ has

$$l\cdot \mathrm{incl}_i = X_i(m,-) \qquad i = 1,2$$

Consider for fixed m M the map $h: D(2) \to V$ given by

$$h(d_1,d_2) = f(L(m,d_1,d_2))$$

We then have (for $i = 1,2$) that $h \circ incl_i : D \to V$ is the tangent vector at $f(m)$ with principal part $X_i(f)(m)$; to see this, for $i = 2$, say

$$\begin{aligned} h(incl_2(d)) &= h(0,d) = f(L(m,0,d)) \\ &= f(X_2(m,d)) \\ &= f(m) + d \cdot X_2(f)(m) . \end{aligned}$$

From the uniqueness assertion in the statement that V is infinitesimally linear, it then follows that

$$h(d_1,d_2) = f(m)+d_1 \cdot X_1(f)(m)+d_2 \cdot X_2(f)(m)$$

We have, for all $d \in D$

$$f((X_1+X_2)(m,d)) = f(m) + d \cdot (X_1+X_2)(f)(m) .$$

On the other hand, for all $d \in D$,

$$\begin{aligned} f((X_1+X_2)(m,d)) &= f(L(m,d,d)) = h(d,d) \\ &= f(m) + d \cdot X_1(f)(m) + d \cdot X_2(f)(m) \end{aligned}$$

Comparing these two espressions for $f((X_1+X_2)(m,d))$ and cancelling the universally quantified d, we get (i), as desired. The proof of (ii) is easier, and omitted. Let us finally prove (iii). For fixed m, d_1, d_2, we consider the circuit (6) (and the elements n, p, q, r described then. We consider $f(r) - f(m)$. First

$$\begin{aligned} f(r) &= f(q) - d_2 \cdot Y(f)(q) \\ &= f(p) - d_1 \cdot X(f)(p) - d_2 \cdot Y(f)(q) \end{aligned}$$

using generalized Taylor (10) twice. Again using generalized Taylor (10) twice. (Noting $m = X(n, -d_1)$ and $n = Y(p_1 - d_2)$ by (4).

$$\begin{aligned} f(m) &= f(n) - d_1 \cdot X(f)(n) \\ &= f(p) - d_2 \cdot Y(f)(p) - d_1 \cdot X(f)(n). \end{aligned}$$

Subtracting these two equations, we get

$$\begin{aligned} f(r) - f(m) &= d_1 \cdot \{X(f)(n) - X(f)(p)\} \\ &\quad + d_2 \cdot \{Y(f)(p) - Y(f)(q)\} \\ &= -d_1 \cdot d_2 \cdot Y(X(f))(p) + d_1 \cdot d_2 \cdot X(Y(f))(p) \end{aligned} \tag{11}$$

using generalized Taylor (10) for the function $X(f)$ and for the function $Y(f)$. Now for any function $g: M \to V$,

$$d_2 \cdot g(n) = d_2 \cdot g(p)$$

and

$$d_1 \cdot g(m) = d_1 \cdot g(m)$$

since

$$\begin{aligned} d_2 \cdot g(p) &= d_2 \cdot g(Y(n, d_2)) \\ &= d_2 \cdot (g(n) + d_2\ Y(g)(n)) \\ &= d_2 \cdot g(n), \end{aligned}$$

the last term vanishing because $d_2^2 = 0$. Similarly for the other equation. Since the terms on the right hand side of (11) occur with both a d_1-factor and a d_2-factor we may apply this principle for the functions $g = X(f)$ and $Y(f)$ to replace the argument p by, first n, and then m. Thus

$$f(r) - f(m) = d_1 \cdot d_2 \cdot (X(Y(f))(m) - Y(X(f))(m)). \tag{12}$$

On the other hand

$$[X,Y](m, d_1 \cdot d_2) = r$$

so that

$$f(r) - f(m) = d_1 \cdot d_2 \cdot [X'Y](f)(m). \tag{13}$$

comparing (12) and (13), we see that for all $(d_1, d_2) \in D \times D$

$$d_1 \cdot d_2 \cdot (X(Y(f))(m) - Y(X(f))(m)) = d_1 \cdot d_2 \cdot [X,Y](f)(m)$$

and cancelling the universally quantified d_i's, we get (iii)

A final useful classical result about Lie brackets of vector fields on M

$$[X, f \cdot Y] = f \cdot [X,Y] + X(f) \cdot Y, \tag{14}$$

(where f is a scalar valued function) is easy to prove if M is an Euclidean module and infinitesimally linear. I do not know how to prove it without the module structure on M.

A function $f: M \to V$ (M and V infinitesimally linear, M satisfying the Requirement, V being a Euclidean module) is called an integral of the vector field X on M if $X(f) \equiv 0$. This is equivalent to saying that for any

$$t: D \to M$$

which is a vector of the field X, i.e. $X(t(0),-) = t$, the function f is constant on t,

$$f \cdot t \equiv f(t(0)).$$

Then the level set $f^{-1}(f(m))$ contains the tangent vector $X(m,-)$ (meaning that $X(m,-): D \to M$ factors through the level set).

An integral $f: M \to V$ of X is called universal if for any other integral $g: M \to W$ of X

$$g = \omega \circ f$$

for some $\omega: V \to W$ (not necessarily linear. This definition shluld really be made a local one, but we are not going very far in this direction anyway. It is reasonable to think of the (level sets of a universal integral of X as being precisely the streamlines of X (viewed as unparametriced 1-manifolds). Here we shall use "level set of universal integral" as definition of "streamline". We then have

Proposition 5. If the proper vector field X admits the vector field Y, in the sense of the conditions of Theorem 3, then for each $d \in D$, the infinitesimal transformation $Y(-,d): M \to M$ permutes the streamlines of X.

Proof. We have by assumption

$$[X,Y] = \rho \cdot X$$

for some $\rho: M \to A$. Assume $f: M \to V$ is a universal integral. We claim $Y(f)$ is an integral also. For

$$\begin{aligned} 0 &\equiv \rho \cdot X(f) = (\rho \cdot X)(f) \\ &= [X,Y](f) = X(Y(f)) - Y(X(f)) \\ &= X(Y(f)) - Y(0) \\ &= X(Y(f)), \end{aligned}$$

using Proposition 4 (ii) and (iii). By universality of f we get $\omega: V \to V$ with

$$Y(f) = \omega \circ f.$$

Now we claim that $Y(-,d)$ takes the level set $f^{-1}(c)$ into $f^{-1}(c+d \cdot \omega(c))$. For, let $f(m) = c$. Then

$$\begin{aligned} f(Y(m,d)) &= f(m) + d\ Y(f)(m) \\ &= f(m) + d \cdot \omega(f(m)) \\ &= f(m) + d \cdot \omega(c). \end{aligned}$$

Since $Y(-,d)$ is bijective, we actually get that it takes the level set $f^{-1}(c)$ <u>onto</u> $f^{-1}(c+d\cdot\omega(c))$.

This proves the Proposition. Of course, we have no way presently of proving existence of universal integrals.

The use of Theorem 3 for differential equations [2] is that for the case $M =$ the plane $A \times A$, if Y permutes X in the sense of Theorem 3 or Proposition 5, then the function which to $m \in M$ associates the reciprocal of the determinant of (the principal parts of) the two vectors $X(m), Y(m)$ in A^2 is an <u>integrating factor</u> for the differential equation, $X(f) = 0$, meaning that

$$\frac{1}{\det(X,Y)} \cdot X$$

is a source-free vector field, and thus an integral for it and thus for X can be found by curve integration (the orthogonal field is a gradient field: its potential function will work).

Lie states [1] that he found these theorems "by synthetic considerations" but found it difficult to write down the proofs synthetically, whence his articles present mainly analytic proofs in coordinates. I believe that the above proofs may be closely related to the synthetic theories of Lie.

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