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TOPOS-THEORETIC FACTORIZATION
OF NON-STANDARD EXTENSIONS

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In the present paper, we attempt to give an analysis (in the form of a factorization theorem for certain functors) of the basic principle of higher order non-standard analysis; this basic principle is the contradiction that "higher order properties are preserved and yet not preserved" by non-standard extensions.

As is well known (see Robinson, [16]), any non-standard argument starts by embedding the structures under investigation in larger ones; in particular, if the argument is a higher order one, so that both a set A and (say) its power-set $\mathcal{P}A$ is considered, one has

$$A \subseteqq A^* \quad ; \quad \mathcal{P}A \subseteqq (\mathcal{P}A)^* \quad ;$$

and in general $(\mathcal{P}A)^*$ will not be the power set $\mathcal{P}(A^*)$, but rather a subset of it

$$(\mathcal{P}A)^* \subseteqq \mathcal{P}(A^*), \tag{0.0}$$

namely that subset which consists of the internal subsets of A^*.

The $*$-operation (which in this note is denoted φ) thus does not preserve power-set formation, and more generally it does not "preserve truth of higher order logic sentences"; yet it does preserve such sentences provided every quantifier ranging over a class of higher order entities is understood to range over internal entities of that kind; for instance if $\mathbb{R}^*$ is a non-standard extension of the reals, the sentence

$$\boxed{\text{"Every bounded subset of real numbers has a sup"}}$$

is true for $\mathbb{R}$ and false for $\mathbb{R}^*$, but true for $\mathbb{R}^*$ if we replace the phrase

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"every bounded subset" by the phrase "every internal bounded subset". So without changing the elements of $\mathbb{R}^*$, a false higher-order sentence becomes true by a change in the ambient logic.

The category theoretic line on logic is that logic (including higher order logic) is built into the category $\mathbb{S}$ of sets (or into any topos $\underline{E}$, in fact). (This line of thought was initiated by Lawvere in 1963, [9]). In particular, one does not need a formalized language to describe the concepts of "first-order logic preserving functor" or "higher order logic preserving functor", (as we will see soon). It furthermore turns out that the idea of "changing the logic without changing the elements" can be made precise, again without introducing any syntactical notions: the toposes themselves play the role of logic, so "changing the logic" means "changing the topos". That this is possible is the content of the main theorem (the notions entering into the statement of the Theorem will be explained later in the paper):

THEOREM. Any first-order logic preserving functor between toposes $\varphi : \underline{E} \longrightarrow \underline{E}_0$ admits a factorization $\underline{E} \longrightarrow \underline{E}^* \longrightarrow \underline{E}_0$ where the first functor preserves higher-order logic and the second functor preserves elements (as well as first-order logic).

The theorem appears as Theorem 3.5 below.

The reader may have the following example in mind for φ; take a set I with a non-principal ultrafilter D on it; define the functor φ from the category $\mathbb{S}$ of sets to itself by the ultrapower

$$\varphi(X) = \prod_I X/D \tag{0.1}$$

for any set $X \in |\mathbb{S}|$. (The factorization given in the theorem seems not in general to yield a known category in the middle, but for this special case, the middle category can be shown to be equivalent to the category $(\prod_I \mathbb{S})/D)$.

The factorization was constructed and some of its properties conjectured by the first named author in [5]. A fuller account was given in our joint preliminary version [6] .

There are three sections. In the first, we recall notions from the theory of elementary toposes, and develop some of the intrinsic logic in a topos. In the second section we describe the notion of first-order-logic-preserving functor between toposes, and develop the notions: standard map, internal map, internal subobject, and related notions needed for defining the middle category $\underline{E}^*$ of the factorization. Finally, the third section contains the proof that $\underline{E}^*$ is a topos, (as well as properties of the functors in the factorization).

Let us remark that from closed category theory one knows that any closed functor admits a factorization into a "residual" closed functor followed by a "normal" closed functor, [4]. The middle category in this factorization will not in general be a topos. But it will sit as a subcategory of our middle topos (as "the category of internal maps").

1. The logic of an elementary topos $\underline{E}$

We consider a category $\underline{E}$ with finite inverse limits, in particular with a terminal object 1. An object Ω of $\underline{E}$ together with a map

$$1 \xrightarrow{true} \Omega$$

(necessarily monic) is said to be a subobject classifier [12],[13], if for any monic map $A' \overset{a}{\rightarrowtail} A$ in $\underline{E}$, there is precisely one map $A \longrightarrow \Omega$ (called the characteristic map ch(a) of a) such that

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A' \arrow[u, tail, "a"] \arrow[r] & 1 \arrow[u, tail, "true"']
\end{tikzcd}$$

is a pull-back. A category with finite inverse limits and a subobject classifier is said to be an (elementary) topos if it further is cartesian closed. This means that there is a natural 1-1 correspondence between maps $A \times B \to C$ and maps $A \to C^B$ for a suitable object C^B (here also denoted $B \pitchfork C$).

The original definition of the concept of elementary topos (as given by Lawvere and Tierney [13], [3]) also required existence of finite colimits, but this can be shown to be redundant, [15].

Further, Cartesian closed-ness calls for existence of exponential objects B^A for any A,B; we show below that exponential objects of the form Ω^A will suffice to get all the others.

One should think of Ω^A as the power-set object of A in the category $\underline{E}$; this is clear when we look at $\underline{E} = \mathfrak{S}$; then $\Omega = 2$ (any two-element set will do), and B^A is the set of mappings from A to B; so Ω^A is the set of mappings from A to 2, which clearly indexes the power set $\mathcal{P}A$ of A. When we illustrate in the category $\mathfrak{S}$ the constructions to come, we shall simply identify Ω^A with $\mathcal{P}A$, thus saying, for instance

"let $A' \subseteq A$ be an element of Ω^A".

From now on, $\underline{E}$ denotes an arbitrary topos, for instance the category $\mathfrak{S}$ of sets.

For any object $A \in |\underline{E}|$, we have the diagonal

$$A \xrightarrow{\Delta_A} A \times A \tag{1.1}$$

which is monic since it has a right inverse: projection to the first (or second) factor (we compose maps from left to right). The subobject of $A \times A$

defined by the monic map (1.1) may be viewed as the extension of the equality predicate for "elements in A". It has a characteristic function

$$A \times A \xrightarrow{\delta_A} \Omega , \tag{1.2}$$

denoted δ_A since it specializes to Kronecker's δ in the case $\underline{E} = \mathfrak{S}$ (and $\Omega = \{1,0\} = \{\text{true, false}\}$). The exponential adjoint of δ_A is a map, "singleton-map",

$$A \xrightarrow{\{\cdot\}_A} \Omega^A \tag{1.3}$$

denoted this way, because in the set case, it assigns to $a \in A$ the element $\{a\} \subseteq A$ of Ω^A.

These three maps appear in the work of Lawvere and Tierney under the same names. They also proved that (1.3) is monic (see e.g. [8] for a proof). Therefore we can take the characteristic function s_A of $\{\cdot\}_A$:

$$\Omega^A \xrightarrow{s_A} \Omega \tag{1.4}$$

In the set-case, it takes $A' \subseteq A$ into "true" iff A' is a singleton $\{a\}$.

For any map $f\colon A \longrightarrow B$, consider the exponential adjoint $\ulcorner f \urcorner$ of the composite

$$1 \times A \cong A \xrightarrow{f} B ;$$

it is a map

$$1 \xrightarrow{\ulcorner f \urcorner} B^A ,$$

which Lawvere in [9] called "the name of f".

For any object A, the characteristic map of the maximal subobject of A (which is A itself) is denoted "true_A"; alternatively

$$\text{true}_A = A \longrightarrow 1 \xrightarrow{\text{true}} \Omega .$$

Thus we have "the name of true_A:

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Recall that Y^X depends functorially on Y (and also contravariant functorially on X). If $g\colon Y \longrightarrow Z$, we denote the resulting map $Y^X \longrightarrow Z^X$ by g^X, or $1 \pitchfork g$. Similarly in the upper variable.

PROPOSITION 1.1. Let A and B be arbitrary objects in $\underline{E}$. Then there is a pull-back diagram of the following form

(1.5)
$$\begin{tikzcd}
B^A \arrow[r] \arrow[d] & (\Omega^B)^A \arrow[d, "(s_B)^A"] \\
1 \arrow[r, "\ulcorner \mathrm{true}_A \urcorner"'] & \Omega^A
\end{tikzcd}$$

We shall not here give the full proof (which may be found in the preliminary version [6]), but rather argue heuristically for the case $\underline{E} = \mathcal{S}$. Clearly $(\Omega^B)^A$ may be identified with $\Omega^{A \times B}$, the power set of $A \times B$, which in turn is the set of relations R between the sets A and B. The upper map associates to an element $\ulcorner f \urcorner \in B^A$ the graph of the corresponding map $f\colon A \longrightarrow B$. To say that the diagram is a pull-back is to say that a relation $R \subseteq A \times B$ is the graph of some map $f\colon A \longrightarrow B$ if and only if

(1.6)
$$(s_B)^A(R) = A$$

(the maximal subset of A). But recalling that s_B is the characteristic map of the singleton-construction $\{\cdot\}_B$, it is not hard to see that

$$(s_B)^A(R) = \{a \in A \mid \{b \mid aRb\} \text{ is a singleton}\} ;$$

So (1.6) says that for every $a \in A$, $\{b \mid aRb\}$ is a singleton; so, for every $a \in A$, precisely one $b \in B$ stands in the relation R to it; so R is (the graph of) a map $A \longrightarrow B$.

COROLLARY 1.2. If a category has finite inverse limits, a subobject classifier Ω, and exponentiation of the form Ω^A for all A, then it is a topos.

<u>Proof</u>. If exponentials of the form Ω^A exist, then also $(\Omega^B)^A$, since this object can be taken to be $\Omega^{B\times A}$. Now one can construct B^A as the pullback of $\ulcorner \mathrm{true}_A \urcorner$ with $(s_B)^A$, as in (1.5). For finite colimits: see Mikkelsen [15]).

We next need topos-theoretic versions of the notion "for all $a \in A$", and "there exists an $a \in A$".

For any map $f\colon A \longrightarrow B$ in $\underline{E}$, "pulling back along f" defines a monotone map

$$\mathcal{P}(B) \xrightarrow{f^{-1}} \mathcal{P}(A)$$

(here $\mathcal{P}(A)$ denotes the set of subobjects of A - a subobject of A being as usual an equivalence class of monic maps with codomain A). We have that $\mathcal{P}(A)$ is a partially ordered set; we denote its elements A', A", etc. Now it is a well-known fact from the theory of (elementary) toposes (see e.g. [8], p.32) that f^{-1} has a right adjoint $\forall_f$ in the sense that for any $B' \in \mathcal{P}(B)$, $A' \in \mathcal{P}(A)$

$$(1.7) \qquad f^{-1}(B') \subseteqq A' \quad \text{iff} \quad B' \subseteqq \forall_f(A') \ ;$$

$\forall_f$ itself is a monotone map $\mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, "universal quantification along f" ; in the set case

$$\forall_f(A') = \{ b \mid f^{-1}(b) \subseteqq A' \} \ ,$$

as the reader may easily check. The fact that "right adjoint for pulling back" is a kind of universal quantification was pointed out by Lawvere in 1965, [10].

Besides the universal quantification $\forall_f$ considered in (1.7), there is

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also an existential quantification $\exists_f$; for an $f\colon A \longrightarrow B$, $\exists_f$ is a monotone map $\mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ satisfying the dual of (1.7): for $A' \in \mathcal{P}(A)$, $B' \in \mathcal{P}(B)$

(1.8) $$A' \subseteq f^{-1}(B') \quad \text{iff} \quad \exists_f(A') \subseteq B'.$$

To construct $\exists_f(A')$, just take the mono part of an epi-mono factorization of the composite $A' \rightarrowtail A \xrightarrow{f} B$. Of course, in a <u>boolean</u> topos (like the category of sets) $\exists_f$ and $\forall_f$ can be constructed in terms of each other, by means of negation.

Besides the $\forall_f$ and $\exists_f$ as given above, we shall need their <u>intrinsic</u> versions. Note that $\mathcal{P}(A)$ is the power <u>set</u> (<u>set</u> of subobjects) of A - but that this also lives intrinsically in $\underline{\underline{E}}$, namely as $A \pitchfork \Omega = \Omega^A$. The functor $\hom_{\underline{\underline{E}}}(1,-)$ takes $A \pitchfork \Omega$ to $\mathcal{P}(A)$. The intrinsic version of $\exists_f$ and $\forall_f$ should be maps in $\underline{\underline{E}}$:

(1.9) $$\exists(f),\ \forall(f)\colon A \pitchfork \Omega \longrightarrow B \pitchfork \Omega$$

which by $\hom_{\underline{\underline{E}}}(1,-)$ go to the previously considered $\exists_f, \forall_f$. Sometimes we write $\exists_f$ instead of $\exists(f)$. We shall need the construction of the intrinsic $\exists(f)$ in Proposition 2.4. We now recall its construction:

For any X, Y the exponential adjointness applied to the identity map of $X \pitchfork Y$ gives rise to an "evaluation" map

$$(X \pitchfork Y) \times X \xrightarrow{ev} Y.$$

In particular, we have

$$(A \pitchfork \Omega) \times A \xrightarrow{ev} \Omega\ ;$$

this map is characteristic map for a subobject

(1.10) $$\in_A \rightarrowtail (A \pitchfork \Omega) \times A\ ;$$

in the set case, $\in_A$ consists of pairs $\langle A' \subseteq A, a \rangle$ such that $a \in A'$. It is an intrinsic version of the $\in$ -relation from set theory.

Now, for $f: A \longrightarrow B$, we get the intrinsic $\exists(f): A\pitchfork\Omega \longrightarrow B\pitchfork\Omega$ as the exponential adjoint of a map $(A\pitchfork\Omega)\times B \longrightarrow \Omega$, which we in turn get as characteristic map of that subobject of $(A\pitchfork\Omega)\times B$ which is the image of

$$\in_A \subseteq (A\pitchfork\Omega)\times A$$

under $1\times f: (A\pitchfork\Omega)\times A \longrightarrow (A\pitchfork\Omega)\times B$.

(This construction is due to Lawvere-Tierney).

A simple construction of $\forall(f)$ may be found in [7].

2. Exact functors which preserve Ω

In this section we shall study functors which preserve smaller or larger parts of the intrinsic logic.

The functors we consider are all left exact functors (that is, finite inverse limit preserving functors)

(2.1) $$\varphi : \underline{E} \longrightarrow \underline{E}_0$$

between two toposes. For such a functor φ, since $\varphi(1) = 1$ (terminal objects are preserved), we will have a monic map

$$\varphi(\mathrm{true}): 1 = \varphi(1) \longrightarrow \varphi(\Omega)$$

which will have a characteristic map

(2.2) $$\gamma : \varphi(\Omega) \longrightarrow \Omega_0$$

where Ω_0 is the subobject classifier of the topos $\underline{E}_0$.

Although it from some examples seems conceivable that one can give the factorization desired under weaker assumptions, we shall in general assume

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that φ preserves Ω in the sense that γ in (2.2) is invertible, and also we shall in general assume that φ is exact in the sense of [1] meaning that it is left exact and preserves epic maps (this does not imply that it preserves coequalizer diagrams).

Because φ is left exact, it will preserve monic maps and thus give rise to a monotone map

$$\varphi'_A: \mathcal{P}(A) \longrightarrow \mathcal{P}(\varphi(A))$$

for each $A \in |\underline{E}|$. Because φ preserves epimorphisms, it will preserve existential quantification in the sense that for each $f: A \to B$ in $\underline{E}$, the following diagram

$$(2.3)\qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "\exists_f"] \arrow[d, "\varphi'_A"'] & \mathcal{P}(B) \arrow[d, "\varphi'_B"] \\ \mathcal{P}(\varphi A) \arrow[r, "\exists_{\varphi(f)}"'] & \mathcal{P}(\varphi B) \end{tikzcd}$$

commutes.

We shall say that "φ <u>preserves universal quantification</u>" if the diagram (2.3) with $\exists$ replaced by $\forall$ commutes. Finally, if φ preserves Ω and existential and universal quantification, it is natural to call φ <u>first-order logic-preserving</u>.

The canonical isomorphism we have because φ preserves products

$$\tilde{\varphi} : \varphi(A) \times \varphi(B) \longrightarrow \varphi(A \times B)$$

will be denoted $\tilde{\varphi}_{A,B}$ or $\tilde{\varphi}$, as indicated.

A functor $\varphi: \mathbb{S} \to \mathbb{S}$ (where $\mathbb{S}$ is the category of sets) preserves first order logic if it preserves finite inverse limits and if $\varphi(2) = 2$. For, φ preserves the notion of complement in the subobject lattices (which are here boolean algebras) because it preserves 2. It preserves existential quantification because it preserves surjective maps (which have left inverses in $\mathbb{S}$).

Finally, universal quantification can in this case be expressed in terms of complement and existential quantification in the familiar way:
$\forall_f(X') = \neg\exists_f(\neg X')$.

Thus the functor "ultrapower" described in (0.1) preserves first order logic.

We now come to the key structure. Namely, for each $A,B \in |\underline{E}|$ we have a canonical map, generalizing (0.0),

$$\hat{\varphi}_{A,B}\colon \varphi(A \pitchfork B) \longrightarrow \varphi(A)\pitchfork\varphi(B). \tag{2.4}$$

This we get by exponential adjointness from the composite

$$\varphi(A\pitchfork B)\times\varphi(A) \xrightarrow{\cong} \varphi((A\pitchfork B)\times A) \xrightarrow{\varphi(ev)} \varphi(B)$$

where the first map is the canonical isomorphism coming from the assumption that φ preserves products. In the language of closed categories, $\hat{\varphi}$ makes into a closed functor. In [7], we proved the following:

THEOREM 2.1. Let $\varphi : \underline{E} \longrightarrow \underline{E}_o$ be a functor between elementary toposes which preserves finite inverse limits and Ω. Then the following two statements are equivalent:

(i) φ preserves universal quantification

(ii) for each A,B, the map $\hat{\varphi}_{A,B}$ in (2.4) is monic.

The theorem will allow one to think of $\varphi(A\pitchfork B)$ as a subset of $\varphi(A)\pitchfork\varphi(B)$ (in case φ preserves universal quantification), namely the "set of <u>internal</u> maps from φA to φB". In particular $\varphi(A\pitchfork\Omega) \subseteq \varphi A \pitchfork\Omega$ can be thought of as the set of <u>internal</u> subsets of φA ($\varphi A\pitchfork\Omega$ being the <u>full</u> power set of φA). (Compare with (0.0) of the Introduction.) The fact that $\hat{\varphi}$ is monic will not be used in an essential way except in the proof of Proposition 3.3. We do not know whether the assumption can be avoided.

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We now come to the construction, and consider a fixed 1st order logic preserving functor $\varphi : \underline{E} \longrightarrow \underline{E}_0$.

DEFINITION 2.2. Call a map $f: \varphi(A) \longrightarrow \varphi(B)$ *internal* if its name factors through $\hat{\varphi}_{A,B}$:

$$\ulcorner f \urcorner = 1 \xrightarrow{f_0} \varphi(A \pitchfork B) \xrightarrow{\hat{\varphi}} \varphi A \pitchfork \varphi B \tag{2.5}$$

for some f_0. Call f a *standard* map if it is of the form $\varphi(g)$ for some $g: A \longrightarrow B$ in $\underline{E}$.

A subobject $A_0 \rightarrowtail \varphi A$ of φA is an *internal subobject* provided its characteristic map $\varphi A \longrightarrow \Omega = \varphi\Omega$ is an internal map. Note that the map a itself is not internal as a *map* since for the notion of internal map to make sense, also the domain must be of form $\varphi(X)$ for some $X \in |\underline{E}|$.

A map which is standard is also internal; if $f = \varphi(g)$, then $\varphi(\ulcorner g \urcorner)$ will do as f_0 in (2.5).

The class of internal maps are closed under composition. This is a standard exercise in closed category theory (see e.g. [2], Theorem 6.6, p. 449). Also, from closed category theory, one derives a canonical 1-1 correspondence between internal maps

$$\frac{\varphi(A) \longrightarrow \varphi(B \pitchfork C)}{\varphi(A \times B) \longrightarrow \varphi(C)} \; , \tag{2.6}$$

both sets of internal maps being indexed by the set of maps

$$1 \longrightarrow \varphi(A \pitchfork (B \pitchfork C)) \cong \varphi((A \times B) \pitchfork C).$$

Since φ preserves pull-backs and internal maps compose, it is easy to see that pulling internal subobjects back along internal maps yields internal subobjects. For existential or universal quantification along internal maps, the same is true, but non-trivial. We shall only use and prove a weaker statement in this direction.

PROPOSITION 2.3. Existential quantification along a standard morphism preserves the notion of internal subobjects. Explicitely, if $f\colon X \longrightarrow Y$ is a map in $\underline{E}$, then the following diagram commutes:

$$\begin{tikzcd} \varphi(X)\pitchfork\Omega \arrow[r, "\exists(\varphi(f))"] & \varphi(Y)\pitchfork\Omega \\ \varphi(X\pitchfork\Omega) \arrow[u, "\hat{\varphi}"] \arrow[r, "\varphi(\exists(f))"'] & \varphi(Y\pitchfork\Omega) \arrow[u, "\hat{\varphi}"'] \end{tikzcd} \tag{2.7}$$

We need a Lemma.

LEMMA 2.4. The functor φ preserves the $\in$-relation in the sense that, inside $(\varphi(X)\pitchfork\Omega)\times\varphi(X)$

$$\varphi(\in_X) = (\varphi(X\pitchfork\Omega)\times\varphi(X)) \cap \in_{\varphi(X)},$$

$\varphi(X\pitchfork\Omega)\times\varphi(X)$ being viewed as a subobject of $(\varphi(X)\pitchfork\Omega)\times\varphi(X)$ by means of $\hat{\varphi}\times 1$.

Proof. This is an easy consequence of the fact that φ preserves pull-backs.

Proof of Proposition 2.3. To prove (2.7) commutative we pass to exponential adjoints; the desired equality is then the total equaltity in the string

$$\begin{aligned}
&\hat{\varphi}\times 1 \cdot \exists(\varphi(f))\times 1 \cdot \mathrm{ev} \\
&= \hat{\varphi}\times 1 \cdot \mathrm{ch}(\exists_{1\times\varphi(f)}(\in_{\varphi(X)})) && \text{by definition of } \exists(\varphi(f)) \\
&= \mathrm{ch}(\exists_{1\times\varphi(f)}(\hat{\varphi}\times 1)^{-1}(\in_{\varphi(X)})) && \text{by pull-back naturality of ch} \\
&= \mathrm{ch}(\exists_{1\times\varphi(f)}\ (\varphi(\in_X))) && \text{by Lemma 2.4} \\
&= \tilde{\varphi}\cdot \mathrm{ch}(\exists_{\varphi(1\times f)}(\varphi(\in_X))) && \text{by pull-back naturality of ch} \\
&= \tilde{\varphi}\cdot \mathrm{ch}(\varphi(\exists_{1\times f}(\in_X))) && \text{since } \varphi \text{ preserves existential quantification} \\
&= \tilde{\varphi}\cdot \varphi(\mathrm{ch}(\exists_{1\times f}(\in_X))) && \text{since } \varphi \text{ preserves pull-backs} \\
&= \hat{\varphi}\cdot \varphi(\exists(f)\times 1 \cdot \mathrm{ev}) && \text{by definition of } \exists(f) \\
&= \varphi(\exists(f))\times 1 \cdot \hat{\varphi}\times 1 \cdot \mathrm{ev}\,.
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A relation from X to Y is a subobject R of $X \times Y$. If S is another relation from Y to Z then the composite relation from X to Z is defined as the $\exists_{\mathrm{proj}_{13}}$ (image along projection to first and third factor) of that subobject of $X \times Y \times Z$ which one gets by intersection $\mathrm{proj}_{12}^{-1}(R)$ with $\mathrm{proj}_{23}^{-1}(S)$. In the set case, this is usual composition of relations. An internal relation from $\varphi(X)$ to $\varphi(Y)$ is a relation $R \rightarrowtail \varphi(X) \times \varphi(Y)$ from $\varphi(X)$ to $\varphi(Y)$ such that

$$R \rightarrowtail \varphi(X) \times \varphi(Y) \cong \varphi(X \times Y)$$

is an internal subobject of $\varphi(X \times Y)$.

Since composing relations involve pull-backs, intersection, and existential quantification along projections, which are of course standard maps, it is easy to conclude from Proposition 2.3 and the fact that φ preserves pull-backs that

PROPOSITION 2.5. The class of internal relations is closed under relational composition.

Those relations we actually are interested in, are certain "partial maps" (the pseudomaps of the definition below). Relational composition of such does not require application of any existential quantification. That existential quantification is preserved may therefore be a redundant hypothesis on φ.

We are now ready to describe the factorization mentioned in the introduction.

We define the category $\underline{E}^*$ as follows:

The objects are triples (A_0, a, A) where $A_0 \in |\underline{E}_0|$, $A \in |\underline{E}|$, and $a \in \mathrm{Hom}_{\underline{E}_0}(A_0, \varphi(A))$ is a monomorphisms such that

$$A_0 \overset{a}{\rightarrowtail} \varphi(A)$$

is an internal subobject of $\varphi(A)$. (One should see A_0 as the main aspect of such an object; $\varphi(A)$ plays only the role of "atlas".)

The morphisms between (A_o,a,A) and (B_o,b,B) in $\underline{E}^*$ are the morphisms in $\underline{E}_o$: $f\colon A_o \to B_o$ with the property that "the graph of a", $\langle a, f \cdot b\rangle \cdot \tilde{\varphi}_{A,B}$ is an internal subobject of $\varphi(A\times B)$. The morphisms of $\underline{E}^*$ are called pseudo-morphisms, and its objects pseudo-objects, in analogy with [14].

It follows from Proposition 2.5 that the composite of two pseudo-morphisms is a pseudomorphism. Also, identity maps of pseudo-objects are pseudo-maps.

If $f \in \mathrm{Hom}_{\underline{E}}(A,B)$ then $\langle 1_{\varphi(A)}, \varphi(f)\rangle \cdot \tilde{\varphi}_{A,B} = \varphi(\langle 1_A, f\rangle)$ is a standard subobject.

Hence we have a well-defined functor

$$\bar{\varphi}\colon \underline{E} \longrightarrow \underline{E}^*$$

$$\bar{\varphi}(A) = (\varphi(A),\ 1_{\varphi(A)}, A)$$

$$\bar{\varphi}(f) = \varphi(f).$$

Also we have a canonical functor

$$\psi\colon \underline{E}^* \longrightarrow \underline{E}_o$$

$$\psi(A_o,a,A) = A_o$$

$$\psi(f) = f.$$

Clearly we have that

$$\varphi \text{ equals } \bar{\varphi} \text{ followed by } \psi.$$

The properties of this factorization will be investigated in the next section. We shall need the following Lemma

LEMMA 2.6. Suppose that we have a commutative square

$$\begin{tikzcd} \varphi(X) \arrow[r, "h"] & \varphi(Y) \\ X_o \arrow[u, "x"] \arrow[r, "h_o"'] & Y_o \arrow[u, "y"'] \end{tikzcd}$$

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where x and y are internal subobjects and where h is an internal map. Then h_o is a pseudomorphism (between the objects $X_1 \in |\underline{E}^*|$ and $Y_1 \in |\underline{E}^*|$ displayed as left and right hand column in the diagram).

The proof is straightforward and omitted.

3. Properties of the factorization

The category of objects of form $\varphi(X)$ and with internal maps $\varphi(X) \longrightarrow \varphi(Y)$ as morphisms can be proved to be cartesian closed. But it is not a topos; it is not even finitely left complete because in general it will have too few equalizers. The introduction of pseudoobjects and pseudomaps remedies this.

PROPOSITION 3.1. The category $\underline{E}^*$ has finite inverse limits.

Proof. By general category theory, it suffices to produce a terminal object, equalizers and binary products. All these things are immediate to construct using Proposition 2.3. - in fact, in such a way that $\psi : \underline{E}^* \longrightarrow \underline{E}_o$ preserves and reflects them.

Also, ψ preserves and reflects isomorphisms and monomorphisms. Any subobject of a pseudo-object $b: B_o \rightarrowtail \varphi B$ can be represented by a monomorphism in $\underline{E}^*$ of the form

$$\begin{tikzcd}
\varphi B \arrow[r, "1"] & \varphi B \\
A_o \arrow[u, "f \cdot b"] \arrow[r, tail, "f"'] & B_o \arrow[u, tail, "b"']
\end{tikzcd}$$

This is used together with Lemma 2.6 in deriving

PROPOSITION 3.2. The category $\underline{E}^*$ has a subobject classifier Ω^*, namely the pseudo-object

$$\Omega \xrightarrow[\gamma]{\cong} \varphi(\Omega),$$

and identifying Ω^* with Ω, true* is identified with true: $1 \longrightarrow \Omega$.

The more difficult thing about the category $\underline{E}^*$ is that it has higher order structure, (that is, exponentials). As we know by Corollary 1.2, it suffices to show that it has intrinsic power "set" formation, that is, if

$$B_1 = B_o \overset{b}{\rightarrowtail} \varphi(B)$$

is an object of $\underline{E}^*$, there is another object in $\underline{E}^*$, $B_1 \overset{*}{\pitchfork} \Omega^*$ with the correct universal property: that subobjects of a product object $A_1 \times B_1$ in $\underline{E}^*$ are in a natural 1-1 correspondence with maps in $\underline{E}^*$

$$A_1 \longrightarrow B_1 \overset{*}{\pitchfork} \Omega .$$

We define $B_1 \overset{*}{\pitchfork} \Omega$ to be the left hand column in the pull-back diagram

(3.1)
$$\begin{tikzcd}
\varphi(B\pitchfork\Omega) \arrow[r, "\hat{\varphi}"] & \varphi B \pitchfork \Omega \\
D \arrow[u] \arrow[r] & B_o \pitchfork \Omega \arrow[u, "\exists_b"']
\end{tikzcd}$$

It is true but not at all obvious that D becomes an internal subobject of $\varphi(B\pitchfork\Omega)$. (The proof of this fact will be sketched below). The idea in proving the universal property consists in the observation (Proposition 3.3 below) that although it is not <u>in general</u> true that pseudo-maps h_o: $X_1 \longrightarrow Y_1$ are restrictions of internal maps,(that is, come about by the procedure given in Lemma 2.6) this is true if Y is of the form $B\pitchfork\Omega$. Our proof of this special kind of injectivity property of $\varphi(B\pitchfork\Omega)$ hinges on φ preserving universal quantification, that is, by Theorem 2.1, on the monicness of the canonical map

$$\hat{\varphi}_{B,\Omega}: \varphi(B\pitchfork\Omega) \longrightarrow \varphi(B)\pitchfork\Omega .$$

For, $\hat{\varphi}_{B,\Omega}$ being monic, the construction of a map into $\varphi(B\pitchfork\Omega)$ is equiva-

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$1 \longrightarrow \varphi(X\pitchfork\Omega)$, lent to the construction of a map into $\varphi B \pitchfork \Omega$ satisfying certain conditions; by the universal properties of $\pitchfork$ and Ω, of course one has good chances of constructing maps ending in $\varphi B \pitchfork \Omega$. We have in fact

PROPOSITION 3.3. Every pseudomap $h: A_1 \rightarrow B_1 \overset{*}{\pitchfork} \Omega^*$ (bold arrows in the diagram below) can be extended to an internal map h' (dotted)

$$\begin{tikzcd}
\varphi A \arrow[r, dashed, "h'"] & \varphi(B\pitchfork\Omega) \\
A_0 \arrow[u, tail] \arrow[r, "h"'] & D \arrow[u, tail]
\end{tikzcd} ,$$

(and conversely, the bottom map in such a square is necessarily a pseudo-map).

The universal property, to be proved for $B_1 \overset{*}{\pitchfork} \Omega^*$ now essentially goes via the chain of relations

$h: A_1 \longrightarrow B_1 \overset{*}{\pitchfork} \Omega^*$

extend to h': $\varphi A \longrightarrow \varphi(B\pitchfork\Omega)$

pass by (2.6) to: $\varphi(A\times B) \longrightarrow \varphi(\Omega) = \Omega$

restrict along $A_0 \times B_0 \longrightarrow \varphi(A\times B)$ to: $A_1 \times B_1 \longrightarrow \Omega^*$.

The passage the other way uses extendability of pseudomaps into Ω^* to internal maps; this extendability also follows from Proposition 3.3.

Of course a good deal of checking is required to actually prove that the relations give rise to a 1-1 correspondence. We omit them.

We are now going to sketch the proof that the "power set pseudo object" $\beta: D \rightarrowtail \varphi(B\pitchfork\Omega)$ constructed above is actually internal. In general, proving $X_0 \rightarrowtail \varphi X$ an internal subobject means displaying its characteristic function $\varphi X \longrightarrow \varphi\Omega = \Omega$ as an internal map, which is again achieved by displaying a map $1 \longrightarrow \varphi(X\pitchfork\Omega)$, a "witness of internalness of the subobject". Now, in the data for construction of D, we have $b\colon B_0 \rightarrowtail \varphi B$, an internal subobject. If the witness of its internalness is $\bar{b}\colon 1 \longrightarrow \varphi(B\pitchfork\Omega)$, then we can consider the composite map

$$1 \xrightarrow{\bar{b}} \varphi(B\pitchfork\Omega) \xrightarrow{\varphi(\mathrm{seg})} \varphi((B\pitchfork\Omega)\pitchfork\Omega). \tag{3.2}$$

where "the segment map" $\mathrm{seg}\colon B\pitchfork\Omega \longrightarrow (B\pitchfork\Omega)\pitchfork\Omega$ is the exponential adjoint of the characteristic map of the inclusion-order-relation on $B\pitchfork\Omega$ (this order-relation can be viewed as a subobject of $(B\pitchfork\Omega)\times(B\pitchfork\Omega)$). Set-theoretically, seg assigns to a subset B' of B a certain <u>family</u> of subsets of B, namely the family of all B'' with $B'' \subseteq B'$. Now, the map (3.2) is a witness of internalness of <u>some</u> subobject of $\varphi(B\pitchfork\Omega)$, and it turns out to be precisely a witness of internalness of the subobject D. To see this requires essentially two "segment" theoretic Lemmas, the one expressing $\varphi(\mathrm{seg})$. in terms of the segment mapping for $\varphi B\pitchfork\Omega$, the other one being the equation $\ulcorner \mathrm{ch}(b)\urcorner \cdot \mathrm{seg} = \ulcorner \mathrm{ch}(\exists_b)\urcorner$. The former is obvious to formulate and prove, the latter is obvious for the category of sets, a little harder for a general topos.

We omit details; they may partly be found in the preliminary versions of the paper; partly the construction is now subsumed under the more general and slightly smoother formulation of Volger [17]. His proof also requires the functor to preserve universal as well as existential quantification.

Combining these remarks with Corollary 1.2 yields

PROPOSITION 3.4. <u>E</u>* has exponentiation.

Recall that we called a functor which preserves finite inverse limits, existential and universal quantification, and finite coproducts a <u>1st order logic preserving functor</u>. We called it <u>higher order logic preserving</u> if it furthermore preserves exponentiation.

THEOREM 3.5. Let $\varphi : \underline{E} \longrightarrow \underline{E}_0$ be a 1st order logic preserving functor between toposes. Then there is a factorization

$$\begin{tikzcd}
\underline{E} \arrow[rr, "\varphi"] \arrow[dr, "\overline{\varphi}"'] & & \underline{E}_0 \\
& \underline{E}^* \arrow[ur, "\psi"'] &
\end{tikzcd}$$

with $\underline{E}^*$ a topos, $\overline{\varphi}$ and ψ both 1st order logic preserving and with the properties

$\overline{\varphi}$ preserves higher order logic

ψ preserves elements (i.e., the map (3.3) below is bijective).

Proof. Most of the work has been done. We take $\underline{E}^*, \overline{\varphi}, \psi$ as described in section 2. It is a topos, by the Propositions 3.2, 3.2, 3.4 and Corollary 1.2. By construction, $\overline{\varphi}$ and ψ preserve Ω, and it is easy to see that they preserve finite inverse limits. The fact that $\overline{\varphi}$ preserves epics follows because φ preserves epics and ψ reflects isos (using also that epi-mono factorizations exist in $\underline{E}^*$ because it is a topos). To see that ψ preserves epics, we note that if h is a pseudomorphism from (A_0,a,A) to (B_0,b,B), then its graph Γ is an internal subobject of $\varphi(A \times B)$. If we apply $\exists_{\varphi(proj_2)}$ to it we get an internal subobject of $\varphi(B)$, according to Proposition 2.3, which is actually the image of $\psi(h): A_0 \longrightarrow B_0$. Thus epi-mono factorization of $\psi(h)$ can be lifted back to a factorization in $\underline{E}^*$. Now we use that ψ preserves isomorphisms.

The fact that $\overline{\varphi}$ preserves exponentiation of the form $B \pitchfork \Omega$ is immediate from the constructing diagram (3.1), with

$$\exists_b = \exists_{id_{\varphi(B)}} = id_{\varphi B \pitchfork \Omega} .$$

From this and the construction (Corollary 1.2) of general exponential objects out of "power-set" objects, it easily follows that $\overline{\varphi}$ preserves all exponential objects.

Next, ψ preserves points. Let $A_1 = (A_o, a, A) \in |\underline{E}^*|$. Then the map given by

$$\hom_{\underline{E}^*}(1, A_1) \longrightarrow \hom_{\underline{E}_o}(1, A_o) \tag{3.3}$$

$(A_o = \psi(A_1))$ is injective since ψ is faithful. It is surjective since every $1 \longrightarrow A_o$ is a pseudo-map; for, its graph is a map

$$\varphi(1) = 1 \longrightarrow \varphi(1 \times A),$$

and every map out of $\varphi(1)$ is internal; this follows from

$$\varphi(1 \pitchfork A) \cong \varphi(A) \cong \varphi(1) \pitchfork \varphi(A),$$

the composite isomorphism being $\hat{\varphi}_{1,A}$. Now we use the general fact that a monic internal map is also an internal subobject. This fact follows because if $\Gamma_{X,Y} \colon X \pitchfork Y \longrightarrow (X \times Y) \pitchfork \Omega$ denotes "graph formation",

$$\varphi(\Gamma_{X,Y}) \cdot \hat{\varphi} = \hat{\varphi} \cdot \Gamma_{\varphi X, \varphi Y} .$$

That $\bar{\varphi}$ and ψ preserve coproducts is easy.

Finally we must prove that $\bar{\varphi}$ and ψ preserve universal quantification. By Theorem 2.1, $\hat{\varphi}$ is monic. Therefore, the constructing diagram (3.1) for power set objects in $\underline{E}^*$ has a monic map as its bottom arrow; but this arrow can easily be seen to be

$$\hat{\psi} \ : \psi(B_1 \overset{*}{\pitchfork} \Omega^*) \longrightarrow \psi B_1 \pitchfork \Omega .$$

From this, one deduces that all instances of $\hat{\psi}$ are monic, and then again by Theorem 2.1 we get that ψ preserves universal quantification. Also, $\hat{\bar{\varphi}}$ has all its instances mono - (even iso-) morphisms, so again by Theorem 2.1, $\bar{\varphi}$ preserves universal quantification. The theorem is proved.

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