CHARACTERIZATION OF STACKS OF PRINCIPAL FIBRE BUNDLES

A. KOCK

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INSTITUT MITTAG-LEFFLER
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Characterization of stacks of principal fibre bundles

Anders Kock

A prime example of a stack, over the category of topological spaces, say, is the category of principal G-bundles, where the group G is fixed, but the base space B varies.

The notion of group, more generally groupoid, makes sense in any category $\mathbf{B}$, and so does the notion of principal fibre bundle, at least if a sufficiently good class $\mathcal{D}$ of effective descent morphisms is specified (e.g. the class of etale surjections, or of open surjections, for the category of topological spaces). For a groupoid object $G_\bullet$ in such $\mathbf{B}$, one has the notion of *principal* $G_\bullet$*-bundle* on an object $B \in \mathbf{B}$, and these principal bundles form a stack $B(G_\bullet)$ over $\mathbf{B}$. (Both notions depend on the class $\mathcal{D}$.) The purpose of the present note is to characterize stacks over $\mathbf{B}$ that arise this way: namely as stacks-in-groupoids $P : \mathbf{X} \to \mathbf{B}$ whose total category $\mathbf{X}$ has binary products, and which admits a sufficiently well-supported object – the latter is also a notion relative to $\mathcal{D}$. (The first two sections below do not depend on $\mathcal{D}$.)

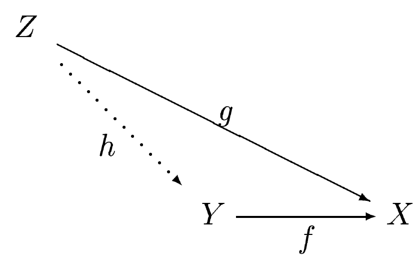
These two properties are essentially an abstraction of the properties which were singled out by Deligne and Mumford; but their notion, corresponding to our " $\mathbf{X}$ has binary products" , namely "representability of the diagonal $\mathbf{X} \to \mathbf{X} \times_{\mathbf{B}} \mathbf{X}$" depends essentially on utilizing the 2-dimensional character of the category of stacks over $\mathbf{B}$, and is much more dependent on cleavage. Our theory is entirely in 1-dimensional category theory.

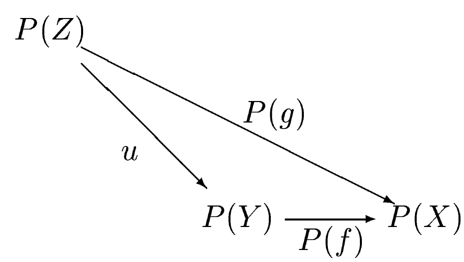
We are not claiming much originality; the results are, in one or the other category, more or less known (possibly except the last Section, whence the title of the Note). I want to acknowledge discussions over the years with Ieke Moerdijk, where I learned about the ubiquity of principal bundles. Also, I acknowledge inspiration from Dorette Pronk's Thesis, [4].

Finally, I want to express my thanks to the Institut Mittag-Leffler, where the present work was carried out.

1 Fibrations

Recall that for any functor $P : \mathbf{X} \to \mathbf{B}$, an arrow $f : Y \to X$ in $\mathbf{X}$ is called *cartesian* (w.r. to P) if it has the following universal property: for any arrow $g : Z \to X$ and any factorization of $P(g)$ over $P(f)$, $P(g) = P(f) \circ u$, there exists a unique $h : Z \to Y$ with $P(h) = u$ and $g = f \circ h$, as displayed in the diagram[1]:





The functor P is called a *fibration* if for every $u : J \to I$ in $\mathbf{B}$ and every $X \in \mathbf{X}$ with $P(X) = I$, there exists a cartesian arrow $f : Y \to X$ with $P(f) = u$ (called a *cartesian lift* of u with codomain X).

One denotes by $\mathbf{X}_I$ the category of those objects and arrows in $\mathbf{X}$ which by P go to I, respectively to the identity arrow on I; such arrows are called *vertical*, over I. It is easy to see that an arrow which is both vertical and cartesian is an isomorphism. More globally, one has the following well known and easy

Proposition 1 *For a fibration $P : \mathbf{X} \to \mathbf{B}$, the following two conditions are equivalent:*

1. All arrows in $\mathbf{X}$ are cartesian

2. All categories $\mathbf{X}_I$ are groupoids

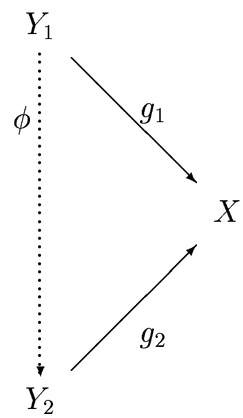
(i.e., all vertical arrows are invertible).

[1] All diagrams are made with Paul Taylor's package.

Fibrations with the equivalent properties of the Proposition are called *fibrations-in-groupoids* or, slightly misleadingly, just groupoids (cf. [6] or [2] for this latter terminology; we are not adopting this wide use of the word 'groupoid'). In this note, we are mainly concerned with fibrations-in-groupoids.

Some of the notions we consider are relevant also for general fibrations, but then the word *cartesian* should be inserted at various places; for fibrations in groupoids, the word has no function, since all arrows are cartesian anyway.

We shall encounter not only lifts of individual arrows, but simultaneous lift of, say, simplicial diagrams. For simplicity, consider a diagram in the base category $\mathbf{B}$ consisting of two parallel arrows $J \to I$, and consider an object X in $\mathbf{X}_I$. Suppose we take a cartesian lift with codomain X of each of the two parallel arrows, then we have a situation as displayed with full arrows in



$$J \underset{f_2}{\overset{f_1}{\rightrightarrows}} I$$

where g_i is a cartesian lift of f_i. There is no privileged vertical comparison map (dotted arrow) since the g_i's are lifts of *different* arrows in the base. So to provide a vertical map, as displayed, is to provide a *structure* to the situation. Note that no vertical map can make the triangle commute unless $f_1 = f_2$. We may replace g_2 by the composite $g_2 \circ \phi$, and get another lift of f_2 with codomain X, but now with the same domain Y_1 as g_1 has; i.e. we get a *parallel* lift. The moral is: *to give a parallel lift of parallel arrows is* *to provide a structure*; in fact, descent data, as we shall meet it, will here be exhibited precisely in terms of *parallel lift of a parallel pair.* – Similarly for more elaborate diagrams in the base. In fact, descent data is better exhibited as simplicial lift of a simplicial diagram, then the cocycle condition is automatically taken care of.

Let $P : \mathbf{X} \to \mathbf{B}$ be a (general) fibration. Given an arrow $u : J \to I$ in $\mathbf{B}$. If one for each $X \in \mathbf{X}_I$ *chooses* a cartesian lift of u with codomain X, and denotes the domain of the chosen arrow u^*X, it is a standard consequence of the universal property of cartesian arrows that u^* extends to a functor $u^* : \mathbf{X}_I \to \mathbf{X}_J$. Sometimes, fibrations are presented in terms of these "transition functors" u^*, and collectively they form a "pseudofunctor" from $\mathbf{B}$ to the category of categories, but we shall nor use this viewpoint here.

The following is an important fact - we refer to [5], Theorem 8.3, or [3], B.1.4.1, for a proof.

Theorem 1 *Assume $P : \mathbf{X} \to \mathbf{B}$ is a fibration, and that $\mathbf{B}$ has pull-backs. Then the following are equivalent:*

1. $\mathbf{X}$ has, and P preserves, pull-backs.

2. each $\mathbf{X}_I$ has pull-backs, and all transition functors $u^* : \mathbf{X}_I \to \mathbf{X}_J$ preserve pull-backs.

Since any groupoid has pull-backs, and any functor between groupoids preserves them, it follows that if $\mathbf{B}$ has pull-backs, and $P : \mathbf{X} \to \mathbf{B}$ is a fibration in groupoids, then $\mathbf{X}$ has, and P preserves pull-backs. But $\mathbf{X}$ may have some finite limits which are not preserved by P; in fact, our main concern here will be binary products in $\mathbf{X}$, and they will only in trivial cases be preserved. Also, the individual groupoids $\mathbf{X}_I$ will of course not have binary products either, unless they are trivial.

2 Groupoids and their actions

In the following, $\mathbf{B}$ denotes a category with pull-backs. If $I \in \mathbf{B}$, we shall say "I is a "set" ". Recall that a groupoid in $\mathbf{B}$ may be presented in terms of its *nerve* $G_\bullet$, with G_0 = "set" of objects, G_1 = "set" of arrows, G_2 = "set" of composable pairs, etc.; jointly, the G_i's form a simplicial "set"

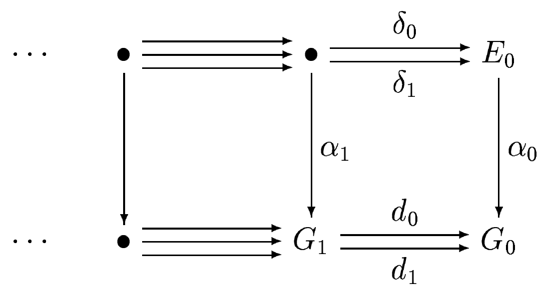
$$\ldots G_2 \overset{d_0}{\underset{d_2}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}} G_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} G_0$$

with d_0 and $d_1 : G_1 \to G_0$ domain- and codomain-formation, etc. The degeneracies like $s_0 : G_0 \to G_1$ pick out identity maps, they are nor exhibited in the figure, in fact, we will often exhibit $G_\bullet$ just by the lowest part $G_1 \rightrightarrows G_0$. (The higher G_i's appear as pull-backs, e.g. $G_2 = G_1 \times_{G_0} G_1$.)

A characteristic property of groupoids (rather than just categories) is that all commutative squares expressing simplicial identities among the face operators are in fact pull-back squares. For instance, the simplicial identity $d_1 \circ d_1 = d_1 \circ d_2$ ("the codomain of a composite is the codomain of the second of the two arrows") is a pull-back, because for two arrows α and β with common codomain, there is a unique composable pair, whose composite is α and whose second component is β, namely $(\beta^{-1} \circ \alpha, \beta)$ (composing from right to left !).

A groupoid is an *equivalence relation* if $d_0, d_1 : G_1 \to G_0$ are jointly mono.

An *action* by a groupoid $G_\bullet$ on $E_0 \to G_0$ may be exhibited in simplicial terms as



(1)

where each of the squares (with indices matching) is a pull-back square. Since pull-backs of pull-backs are pull-backs, it follows that the commuting squares expressing the simplicial identities in the upper row are pull-backs, and so the upper row $E_\bullet$ is a groupoid in its own right, called the *action* groupoid of the action. In fact, by drawing suitable boxes with three sides being pull-backs to conclude that so is the fourth, using the familiar pasting properties of pull-back diagrams, one may formulate matters more completely:

Proposition 2 *Let $f_\bullet : E_\bullet \to G_\bullet$ be a simplicial map, (as exhibited partially in (1) above), and assume that $G_\bullet$ is a groupoid. Then the following are equivalent:*

1) all matching squares in (1) are pull-backs

2) one (hence both) of the two squares in the right hand part of (1) are pull-backs, and $E_\bullet$ is a groupoid

3) $f_\bullet : E_\bullet \to G_\bullet$ is an action groupoid.

The way to reconstruct a right action from this simplicial data is: given $e \in E_0$ and $g \in G_1$ with $\alpha(e) = d_1(g)$; then since the lower square with the d_1 is a pull back, there is a unique a in the upper left corner with $\delta_1(a) = e$ and $\alpha_1(a) = g$; then $\delta_0(a)$ goes by α to $d_0(g)$ and may be considered the effect of acting by g on e. One could of course equally have reconstructed a left action.

A functor between groupoids is, in simplicial terms, just a simplicial map between their nerves. So an action groupoid comes equipped with a functor to the groupoid that acts, but a functor with the special property that all the squares (as exhibited in the diagram above) are pull-backs. Given a functor $E_\bullet \to G_\bullet$ with this property, we say that it exhibits $E_\bullet$ as an *action groupoid* over $G_\bullet$.

We say that the action is *principal homogeneous* if the action groupoid is an equivalence relation.

It is easy to see that if $G_\bullet$ is an equivalence relation, then so is also any action groupoid $E_\bullet$ for any $G_\bullet$-action. In other words, any action by an equivalence relation is automatically principal homogeneous.

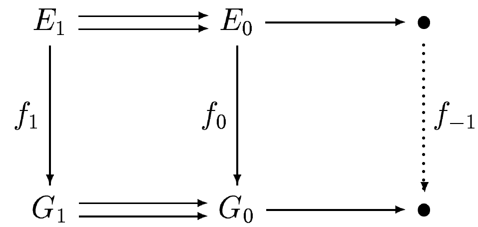
3 Descent, and principal bundles

From now, the notions and results depend on the choice of a class $\mathcal{D}$ of "good epis" in the category $\mathbf{B}$. In an exact category (in the sense of Barr; "effective regular" in [3] A.1.3), e.g. a topos, one could take all regular epis, in topological spaces one could take the class of open surjections, or the class of surjections that are local homeomorphisms – this is the classical choice. In order not to go into technicalities, we present axiomatically the properties we need of $\mathcal{D}$. There may be some redundancy in this list of Axioms.

The $\mathcal{D}$-Axioms

1. All $q \in \mathcal{D}$ are regular epis (coequalizer of their kernel pair).
2. The pull-back of a $q \in \mathcal{D}$ is again in $\mathcal{D}$
3. If $G_1 \rightrightarrows G_0$ is an equivalence relation, with one (hence both) of the exhibited maps in $\mathcal{D}$, then the coequalizer exists and is in $\mathcal{D}$.
4. (The descent property). If $f_\bullet : E_\bullet \to G_\bullet$ is an action groupoid, and $G_\bullet$ (and hence $E_\bullet$) is an equivalence relation, with structural maps in $\mathcal{D}$, as

described in 3., then the resulting right hand square in the diagram



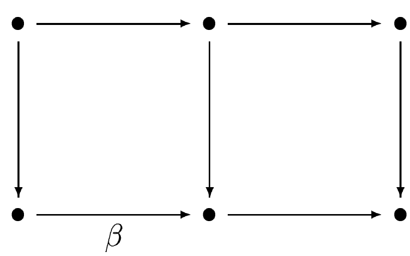
is a pull-back (where the two exhibited horizontal maps are the respective coequalizers, and where the dotted arrow is the unique one making the square commute).

5. If three sides in a pull-back square are in $\mathcal{D}$, then so is the fourth.

6. Pulling back along an epi in $\mathcal{D}$ reflects the property of being an isomorphism.

It is moderately easy to prove (using Axiom 2 and 6)

Lemma 1 *Consider a commutative diagram*



with $\beta \in \mathcal{D}$. If the total diagram and the left hand square are pull-backs, them so is the right hand square. Also, pulling back along a $\beta \in \mathcal{D}$ reflects the property of being a pull-back square.

We consider a principal homogeneous action $E_\bullet \to G_\bullet$ by a groupoid $G_\bullet$. If the equivalence relation $E_\bullet$ is effective, with quotient $\beta : E_0 \to B$, say, (and that will be the case if the structural map $d_0 : G_1 \to G_0$ is in $\mathcal{D}$), the principal homogeneous action by the groupoid $G_\bullet$ gives rise to a *principal* $G_\bullet$-bundle on B, by which one means a diagram like (1) but augmented by $\beta : E_0 \to B$, and such that the augmented top row is exact: β is coequalizer of δ_0, δ_1, which in turn is kernel pair for β.

Let $u : B' \to B$ be a map in $\mathbf{B}$, and let E', β', α', resp. E, β, α be principal $G_\bullet$ bundles on B' and B, respectively. A *morphism* of principal bundles, above u is then given by a map $f : E' \to E$ making the square

$$\begin{array}{ccc} E'_0 & \xrightarrow{\beta'} & B' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ E_0 & \xrightarrow[\beta]{} & B \end{array} \tag{2}$$

commutative, and commuting with the action of $G_\bullet$; the latter condition can be expressed by saying that f extends to a simplicial map from the row to the left of E' to the row to the left of E. Since the maps from these rows to the $G_\bullet$-row form pull-back squares with the d_i's, and since the commutaive squares expressing the simplicial identities in the E' as well as in the E-rows are pull-backs, it follows that even the commutative square (2) is a pull-back, provided β is in $\mathcal{D}$, which in turn will follow if we assume that $d_0 : G_1 \to G_0$ is a $\mathcal{D}$ epi.

We let $\mathbf{X}$ denote the category of principal $G_\bullet$-bundles, with respect to the class $\mathcal{D}$; the functor $P : \mathbf{X} \to \mathbf{B}$ associates to such a $G_\bullet$-bundle

$$\begin{array}{ccccc} \cdots & \rightrightarrows & E & \longrightarrow & B \\ & & \downarrow & & \\ \cdots & \rightrightarrows & G_0 & & \end{array}$$

the "set" B. The following is then well known, and, as mentioned in the introduction, a main example for the notion of stack.

Proposition 3 *Assume that the groupoid $G_\bullet$ is a $\mathcal{D}$-groupoid. Then the functor $P : \mathbf{X} \to \mathbf{B}$ is a fibration, in fact a fibration-in-groupoids.*

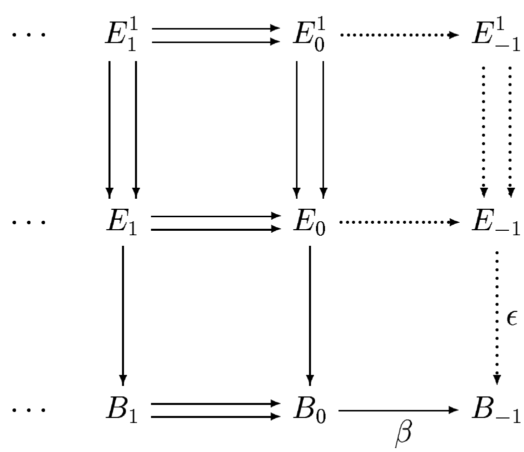
The last assertion is essentially the assertion that the square exhibited in (2) above is a pull-back, and therefore easily seen to be a *cartesian* morphism over u; thus all morphisms are cartesian. (Equivalently, by Proposition 1, a morphism of principal bundles over the same base object B is necessarily an isomorphism).

4 Stacks of principal bundles

Recall that a *stack* is a fibered category $P : \mathbf{X} \to \mathbf{B}$ with a descent property with respect to some covering notion in $\mathbf{B}$ (which has to be specified for the notion of stack to make sense). In our case, we shall consider the covering notion given by a class $\mathcal{D}$ of "descent" epis. Also, we shall only consider stacks-in-groupoids. We shall simply say *stack* to mean a fibration in groupoids which has the descent property with respect to the class $\mathcal{D}$.

We now prove that for a $\mathcal{D}$-groupoid $G_\bullet$, the fibered category $P : \mathbf{X} \to \mathbf{B}$ of principal $G_\bullet$-bundles is a stack with respect to the coverings given by $\mathcal{D}$. The non-trivial part is to show that for $\beta : B_0 \to B_{-1}$ a $\mathcal{D}$-epi, the functor $\mathbf{X}_{B_{-1}} \to Desc(\beta)$ is essentially surjective on objects.

So let $B_\bullet$ be the (nerve of) the kernel pair of β, and let there be given a simplicial object $E_\bullet$ of principal $G_\bullet$ bundles, mapping by $P : \mathbf{X} \to \mathbf{B}$ to $B_\bullet$. We then have a bisimplicial diagram, partly exhibited by the full arrows in



Each of the columns (not, yet, the rightmost one) are the (nerves of) the various action groupoids; together they constitute the various principal bundles in the simplicial set of that in turn constitutes a descent datum in $\mathbf{X}$. And, being principal bundles, they come with an augmentation (quotient map) to their respective base spaces, $E_n \to B_n$.

The E-objects of the rightmost column are produced using that β has the descent property assumed for the class $\mathcal{D}$. So the map $E_0 \to E_{-1}$, as well as $E^1_0 \to E^1_{-1}$, etc. are therefore coequalizers. By the pull-back property in Axiom 4 for the class $\mathcal{D}$, all squares ending in β in the right hand column are pullbacks, hence so are *all* matching squares in this column. The arrows in the rightmost column satisfy the simplial identities, since the ones in the second but last column do, and the maps $E_0^i \to E_{-1}^i$ are epimorphisms. Except for the rightmost one, all columns (stripped of their augmentations to the B_i's), are (nerves of) action groupoids for $G_\bullet$-actions, and therefore they all are equipped with "structural" functors $q_\bullet$ (simplicial maps) to (the nerve of) $G_\bullet$, in a way compatible with with the horizontal maps between the E_i^j's. By the coequalizer property of the $E_0^j \to E_{-1}^j$, the individual maps of the structural functors produce maps $q_j' : E_{-1}^j \to G_j$, and evidently these q_j's form a simplicial map. From Lemma 1 follows that the squares from the rightmost column to the nerve of $G_\bullet$ are pull-backs, and from Proposition 2 then follows that the maps q' make the right hand column into an action groupoid for $G_\bullet$. So the right hand column is a groupoid (in fact a $\mathcal{D}$-groupoid, since $G_\bullet$ is); but it is even the kernel pair groupoid of the map ϵ. For, pulling it back along β yields the column above B_0, which is the kernel pair of $E_0 \to B_0$, but by Lemma 1, pulling back along the $\mathcal{D}$-epi β reflects the property of being a pull-back (hence of being a kernel pair). Finally, ϵ is the coequalizer of the vertical maps above it; for, the coequalizer exists in any case by $\mathcal{D}$-Axiom 3. Then the comparison map from this coequalizer to ϵ pulls back along β to an isomorphism, hence is an isomorphism.

5 Binary products in $BG_\bullet$

Recall that the total category $\mathbf{X}$ of a fibration $\mathbf{X} \to \mathbf{B}$ always has pull-backs (assuming, as we do throughout, that $\mathbf{B}$ does, and that the fibration is a fibration in groupoids). What about other finite limits? We shall prove

Proposition 4 *Let $G_\bullet$ be a $\mathcal{D}$-groupoid in $\mathbf{B}$. Then the Category $BG_\bullet$ of principal $G_\bullet$-bundles has binary products.*

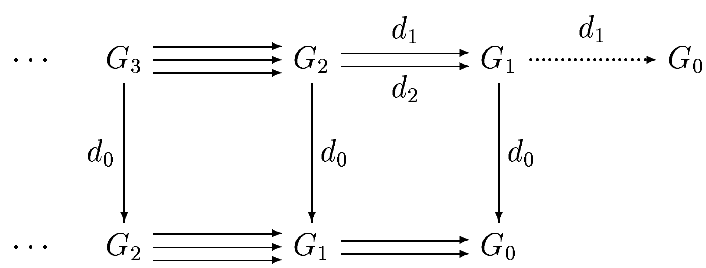
Note: only in trivial cases will there be a terminal object, see below. So the word "binary" cannot be replaced by "finite". Also, only in trivial cases will the functor $P : BG_\bullet \to \mathbf{B}$ preserve the binary products.

Proof. The way we have set things up, there is hardly anything to prove. A principal bundle is an action groupoid over $G_\bullet$, equipped with an "exact" augmentation. The product of two such groupoids is just taken to be their pull-back over $G_\bullet$. It is clear that the pull-back of two action groupoids over $G_\bullet$ is again an action groupoid over $G_\bullet$. We just have to make sure that this pull-back groupoid aquires an augmentation, suitably compatible with the given ones.

In more detail, let the given principal bundles be $E_\bullet \to G_\bullet$ and $F_\bullet \to G_\bullet$ respectively, and with augmentations (quotient maps) $E_0 \to E_{-1}$ and $F_0 \to F_{-1}$, respectively. Let $H_\bullet := E_\bullet \times_{G_\bullet} F_\bullet$. Then since $F_\bullet \to G_\bullet$ is an action groupoid, then so is its pull-back, $H_\bullet \to E_\bullet$, and since $E_\bullet$ is an equivalence relation, then so is $H_\bullet$, and since $G_\bullet$ is a $\mathcal{D}$-groupoid, then so is $H_\bullet$. But an equivalence relation whose structural maps are in $\mathcal{D}$ is an effective equivalence relation, with quotient map in $\mathcal{D}$, by the third $\mathcal{D}$-axiom. Denote this quotient map $H_0 \to H_{-1}$. Since $H_\bullet \to E_\bullet$ is an action groupoid, and $E_\bullet$ is an effective $\mathcal{D}$-equivalence relation, it follows from the fourth (descent) $\mathcal{D}$-axiom that $H_\bullet$ also is effective and that the quotient map $H_0 \to H_{-1}$ fits into a pull-back square with $E_0 \to E_{-1}$. Similarly for $H_\bullet \to F_\bullet$ and the resulting $H_{-1} \to F_{-1}$. So we have constructed a principal $G_\bullet$-bundle on the base space H_{-1}, and equipped with projections to the two given principal bundles (on the base spaces E_{-1} and F_{-1}, respectively). Its universal property is almost immediate from the universal property of $H_\bullet = E_\bullet \times_{G_\bullet} F_\bullet$ in the category of groupoids over $G_\bullet$.

If there is a terminal object in $BG_\bullet$, it has to be the identity map on $G_\bullet$, and if that is a principal bundle, $G_\bullet$ is an effective equivalence relation with quotient map $G_0 \to G_{-1}$ in $\mathcal{D}$, and the fibration $BG_\bullet$ is then equivalent to the domain-formation $\mathbf{B}/G_{-1} \to \mathbf{B}$, essentially by the descent property of $G_0 \to G_{-1}$.

Among the objects in $BG_\bullet$, there is a particular important one, called $Dec(G)$, (following Illusie, Duskin,...); we exhibit it simplicially:



where the full arrows form an action groupoid over $G_\bullet$, and the dotted one is the augmentation witnessing that the top line is a principal $G_\bullet$-bundle with base G_0. The fact that the squares are pull-backs follow because the squares representing the simplicial identities in the (nerve of) a groupoid are pull-backs, and for the same reason the top line is itself a groupoid. Also, for the same reason again, the augmentation (dotted arrow) has the top line as its kernel pair. Finally, the fact that the augmentation is coequalizer of this kernel pair follows because it is a $\mathcal{D}$ epi, by assumption on $G_\bullet$.

A special property which $Dec(G)$ has, as an object in $BG_\bullet$, is that it is an *atlas* object, in the sense of the following definition. We assume $\mathcal{D}$ is a class of descent epis, i.e. satisfying the $\mathcal{D}$-axioms in Section 3.

Definition 1 *Let $P : \mathbf{X} \to \mathbf{B}$ be a fibration in groupoids. An object $U \in \mathbf{X}$ is called an* atlas *if for every $X \in \mathbf{X}$, the binary product $X \times U$ exists in $\mathbf{X}$, and the projection $X \times U \to X$ goes to a $\mathcal{D}$-epi by P.*

One half of the reason for the definition is, as we claimed,

Proposition 5 *If $G_\bullet$ is a $\mathcal{D}$-groupoid, then $Dec(G)$ is an atlas in the fibered category $BG_\bullet \to \mathbf{B}$.*

Proof. Given a principal $G_\bullet$-bundle $E_\bullet \to G_\bullet$, with augmentation (quotient) $E_0 \to E_{-1}$, say. The projection map $\gamma_\bullet$ from $E_\bullet \times_{G_\bullet} Dec(G)$ to $E_\bullet$ has for its lowest dimensional part (dimension 0 and -1) the square

$$\begin{array}{ccc} E_0 \times_{G_0} G_1 & \longrightarrow & \bullet \\ \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \gamma_{-1}} \\ E_0 & \longrightarrow & E_{-1} \end{array}$$

and this square is a pull-back diagram, for descent reasons encountered several times. Since γ_0 is a $\mathcal{D}$-epi (it comes about by pulling back $d_0 : G_1 \to G_0$), we have that three of the maps in the displayed pull-back squares are $\mathcal{D}$-epis, hence so is the fourth γ_{-1}. But applying $P : BG_\bullet \to \mathbf{B}$ to $\gamma_\bullet$ gives γ_{-1}.

6 Characterization of stacks of principal bundles

We fix a base category $\mathbf{B}$ with pull-backs, and a class $\mathcal{D}$ of descent epis, i.e., satisfying the axioms mentioned in Section 3. With respect to this class, we have the notion of *stack* over $\mathbf{B}$. We know already from the previous section that if $G_\bullet$ is a $\mathcal{D}$-groupoid in $\mathbf{B}$, then the category of its principal bundles is a stack over $\mathbf{B}$, which has binary products, as well as an atlas. But conversely, we have

Theorem 2 *Let $\mathbf{X} \to \mathbf{B}$ be a fibration in groupoids, and assume that the total category $\mathbf{X}$ has binary products, as well as an atlas. Then $\mathbf{X}$ is equivalent (as a category fibered over $\mathbf{B}$) to a stack of principal $G_\bullet$-bundles, for some $\mathcal{D}$-groupoid $G_\bullet$ in $\mathbf{B}$.*

Proof. Let U be an atlas. We have the following groupoid $U_\bullet$ in $\mathbf{X}$

$$\ldots U \times U \times U \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; U \times U \; \substack{\longrightarrow \\ \longrightarrow} \; U,$$

i.e. $U_n = U^{n+1}$, and the simplicial d_i's are just the various maps arising from the projections. In set-theoretic terms, $U_\bullet$ is the codiscrete groupoid on the "set" U of objects. Since P preserves pull-backs, it follows that $P(U_\bullet)$ is a groupoid in $\mathbf{B}$, which is to be our $G_\bullet$, $G_0 = P(U)$, $G_1 = P(U \times U)$, etc. Furthermore, if $X \in \mathbf{X}$, we have a "principal $U_\bullet$ bundle" in $\mathbf{X}$, namely $X \times U$, more precisely, $X \times U^n$, with the projections to the U^n's as action groupoid functor to the groupoid $U_\bullet$, and the projection $X \times U \to X$ as augmentation. We put the word "principal bundle" in quotation marks, because we don't have such notion in $\mathbf{X}$, because we don't there have a class of descent epis. But precisely the facts that U is an atlas, and that P preserves pull-backs give that P takes this "principal bundle" into a genuine principal bundle, (over the groupoid $P(U_\bullet) = G_\bullet$, and with respect to the class $\mathcal{D}$ in $\mathbf{B}$). Thus we have a functor $X \mapsto P(X \times U \to U)$ from $\mathbf{X}$ to $BG_\bullet$. We shall prove that this functor is an equivalence. We produce a quasi-inverse, and not surprisingly, this quasi-inverse is going to involve some choices, here in the form of some choice (cleavage) of cartesian arrows in $\mathbf{X}$.

So we suppose a principal $G_\bullet$-bundle $E_\bullet$ given, partially displayed in

$$\begin{array}{ccccc} \cdots & E_1 & \substack{\longrightarrow \\ \longrightarrow} & E_0 & \longrightarrow & E_{-1} \\ & \big\downarrow & & \big\downarrow & & \\ \cdots & G_1 & \substack{\longrightarrow \\ \longrightarrow} & G_0 & & \end{array} \qquad (3)$$

This is a principal bundle on the base object E_{-1}. To produce an object in $\mathbf{X}_{E_{-1}}$, we note that we have a simplicial object $U_\bullet$ in $\mathbf{X}$ over $G_\bullet$ (this is how we constructed the latter). Choosing a lift of $E_0 \to G_0$ with codomain U, say $X_0 \to U$, and choosing a simplicial lift of $E_1 \to G_1$ with codomain $U \times U = U_1$, say $X_1 \to U_1$ etc., we get by utilizing the universal property of cartesian arrows a simplicial lift of the whole diagram (3). The part of this lift which is over the E-part of the diagram can be seen as descent data on the object X_0. We have thus constructed a functor from the category (groupoid) of principal $G_\bullet$-bundles on base object E_{-1}, to the category of descent data for descent in $\mathbf{X}$ along $E_0 \to E_{-1}$. But since $\mathbf{X}$ was assumed to be a *stack* over $\mathbf{B}$, the category of such descent data is equivalent to $\mathbf{X}_{E_{-1}}$. Thus we get a functor from principal $G_\bullet$- bundles on base object E_{-1} to the fibre of $\mathbf{X}$ above E_{-1}, providing a (fibrewise) quasi-inverse to the (more canonical) functor from $\mathbf{X}$ to $BG_\bullet$. It is easy to see that the processes described are mutually inverse, up to isomorphism.

Fine Moduli Spaces. We conclude with a remark about the possibility of a *terminal object* in the total category $\mathbf{X}$ of a stack $P : \mathbf{X} \to \mathbf{B}$ in groupoids. We already saw that if $\mathbf{X}$ satisfy the conditions of the Theorem above, then this can only happen in the trivial case where $\mathbf{X}$ is equivalent to $\mathbf{B}/B$. But if we don't assume binary products or an "atlas" object, then a terminal object in $\mathbf{X}$ does not force triviality; rather, this is an abstraction of the situation where there is a "fine moduli space for the stack". It is easy to see that for *any* object $U \in \mathbf{X}$, we have a fibration of the slice category $\mathbf{X}/U$ over $\mathbf{B}$, and this category has "quasi discrete" fibres, i.e., the fibres are equivalent to a discrete categories; in fact

$$(\mathbf{X}/U)_I \to Hom_{\mathbf{B}}(I, P(U))$$

is an equivalence of the fibre with a discrete category, namely with the set $Hom_{\mathbf{B}}(I, P(U)$. If now U happens to be terminal in $\mathbf{X}$, we have equivalences

$$\mathbf{X}_I \sim (\mathbf{X}/U)_I \sim Hom_{\mathbf{B}}(I, P(U)).$$

So one may consider $P(U)$ as a fine moduli space for $\mathbf{X}$: there is a bijective correspondence between isomorphism classes of $\mathbf{X}$-objects above I, and the set of maps from I into the moduli space.

As an example from outside algebraic geometry: Let $\mathbf{B}$ be an elementary topos, and let $\mathbf{X}$ have as objects all monomorphisms, and as arrows are pull-back squares, with monos in the left and right end. For $P : \mathbf{X} \to \mathbf{P}$ is codomain-formation. Then $\mathbf{X}$ has a terminal object, namely true : $1 \to \Omega$.

So Ω is a fine moduli space for the notion of monic map. Note that this viewpoint on the classifying property of Ω in one respect is simpler than the one "Ω as a subobject classifier"; for, with this latter viewpoint one first has to *collect* a proper class of monomorphisms into *one* subobject. Set theoretically, the latter procedure means that one is considering a set (the set of subobjects) whose elements are proper classes. (I think this point has been made by Benabou.)

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Permanent address of author:
Mathematics Institute, University of Aarhus, DK 8000 Aarhus C, Denmark,
kock@imf.au.dk