Closed limit formula for higher derivatives

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Let R denote the number line (say, the real numbers $\mathbb{R}$, or the basic ring in a model of synthetic differential geometry (SDG)). Let $T(R)$ denote the distributions of compact support on R. Then $T(R)$ carries structure of a commutative R-algebra, with convolution $*$ as multiplication; the unit is the Dirac distribution δ_0 at $0 \in R$. Recall that $\delta_x * \delta_y = \delta_{x+y}$, where for any x, δ_x denotes the Dirac distribution at x.

Recall binomial expansion in a commutativer ring:

$$(a-b)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^{n-k} b^k.$$

In particular, in the convolution ring $(T(R), *)$ for the number line R,

$$(\delta_\omega - \delta_0)^{*n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \delta_\omega^{*(n-k)} \delta_0^{*k}.$$

Since δ_0 is the multiplicative unit in $T(R)$, and $\delta_x^{*k} = \delta_{kx}$, this may be rewritten as the identity

$$(\delta_\omega - \delta_0)^{*n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \delta_{(n-k)\omega} \tag{1}$$

For each invertible ω, we have the basic difference quotient distribution $\Delta(\omega) := \omega^{-1} \cdot (\delta_\omega - \delta_0) \in T(R)$. As a function of ω, it defines a map $\Delta : R^\times \to T(R)$ (where $R^\times$ denotes the set of invertible elements in R). It extends uniquely[1] to a function, likewise denoted Δ,

$$\Delta : R \to T(R),$$

and $\Delta(0) = (\delta_0)'$. Composing with the map $(-)^{*n} : T(R) \to T(R)$ (n-fold convolution power) gives a map $\Delta_n : R \to T(R)$, whose value at an invertible $\omega \in R$ by (1) is

$$\omega^{-n} \cdot \sum_{k=0}^{n} \binom{n}{k} (-1)^k \delta_{(n-k)\omega}$$

[1]We assume tacitly that all functions considered are smooth, or that we are in the context of SDG.

and whose value at 0 is $((\delta_0)')^{*n}$.

Recall that the derivative P' of a distribution $P \in T(R)$ may be defined in terms of convolution, namely $P' := (\delta_0)' * P$, and thus the n-fold derivative $P^{(n)}$ of P is $P^{(n)} = ((\delta_0)')^{*n} * P$. In particular, $\delta_0^{(n)} = ((\delta_0)')^{*n}$.

Therefore, there is a function of $\omega \in R$, which for $\omega \in R^\times$ is given by the expression

$$\delta_0^{(n)} - \omega^{-n} \cdot \sum_{k=0}^{n} (-1)^k \binom{n}{k} \delta_{(n-k)\cdot\omega}$$

and which vanishes for $\omega = 0$; therefore (Hadamard), there is a function $\rho_n : R \to T(R)$, such that for $\omega \in R^\times$

$$\delta_0^{(n)} - \omega^{-n} \cdot \sum_{k=0}^{n} (-1)^k \binom{n}{k} \delta_{(n-k)\cdot\omega} = \omega \cdot \rho_n(\omega)$$

for all $\omega \in R$. Multiplying this equation by ω^n, we get that

$$\omega^n \cdot \delta_0^{(n)} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \delta_{(n-k)\cdot\omega} = \omega^{n+1} \cdot \rho_n(\omega) \tag{2}$$

for all $\omega \in R$.

In terms of SDG, we have in particular for all ω with $\omega^{n+1} = 0$ that

$$\omega^n \cdot \delta_0^{(n)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \delta_{(n-k)\cdot\omega}, \tag{3}$$

and by the cancellation principles of SDG, this characterizes $\delta_0^{(n)}$. Thus for instance $\delta_0^{(2)}$ is characterized by

$$\omega^2 \cdot \delta_0^{(2)} = \delta_{2\omega} - 2\delta_\omega + \delta_0 \text{ for all } \omega \text{ with } \omega^3 = 0.$$

By convoluting (3) by $P \in T(R)$, we therefore have

Theorem 1 *For any $P \in T(R)$, and for any ω with $\omega^{n+1} = 0$, we have*

$$\omega^n P^{(n)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \delta_{(n-k)\cdot\omega} * P.$$

Recall ([4], Prop. 17.4) that for any test function $\phi : R \to R$, we have[2] $\langle P', \phi \rangle = \langle P, \phi' \rangle$, hence by iteration $\langle P^{(n)}, \phi \rangle = \langle P, \phi^{(n)} \rangle$. In particular,

$$\langle \delta_0^{(n)}, \phi \rangle = \langle \delta_0, \phi^{(n)} \rangle = \phi^{(n)}(0).$$

So by applying the two sides of (3) to an arbitrary test function $\phi : R \to R$, we get

[2]this is the opposite sign convention than the classical one of Schwartz; see the footnote 7 in [4]

Theorem 2 *For any $\phi : R \to R$, and for any ω with $\omega^{n+1} = 0$, we have*

$$\omega^n \phi^{(n)}(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi((n-k)\omega).$$

More generally, by translation: for any $x \in R$,

$$\omega^n \phi^{(n)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi(x + (n-k)\omega). \tag{4}$$

One may alternatively proceed from (2) without using the language of SDG, and arrive at classical formulations of the two theorems presented; for instance, the latter may be formulated

Theorem 3 *For any $\phi : R \to R$, the function of $\omega \in R$ given by*

$$\omega^n \phi^{(n)}(0) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi((n-k)\omega)$$

vanishes to order $n+1$.

or, for the case of $R = \mathbb{R}$

$$\phi^{(n)}(0) = \lim_{\omega \to 0} \omega^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi((n-k)\omega).$$

An application. Let us consider the right hand side of (4), for $n = 2$ and $y = x + 2\omega$; it is

$$\phi(y) - 2\phi(\mu(x,y)) + \phi(x) \quad \text{which we write} \quad d^{(2)}\phi(x,y), \tag{5}$$

where $\mu(x,y)$ denotes the *midpoint* of x and y; since $y = x + 2\omega$, this midpoint is $x + \omega$. Since $\omega^3 = 0$, the points x and y are second order neighbours in R. In [2] (see also [3] 8.2), it is shown that a symmetric affine connection on a manifold M can equivalently be encoded in terms of such "midpoint formation μ for second order neighbours", leading us to consider the expression (5) for any (smooth) R-valued function ϕ on a manifold with such a midpoint formation structure μ. The expression (5), as a function of x and y, therefore defines a map $d^{(2)}\phi : M_{(2)} \to R$, where $M_{(2)} \subseteq M \times M$ is the set of pairs of points which are second order neighbours. Furthermore, it can be proved that this $d^{(2)}\phi$ vanishes on $M_{(1)} \subseteq M_{(2)}$ (= the set of pairs of points which are first order neighbours). Thus, $d^{(2)}\phi$ is a (combinatorial) quadratic differential form on M, in the sense of [3], 8.1.[3]

[3]Symmetric affine connections are also equivalent to sprays; probably, the construction of $d^{(2)}\phi$ is well known analytically for manifolds equipped with sprays. However, affine connections, and in particular midpoint formation, seems to me to have more immediate geometric content than sprays.

Let us calculate $d^{(2)}\phi$ for the case where $M = R$ with its standard affine connection; the corresponding μ is the standard arithmetic midpoint. Then

$$d^{(2)}\phi(x,y) = d^{(2)}\phi(x, x+2\omega) = \phi(x+2\omega) - 2[\phi(x+\omega)] + \phi(x).$$

Using $\phi(x+\omega) = \phi(x) + \omega\phi'(x) + \frac{1}{2}\omega^2\phi''(x)$ and $\phi(x+2\omega) = \phi(x) + 2\omega\phi'(x) + \frac{1}{2}(2\omega)^2\phi''(x)$ (by Taylor expansion and using $\omega^3 = 0 = (2\omega)^3$), simple arithmetic gives $(d^{(2)}\phi)(x,y) = \omega^2\phi''(x)$, or

$$d^{(2)}\phi(x,y) = \tfrac{1}{4}(y-x)^2 \cdot \phi''(x).$$

References

[1] A. Kock, *Synthetic Differential Geometry*, London Math. Soc. Lecture Notes 51, Cambridge U. P. 1981 (2nd ed. London Math. Soc. Lecture Notes 333, Cambridge U. P. 2006).

[2] A. Kock, Affine connections and midpoint formation, in *Discrete Geometry for Computer Imagery*, 15th IAPR International Conference, Montreal 2009, Springer Lecture Notes in Computer Science 5810, 13-21.

[3] A. Kock, *Synthetic Geometry of Manifolds*, Cambridge Tracts in Math. 180, Cambridge U.P. 2010.

[4] A. Kock, Commutative monads as a theory of distributions, *Theory and Applications of Categories* 26 (2012), 97-131.

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