Natural bundles over smooth etendues

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On the occasion of the 25th anniversary of the Halifax category years, where I had the privilege of participating, I would like to dedicate this talk to The Internationalists.

The content of the talk is meant as a contribution to the program of investigating the differential geometry of smooth etendues by category theoretic (and groupoid theoretic) means. It represents reflections based on long term joint work with Ieke Moerdijk on this topic.

On etendues

The category of (topological) etendues was introduced by SGA4 in the 1960's, as a category in between the category of topological spaces, and the category of (Grothendieck) toposes and geometric morphisms. Nowadays, we may replace the category of topological spaces with the category of locales, and then the notion of topological etendue gets replaced by that of localic etendue.

Whether we by 'space' understand a (sober) topological space or a locale, we get in any case a full embedding

$$\underline{\text{Spaces}} \xrightarrow{sh} \underline{\underline{\text{Top}}} ,$$

where sh denotes the functor "take the topos of of sheaves on", and $\underline{\underline{\text{Top}}}$ denotes the category of toposes and geometric morphisms. We put a double bar in the notation here to remind us that it is really a 2-category; this aspect is going to be crucial for etendue theory.

If M is a smooth manifold, $sh(M)$ is not just a topos, but a *smooth* topos, meaning that it is equipped with a suitable ring object (structure sheaf): $sh(M)$ carries the sheaf $\mathcal{O}_M$ of germs of smooth real valued functions as structure sheaf.

The general picture now is, omitting the full and faithful functor sh from notation,

$$\underline{\text{Spaces}} \subseteq \underline{\underline{\text{Etendues}}} \subseteq \underline{\underline{\text{Top}}} ,$$

both for the topological, localic, and smooth case; in the latter case, 'space' means *smooth manifold*, and $\underline{\underline{\text{Top}}}$ then denotes the 2-category of *smooth* toposes, with the geometric morphisms in question appropriately interlinked with the smooth structures; and similarly, the middle category is the 2-category of *smooth* etendues (we recall this notion below). We shall often leave it to the reader (helped by the context) to interpret the words space, etendue, or topos, as either referring to the spatial, localic, or smooth situation. So recall that an etendue is a topos $\underline{E}$, which has an object X with full support, such that the slice topos $\underline{E}/X$ is equivalent to $sh(M)$ for M a space. If $\underline{E}$ is a ringed topos (cf. e.g. [4]), and $\underline{E}/X$ as a ringed topos is equivalent to $sh(M), \mathcal{O}_M$, we say that it is a *smooth* etendue. There is an evident notion of when a geometric morphism between smooth etendues is smooth, i.e. compatible with the smooth structures $\mathcal{O}$.

There are alternative characterizations of the notion of etendue: First, the groupoid theoretic one (SGA4) says that $\underline{E}$ is equivalent to the topos $B\mathbf{G}$ of $\mathbf{G}$-sheaves, where $\mathbf{G}$ is an etale groupoid. This means that

$$\mathbf{G} = (G_1 \rightrightarrows G_0),$$

is a groupoid, such that the structural maps $d_0 : G_1 \to G_0$ and $d_1 : G_1 \to G_0$, from the space G_1 of arrows to the space G_0 of objects, are *etale*, i.e. local homeomorphisms, (resp. local diffeomeorphisms, in the smooth case). A $\mathbf{G}$-sheaf may be identified with an etale space $E \to G_0$ equipped with an action by $\mathbf{G}$.

There is a canonical geometric morphism $\pi : sh(G_0) \to B\mathbf{G}$; its inverse image functor is simply: forget the $\mathbf{G}$-action. In the notation we use throughout, the symbol sh is omitted, so we have $\pi : G_0 \to B\mathbf{G}$.

(Another characterization of etendues, for the localic case, is that an etendue is a topos that can be presented by a site all of whose maps are monomorphisms (cf. [20] for one direction, [9] for the other). And finally, for the spatial and localic case, there are also characterizations in terms of spaces equipped with local equivalence relations, cf. [11], [10].)

The groupoid theoretic characterization of etendues from SGA 4 may be seen as a precursor of the Joyal-Tierney theory [7], according to which *every* Grothendieck topos may be presented as the topos of $\mathbf{G}$-sheaves for a suitable localic groupoid. The functor B from the 2-category of localic groupoids to $\underline{\underline{\text{Top}}}$, assigning to $\mathbf{G}$ the topos $B(\mathbf{G})$ of $\mathbf{G} - \textit{sheaves}$ is thus essentially surjective on objects; and it restricts to an essentially surjective

$$\underline{\underline{\text{Etale groupoids}}} \to \underline{\underline{\text{Etendues}}} \,. \tag{1}$$

The functor in (1) makes sense also in the spatial or smooth case. The 2-category $\underline{\underline{\text{Etale groupoids}}}$ has for its morphism continuous, (respectively smooth), but not necessarily etale, functors between etale groupoids. Its 2-cells are of course all invertible, so the category is what sometimes is called an *iso*-2-category.

In so far as the 2-category of toposes or etendues are concerned, we shall also consider them as iso-2-categories, in other words, we disregard 2-cells which are not invertible.

Category-of-Fractions aspects of B have been studied in [13], and in [19].

Some differential geometry

One reason why differential geometers for almost 50 years have been interested in smooth etale groupoids, is that these may be seen as a very general notion of *atlas* for a smooth manifold. In fact, an "ordinary" atlas for a manifold M is given by a collection $(U_i)_{i \in I}$ of coordinate patches U_i, and smooth patching maps on their overlaps, and this data can be summarized into one smooth etale groupoid

$$U_\bullet = (\coprod U_i \cap U_j \rightrightarrows \coprod U_i).$$

(And $B(U_\bullet) \simeq sh(M)$.) Two atlases may give rise to the same manifold. The calculus-of-fractions notions thus provide a more categorical way of expressing when two atlases should be considered equivalent, and it generalizes to less simple smooth etale groupoids.

So the study of smooth etendues may be construed as the study of smooth etale groupoids. We shall give two further examples of such groupoids.

One is Haefliger's Γ^n. Its space of objects is $\mathbf{R}^n$; an arrow from $a \in \mathbf{R}^n$ to $b \in \mathbf{R}^n$ is a *germ* of a diffeomorhism $a \rightarrowtail b$, so is represented by a diffeomorphism $U \to V$ where U is an open neighbourhood around a and V is an open neighbourhood around b. The space of arrows, being a set of germs, is a sheaf over $\mathbf{R}^n$, thus equipped with an etale map d_0 to $\mathbf{R}^n$, hence is also a manifold (not Hausdorff, though). Taking codomain of germs defines another etale map to $\mathbf{R}^n$, denoted d_1. (Of course, the same construction works with any other manifold M instead of $\mathbf{R}^n$. It can even be carried out for locales M, see [8].)

If a smooth groupoid $\mathbf{G} = (G_1 \rightrightarrows G_0)$ acts on a manifold $F \to G_0$ over G_0, (and if the structural map is a submersion, so that pull-backs work well), we may form the usual "action groupoid", whose space of objects is F, and where arrows consist of pairs (γ, f) where $f \in F$ lives over $d_0(\gamma)$ $(\gamma \in G_0)$. Let us denote this groupoid $\mathbf{G} \propto F$. It comes equipped with a smooth functor p to $\mathbf{G}$, which from the algebraic point of view is a discrete (op-)fibration. If $\mathbf{G}$ is etale, then so is $\mathbf{G} \propto F$. We shall here be specifically interested in the case where $\mathbf{G}$ is Γ^n, and F is the bundle $Fr(\mathbf{R}^n)$ of first-order frames at points of $\mathbf{R}^n$, meaning invertible 1-jets j from $\underline{0} \in \mathbf{R}^n$ to a point of $\mathbf{R}^n$. Such j is in coordinate terms given by $n + n^2$ coordinates: the first n to pick the point a of $\mathbf{R}^n$ where the frame sits, the second n^2 coordinates to give the invertible Jacobian $n \times n$ matrix of the jet. An invertible 1-jet $j : \underline{0} \rightarrowtail a$ may be post-composed with an invertible germ $\gamma : a \rightarrowtail b$ to produce an invertible 1-jet $\underline{0} \rightarrowtail b$. This describes the (left) action of Γ^n on the frame bundle $Fr(\mathbf{R}^n) \to \mathbf{R}^n$, and thus a smooth etale grupupoid $\Gamma^n \propto Fr(\mathbf{R}^n)$ of dimension $n + n^2$. The functor (algebraically discrete fibration) $p : \Gamma^n \propto Fr(\mathbf{R}^n) \to \Gamma^n$ defines a map between smooth etendues

$$Bp : B(\Gamma^n \propto Fr(\mathbf{R}^n)) \to B(\Gamma^n). \tag{2}$$

Many notions and constructions in differential geometry can be extended to smooth etendues; often, but not always, this is straightforward. For instance, the etendues appearing in (2) are $n+n^2$ and n-*dimensional*, respectively, and the map Bp makes $B(\Gamma^n \propto Fr(\mathbf{R}^n))$ into a principal fibre bundle over $B(\Gamma^n)$, (its *frame* bundle) with group $Gl(n)$. (It is not in general immediate to lift notions from fibre bundle theory to the etendue context, because of the 2-dimensional nature of the category of etendues.) Also, the map Bp is a *submersion* of smooth etendues. The meaning of this can be made explicit in many ways; in particular, once we have introduced the notion of *tangent bundle* of a smooth etendue, we can give the natural invariant formulation of the submersion notion. But in less invariant terms, one may say that a map $\underline{E} \to \underline{E}'$ of smooth etendues is a submersion if it can be *covered* by a submersion between manifolds, meaning that there exists a square, commuting up to isomorphism,

$$\begin{array}{ccc} M & \longrightarrow & M' \\ \circ\big\downarrow & & \big\downarrow\circ \\ \underline{E} & \longrightarrow & \underline{E}' \end{array}$$

with M and M' ordinary manifolds, with $M \to M'$ a submersion, and $M \to \underline{E}$ and $M' \to \underline{E}'$ *etale* surjections; to say that $M \to \underline{E}$ is an *etale surjection* in turn means that there is a fully supported object X in the topos $\underline{E}$ such that $M \to \underline{E}$ is equivalent to $\underline{E}/X \to \underline{E}$. (We generally indicate etale maps by decorating them with a circle $\circ$.) When $\underline{E} = B\mathbf{G}$, for a smooth etale groupoid $\mathbf{G}$, the etale map $d_1 : G_1 \to G_0$ defines a sheaf over G_0 with a (left) $\mathbf{G}$-action, hence an object X in $\underline{E} = B\mathbf{G}$, and with this X, $\underline{E}/X \simeq sh(G_0) = G_0$ (recalling that we do not distinguish between a manifold and the (ringed) topos of sheaves on it). So $\underline{E}/X \to \underline{E}$ gets identified with $\pi : G_0 \to B\mathbf{G}$.

Any submersion $f : M \to \mathbf{R}^n$ (where M is a manifold) defines a *foliation* on M of codimension n. E.g if M is an open subset of $\mathbf{R}^2$ and $n = 1$, the foliation is the partition of M into the *level curves* of f, as depicted in any Calculus book. Of corse one may have several submersions $M \to \mathbf{R}$ defining the same foliation by level sets; and also, there may be a foliation on M which does not come about as the level set foliation for any $f : M \to \mathbf{R}$. An example (which I cannot draw in TeX) is the foliation of an annulus in the plane, rotationally symmetric, which you find as the cover page logo on the (Ehresmann) journal "Cahiers de Topologie et Geometrie Differentielle Categoriques". The "leaves" of this foliation could never be the level sets of a function to $\mathbf{R}$; if they were, you would have a perpetuum mobile, since by steady descent along the gradients (i.e. perpendicular to the level curves) you would soon come to cross the level curve on which you started, so be back at the same altitude.

So in this sense, $\mathbf{R}^n$ only classifies codimension n foliations in a very weak local sense. However, this picture changes when we go into the category of smooth etendues, where many differential geometric notions become classifiable. In particular, we shall see that *every* foliation comes about as the level set foliation for a submersion into a suitable etendue. This follows from Theorem 1 below.

Let me as examples of such general classification results quote the following two Theorems from the ongoing collaboration between Ieke Moerdijk and myself. Let M be a manifold.

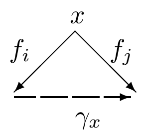
Theorem 1 *(KM) There is a bijective correspondence between foliations of codimension n on M and isomorphism classes of submersions $M \to B\Gamma^n$.*

Theorem 2 *(KM) There is a bijective correspondence between integrable non-singular 1-forms on M,and isomorphism classes of morphisms $M \to B(\Gamma^1 \propto Fr(\mathbf{R}^1))$ with the property that its composite with $B(\Gamma^1 \propto Fr(\mathbf{R}^1)) \to B(\Gamma^1)$ is a submersion.*

Recall that a 1-form ω on M is integrable if it locally can be written $g \cdot df$, for smooth functions $f, g : M \to \mathbf{R}$. The integrability condition implies that the hypeplanes in the tangent spaces, defined as the null spaces of ω, fit together as the tangent planes of a unique codimension 1 foliation on M. From the viewpoint of the classification, composing the classifying map $M \to B(\Gamma^1 \propto Fr(\mathbf{R}^1))$ for an integrable 1-form ω with the canonical $Bp : B(\Gamma^1 \propto Fr(\mathbf{R}^1)) \to B(\Gamma^1)$ gives the classifying map for the foliation determined by ω.

I shall sketch a proof of Theorem 1.

Let $\mathcal{F}$ be a codimension n foliation on M. There is a cover $\{U_i \subseteq M \mid i \in M\}$ by open sets, and submersions $f_i : U_i \to \mathbf{R}^n$, whose level sets are (the connected components of) the $\mathcal{F}$-leaves intersected with U_i. If $x \in U_i \cap U_j$, the diagram of germs (straight arrows)



(3)

admits a unique commutative filling $\gamma_x : f_i(x) \to f_j(x)$, since the kernel pairs of f_i and f_j agree in a suitable small neighbourhood of x (since both "agree with $\mathcal{F}$"). Putting these γ_x's together, we get a smooth map $U_i \cap U_j \to \Gamma^n$, and putting these together again, we get a smooth map $F_1 : \coprod U_i \cap U_j \to \Gamma^n$, which, together with the map $F_0 : \coprod U_i \to \mathbf{R}^n$ obtained by putting the f_i's together, provides us with a smooth functor F between smooth etale groupoids, as exhibited in the upper half of he diagram

$$\begin{array}{ccc} \coprod U_i \cap U_j & \xrightarrow{F_1} & \Gamma^n \\ {\scriptstyle\circ}\downarrow\;\downarrow{\scriptstyle\circ} & & {\scriptstyle\circ}\downarrow\;\downarrow{\scriptstyle\circ} \\ \coprod U_i & \xrightarrow{F_0} & \mathbf{R}^n \\ {\scriptstyle\circ}\downarrow & & \downarrow{\scriptstyle\circ} \\ M \simeq B(U_\bullet) & \xrightarrow[BF]{} & B(\Gamma^n) \end{array} \tag{4}$$

(We also use Γ^n to denote the space of arrows of the Haefliger groupoid Γ^n.) Then $B(F) : M \simeq B(U_\bullet) \to B(\Gamma^n)$ is the desired classifying map for the foliation $\mathcal{F}$, and it makes the lower square in (4) commute (up to isomorphism). This square then witnesses that $B(F)$ is in fact a submersion of etendues, being covered by the submersion F_0.

Conversely, given any map $E \to E'$ in the category of etendues, it may be exhibited in the form $B(F)$ for some functor $F : \mathbf{G} \to \mathbf{G}'$ between suitably chosen representing etale groupoids. The groupoid representing the codomain may even be chosen in advance. So it may be seen as being of form BF, as displayed in

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & G_1' \\ {\scriptstyle\circ}\downarrow\;\downarrow{\scriptstyle\circ} & & {\scriptstyle\circ}\downarrow\;\downarrow{\scriptstyle\circ} \\ G_0 & \xrightarrow{F_0} & G_0' \\ {\scriptstyle\circ}\downarrow & & \downarrow{\scriptstyle\circ} \\ B\mathbf{G} & \xrightarrow[BF]{} & B\mathbf{G}' \end{array}$$

In particular, a submersion $M \to B(\Gamma^n)$ may be exhibited in this form with $F_0 : G_0 = N \to \mathbf{R}^n$ a submersion. Since $N \to M$ is etale, we may even replace N by an N of form $\coprod U_i$ with each U_i open in M. We then have a situation like the one in (4). It is now clear that the f_i's which constitute F_0, form a "submersion atlas" for a foliation $\mathcal{F}$ on M, i.e. that in a neighbourhood around each $x \in U_i \cap U_j$, f_i and f_j define the same foliation, i.e. have the same kernel pair (=partition into level sets); this follows simply from the commutativity of(3), where the germ $\gamma_x \in \Gamma^n$ is now provided by $F_1(x)$. An isomorphism between morphisms of etendues may be represented by a natural transformation τ between functors between representing groupoids, and now the τ will provide germs γ_x, like in (3), to prove that the two foliations defined are the same.

(A different argument for Theorem 1, not involving groupoids, is presented in [15].)

The proof of the second theorem is in the same spirit: given a non-singular integrable 1-form ω on M, it may on a cover $\{U_i\}$ be exhibited in the form $g_i \cdot df_i$ with f_i a submersion $U_i \to \mathbf{R}$, and with $g_i : U_i \to \mathbf{R}$ nowhere vanishing; out of this data, one constructs a smooth functor F from the groupoid $U_\bullet$ to the groupoid $\Gamma^1 \propto Fr(\mathbf{R}^1)$; its object function associates to $x \in U_i$ the element $(f_i(x), 1/g_i(x)) \in Fr(\mathbf{R}^1)$. (We coordinatize $Fr(\mathbf{R}^1)$ by pairs of real numbers (a, α) with $\alpha \neq 0$; (a, α) denotes the frame at a given as the 1-jet of the function $t \mapsto a + \alpha \cdot t$.) The action of Γ^1 on it by post-composition then is described by

$$\gamma \cdot (a, \alpha) = (b, \gamma'(a) \cdot \alpha) \tag{5}$$

for $\gamma : a \rightarrowtail b$ a germ of a diffeomorphism taking a to b; this is just the chain rule. To extend the described object function $\coprod U_i \to Fr(\mathbf{R})$ to a functor into $\Gamma \propto Fr(\mathbf{R})$ then means the following (since the latter is an action groupoid): for $x \in U_i \cap U_j$ with $f_i(x) = a$, $f_j(x) = b$, say, we should find a germ $\gamma : a \rightarrowtail b$ with

$$\gamma \cdot (a, 1/g_i(x)) = (b, 1/g_j), \text{ i.e. with } \gamma'(a) \cdot 1/g_i(x) = 1/g_j(x). \tag{6}$$

But the equation

$$g_i\, df_i = g_j\, df_j (= \omega) \tag{7}$$

implies that f_i and f_j have the same kernel pair in a neighbourhood of x, so that we may find a unique germ $\gamma : a \rightarrowtail b$ making (3) commute, just as in the proof of Theorem 1. The chain rule applied to (3) gives

$$df_j(x) = \gamma'(a) \cdot df_i(x).$$

On the other hand (7) implies that we in all of $U_i \cap U_j$ have

$$df_j = \frac{g_i}{g_j} df_i,$$

and these two equations imply (6) since the form df_i is nowhere vanishing (f_i being a submersion).

There is a rather evident way of defining a notion of differential form, say, on a smooth etendue $\underline{E}$ (and it has been studied by several authors, see e.g. [22]). Namely, take a representing smooth etale groupoid $\mathbf{G} = (G_1 \rightrightarrows G_0)$; then any $\mathbf{G}$-invariant form on G_0 represents a form on $\underline{E}$. This is an immediate generalization of the idea of defining a differential form on a manifold in terms of differential forms on the charts of a coordinate atlas, behaving properly with respect to coordinate change (which is what the $\mathbf{G}$-invariance amounts to in this case). For instance, the etendue $B(\Gamma^1 \propto Fr(\mathbf{R}^1))$ carries a non-singular integrable 1-form, which is in fact *generic*, essentially by Theorem 2. This 1-form is just $1/y\ dx$ or $1/\alpha\ da$ (with coordinatization of $Fr(\mathbf{R})$ as above).

However, one would like also "invariant" descriptions; one would like, say, to be able to understand a differential form on M as a cross section of a certain fibre bundle over M, the cotangent bundle $T^*(M)$; a vector field as a cross section of the tangent bundle $T(M)$; and a framing as a cross section of the frame bundle $Fr(M)$, etc.[1] This led in the 70's to a general study of "bundles of geometric objects" or "natural bundles" (see e.g. [18], [21]), and of their functorial properties with respect to diffeomorphisms. In a recent book [12], this study has been systematized and extended, and notably the functorial properties with respect to arbitrary smooth maps considered. Of course the cotangent bundle construction and the frame bundle construction are not functors with respect to *all* smooth maps, but at least the cotangent bundle is an example of a functorial structure which the authors of loc.cit. call **-naturality*; this is the bundle version of the fact that one may pull a differential form back along a map. My aim is now to recall some of this theory, and to extend the scope of this theory to the category of smooth etendues.

Natural bundles

By $\underline{Mf}_m$ we understand the category of m-dimensional manifolds and the smooth etale maps (local diffeomorphisms) in between; similarly $\underline{\underline{Et}}_m$ for the 2-category of m- dimensional smooth etendues. Recall that a *natural bundle* on $\underline{Mf}_m$ is a functor $F : \underline{Mf}_m \to \underline{Mf}_{m+k}$, together with, for each $M \in \underline{Mf}_m$, a surjective submersion $p_M : F(M) \to M$, natural in M. This data should satisfy the requirement that all naturality squares

$$\begin{array}{ccc} F(M) & \xrightarrow{\circ} & F(M') \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_{M'}} \\ M & \xrightarrow{\circ} & M' \end{array} \tag{8}$$

are pull-backs. Recall that a circle on an arrow is meant to indicate that the map is etale. Some such functors, like the tangent bundle functor T, extend to the category of *all* manifolds and *all* smooth maps, but the naturality square

[1] The examples quoted here all happen to be "1st order". There are also natural 2nd or higher order examples, e.g. the bundle over M whose cross sections correspond to affine connections on M.

w.r. to a non- etale map is then not a pull-back. Such a natural bundle like T is called in loc.cit. a *bundle functor*. More challenging are what loc.cit. calls a "*-*bundle functor*", like T^*, and on these, we shall elaborate below.

First, however, we shall extend, in a straightforward way, natural bundles om $\underline{Mf}_m$ to natural bundles on $\underline{\underline{Et}}_m$.

Recall the fundamental "pasting lemma" for pull-backs: if the right hand square in

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

is a pull-back, then the total square is a pull-back if and only if the left hand square is a pull-back. From this, and the fact that the square (8) is a pull-back, it immediately follows that if $F : \underline{Mf}_m \to \underline{Mf}_{m+k}$ is a natural bundle, then F applied to a pull-back of in $\underline{Mf}_m$ yields a pull-back in $\underline{Mf}_{m+k}$. So in particular, if $\mathbf{G} = (G_1 \rightrightarrows G_0)$ is an m-dimensional etale groupoid, then we may define an $m+k$ dimensional etale groupoid structure on $F(G_1) \rightrightarrows F(G_0)$, denoted $F(\mathbf{G})$. The data p which is part of the natural bundle then immediately provides us with a smooth functor between groupoids $p : F(\mathbf{G}) \to \mathbf{G}$, and because the squares (8) are pull-backs, this functor is, algebraically, a discrete fibration and a discrete op-fibration.

(In particular, it may be viewed as coming about as the action groupoid of $\mathbf{G}$ by a left action on the bundle $\pi : F(G_0) \to G_0$; the reader may prefer to make this action explicit, in terms of germs, using the functorality of F, and think of $F(\mathbf{G})$ in this way. We shall give this explicit form later, cf. the proof of Lemma 1.)

Roughly speaking, we may for any m-dimensional etendue $\underline{E}$ define an $m+k$-dimensional one, $F(\underline{E})$, by taking a repesenting etale groupoid $\mathbf{G}$ for $\underline{E}$ (so $\underline{E} \simeq B(\mathbf{G})$) and putting $F(\underline{E}) = B(F(\mathbf{G}))$. Also $p : F(\mathbf{G}) \to \mathbf{G}$ defines $B(F\mathbf{G}) \to B(\mathbf{G})$, thus a morphism $p_{\underline{E}} : F\underline{E} \to \underline{E}$. This well defines $F(\underline{E})$, not modulo isomorphism, but only modulo equivalence. So when extended to the 2-category $\underline{\underline{Et}}_m$ in this way, by making choices of representing groupoids, F becomes a functor-up-to-isomorphism, i.e. a pseudo-functor, and p becomes pseudo-natural, and you cannot do better than that. The following considerations, however, deal only with one morphism of etendues at a time, and with the etendues in question provided with a given representing groupoid. So the (unavoidable) looseness in F as a pseudofunctor does not matter here.

To say that a natural bundle F is a *bundle functor* is just to say that it is a functor defined on all smooth maps between manifolds, whose restriction to Mf_m is a natural bundle for each m, and such that p is natural with respect to all smooth maps. (Example: the tangent bundle functor T.) For such F, a functor $\phi : \mathbf{G} \to \mathbf{G}'$ between smooth etale groupoids immediately gives rise to a functor $F(\phi) : F(\mathbf{G}) \to F(\mathbf{G}')$, and hence to a morphism $F(B\mathbf{G}) \to F(\mathbf{G}')$ of etendues. In this way, F becomes a pseudofunctor $\underline{\underline{Et}} \to \underline{\underline{Et}}$, and p becomes pseudo-natural. - In particular, we may now talk about the *tangent bundle* $T(\underline{E}) \to \underline{E}$ of any (smooth) etendue $\underline{E}$; $T(\underline{E})$ is of dimension $2n$ if $\underline{E}$ is of dimension n. We shall not elaborate further on this here; there is a nice infinitesimal characterization of bundle functors, in terms of actions by algebraic theories of jets, in [5] and [12].

We now turn to the *-bundle functors, like the cotangent bundle construction T^*. According to [12], a *-bundle functor F consists of a natural bundle F for each dimension m, together with the following data: for each smooth map $f : M \to M'$, there is given a smooth map

$$\overline{f} : f^*(F(M')) \to F(M)$$

over M, satisfying three compatibility conditions (see below); here $f^*(F(M'))$ denotes the pull-back of $p_{M'} : FM' \to M'$ along f. We shall, for a fixed *-bundle functor, use a subscript notation for the data given by the $\overline{f}$'s, as follows. Given $f : M \to M'$. Let us identify $f^*(F(M'))$ with $M \times_{M'} F(M')$. If $(a, z) \in M \times_{M'} F(M')$, we shall write $f_a(z)$ for $\overline{f}(a, z) \in F(M)$.

The three compatibility conditions are a unit- and an associativity condition, and a compatibility for the already assumed functoriality of F with respect to etale maps. The unit condition is just that $\overline{f}$ is an identity map if f is; the associativity condition is that for for any composable pair

$$L \xrightarrow{g} M \xrightarrow{f} N,$$

if $e \in L$ and $z \in F(N)$ over $f(g(e))$ we have

$$(f \circ g)_e(z) = g_e(f_{g(e)}(z)). \tag{9}$$

The third condition finally says that if $h : M \to M'$ is etale, then $\overline{h}$ is invertible with

$$FM \xrightarrow{< F(h), p_M >} M \times_{M'} FM'$$

as inverse. In subscript notation, this reads that: for $h : U \to V$ *etale*, $c \in U$ and $z \in F(V)$ over $h(c)$, we have

$$F(h)(h_c(z)) = z. \tag{10}$$

We shall now embark on the extension of any *-bundle functor F on $\underline{Mf}$ to a (pseudo-) functor F on $\underline{\underline{Et}}$, i.e. to construct comparison maps $\overline{f}$ for morphisms f between etendues (the functorality of F w.r.to *etale* maps between etendues follows the same pattern as the one sketched above for bundle functors, by representing such etale map by an etale morphism of etale groupoids, and utilizing the functorality of F w.r. to etale maps.) The problem in describing $\overline{f}$ is that its domain is a pull-back, forcing us to consider "pull-backs" of etendues. Now etendues form a 2-category, and strict pull-backs are usually not available, and if they are, they are not what you want anyway. Nor are pseudo-pullbacks always available; the right notion is that of *bi-pull-back*. We shall make a short digression into some notions from 2-dimensional category theory (using terminology as codified by Kelly and his collaborators).

Some 2-dimensional category theory, kernel groupoids, and descent coequalizers

I believe that (iso-)2-dimensional category theory lies at the heart of etendue theory. Recall that a *2-category* is a category enriched in the category of categories, and that an *iso-2-category* is a catgory enriched in the category of groupoids, i.e. a 2-category, where all 2-cells are invertible. For manifolds and smooth etendues, we only consider (or only have ?) 2-cells which are invertible, so we consider only iso-2-categories. *Bicategories* means that composition of arrows is only associative up to isomorphism. It is well known that one may replace any bicategory by an "equivalent" 2-category (strictly associative composition), so we shall forget about the weaker notion here. However, for the various limit notions, one cannot freely "strictify".

Recall that when one has two functors $\mathbf{A} \to \mathbf{C}$, $\mathbf{B} \to \mathbf{C}$ with common codomain, one may form their comma category $\mathbf{A} \downarrow_{\mathbf{C}} \mathbf{B}$ (which is a groupoid if $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are). Its objects are triples (A, ϕ, B) where A and B are objects in $\mathbf{A}$ and $\mathbf{B}$, respectively, and ϕ is an arrow between their images in $\mathbf{C}$.

A *pseudo-pullback* of two arrows in an iso-2-category is an object which represents the comma category formed of the hom categories *up to isomorphism*; it is thus itself determined up to isomorphism. Whereas a *bi-pull-back* represents the same comma-category-of hom-categories *only up to equivalence*; it is determined up to equivalence only. If a pseudo-pull-back happens to exist, it may serve as a bi-pull-back. A *strict* or 2- pull-back classifies up to isomorphism, not the comma-category, but the pull-back of hom-categories, and usually does not serve as a pseudo-pull-back (under some conditions, it does; see [6]). Bi-pull-backs were a main tool in [4].

In 1-dimensional category theory, besides pull-backs, one considers coequalizers of equivalence relations. And the pull-back of a map p along itself is an equivalence relation, called the kernel pair of p. A fundamental notion is that of short exact sequence

$$\bullet \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bullet \xrightarrow{\;p\;} \bullet$$

meaning that the equivalence relation d_0, d_1 is the kernel pair of p and p is the coequalizer of d_0, d_1.

In (iso-)2-dimensional category theory, the notion of short exact sequence: (kernel pair)/(coequalizer) is replaced by (kernel-groupoid)/(descent-coequalizer). More precisely, given a morphism $\pi : G_0 \to B$ in a an iso-2-category $\underline{\underline{E}}$ with sufficiently many bi-pull-backs, one may form the twofold bi-pull-back of π with itself $G_1 \rightrightarrows G_0$, and the three-fold bi-pull-back $G_2 \Rrightarrow G_0$ of π with itself, etc., and this whole structure forms, up to coherent isomorphisms, a simplicial object $G_\bullet$ in $\underline{\underline{E}}$, which is in fact in a suitable sense (the nerve of) a groupoid in $\underline{\underline{E}}$. (For our purpose, we don't have to worry about this "up-to-isomorphism" complication, because for a particular reason, the isomorphisms are identities, see below).

Given, conversely, such a groupoid (up-to...) in $\underline{\underline{E}}$, $G_2 \Rrightarrow G_1 \rightrightarrows G_0$, we understand by a *descent coequalizer* of it an arrow $p : G_0 \to B$ together with an isomorphism $\theta : p \circ d_0 \cong p \circ d_1$, satisfying the standard cocycle condition w.r.to $G_2 \Rrightarrow G_1$, and also a unit condition. There is an evident notion of arrow between such B-valued "descent"data (p, θ) making these data into a groupoid, and to say that p is a descent coequalizer is to say that it is a couniversal descent data for the gropuoid, up to equivalence of categories: the hom-groupoid $\underline{\mathrm{Hom}}(B, X)$ is equivalent to the groupoid of descent data with value in X.

For the iso-2-category of toposes and geometric morphisms, these notions have been studied in [16] under the names *groupoid topos* and *coequalizer*, respectively. In the present context, I believe that the latter terminology will lead to confusion with ordinary coequalizers, whence the prefix "descent".

The particular reason why the kernel-"groupoids-up-to..." in our context become plain groupoids, in the usual 1-dimensional sense, is essentially the same as that considered in [7]: we consider a situation which we may axiomatize as follows. The iso-2-category $\underline{\underline{E}}$ comes equipped with a full subcategory $\underline{M}$ (full both with respect to 1-cells and with respect to 2-cells), which is "strongly 1-dimensional" in the sense that all the hom-groupoids $\underline{\mathrm{hom}}(X, M)$, for $M \in \underline{M}$ and for X arbitrary, are *discrete*. Also we assume that in any[2] bi-pull-back

$$\begin{array}{ccc} ? & \longrightarrow & M' \\ \downarrow & \cong & \downarrow \\ M & \longrightarrow & X \end{array}$$

with M and $M' \in \underline{M}$, ? may be chosen to be in $\underline{M}$ also. -Note that in particular, $\underline{M}$ is really only an ordinary 1-category. The example we are having in mind is when $\underline{M}$ is the category of manifolds, and $\underline{\underline{E}}$ is the category of smooth etendues. So for a map $p : G_0 \to E$ with $E \in \underline{\underline{E}}$, the kernel groupoid of p is an honest

[2] Really, one should describe a subclass of the class of bi-pull-backs, consisting of the *good* ones -like for manifolds, where the only pull-backs that have good beaviour are the transversal ones, in particular the ones in which one of the given maps is a submersion

groupoid object in the 1-category $\underline{M}$. As usual, we often exhibit a groupoid by exhibiting only its G_0 and G_1.

A more familiar example of an iso-2-category $\underline{\underline{E}}$ with a strictly 1-dimensional subcategory $\underline{M}$ is the category of ordinary groupoids with its subcategory of *discrete* groupoids (sets). It is good to get some wisdom from this example; it will throw light on the category of etendues as well. (In this particular $\underline{\underline{E}}$, bi-pull-backs may be computed as pseudo-pull-backs, in fact, as (iso-) comma squares.) For instance, let $\mathbf{G} = (G_1 \rightrightarrows G_0)$ be any groupoid, i.e. an object in $\underline{\underline{E}}$ (leaving G_2 etc. implicit). The set G_0 is then an object in $\underline{M} \subseteq \underline{\underline{E}}$, and there is a canonical map $\pi : G_0 \to \mathbf{G}$ in $\underline{\underline{E}}$, the inclusion of the set of objects into the groupoid. Then we have a bi-pull-back

$$\begin{array}{ccc} G_1 & \xrightarrow{d_1} & G_0 \\ {\scriptstyle d_0}\downarrow & \cong & \downarrow{\scriptstyle \pi} \\ G_0 & \xrightarrow[\pi]{} & \mathbf{G} \end{array}$$

where the 2-cell (natural transformation) exhibited is the one which to an *object* (element) g in the set G_1 associates an *arrow* in the groupoid $\mathbf{G}$ - namely g itself. (Similarly, the three-fold bi-pull-back of p with itself gives G_2, the set of composable pairs of arrows in $\mathbf{G}$, etc.) Thus, the kernel groupoid of $p : G_0 \to \mathbf{G}$ is just $G_1 \rightrightarrows G_0$, i.e. is $\mathbf{G}$ itself (or, more precisely perhaps, the nerve $G_\bullet$ of $\mathbf{G}$). And it is easy to prove that the descent coequalizer of $G_\bullet$ is exactly $G_0 \to \mathbf{G}$ with the 2-cell exhibited above. So with these data implicit, we have for any groupoid $\mathbf{G}$ a short exact sequence in $\underline{\underline{E}}$,

$$\ldots G_1 \rightrightarrows G_0 \xrightarrow{\pi} \mathbf{G}.$$

(In particular, for any group $\mathbf{G}$, with neutral element e, the map $e : 1 \to \mathbf{G}$ is a descent-coequalizer of the two equal maps $G \to 1$. Also we see that a bi-pull-back does not define a pull-back in the 1-category obtained by identifying isomorphic 1-cells.)

So in this sense, *every* groupoid $\mathbf{G}$ is *effective*[3], i.e. is a kernel groupoid (namely of $\pi : G_0 \to \mathbf{G}$).

Not every localic groupoid is effective in the iso-2-category of toposes. Those that are were called *etale-complete* in [13]. In [9], Proposition 3.1, we proved that every etale localic groupoid is etale complete. Since smooth structure automatically transports along etale maps, it follows that any smooth etale groupoid $\mathbf{G}$ is effective in the iso-2-category of smooth etendues. And its descent

[3]recall that an equivalence relation is called effective it it the kernel pair of something

coequalizer is $\pi : G_0 \to B(\mathbf{G})$, the geometric morphism whose inverse image just forgets the $\mathbf{G}$-action of a $\mathbf{G}$-sheaf $E \to G_0$.

So for smooth etale groupoids $\mathbf{G}$, we have a short exact sequence in the 2-dimensional sense explained above, and with some of the constituent data not exhibited:

$$\ldots G_1 \rightrightarrows G_1 \to B\mathbf{G}, \tag{11}$$

(commuting up to a specific isomorphism θ); in particular, we have a bi-pull-back in $\underline{\underline{\underline{Et}}}$

$$\begin{array}{ccc} G_1 & \longrightarrow & G_0 \\ \downarrow & \overset{\theta}{\cong} & \downarrow \\ G_0 & \longrightarrow & B\mathbf{G} \end{array} .$$

For the purpose of natural bundles, we are interested in action groupoids $\mathbf{G} \propto F$ (discrete (op-)fibrations), for actions of etale groupoids $\mathbf{G}$ on manifolds over G_0, $F \to G_0$. Writing H_0 for F, H_1 for $G_1 \times_{G_0} F$, we thus have an etale groupoid $\mathbf{H} = (H_1 \rightrightarrows H_0) = \mathbf{G} \propto F$, equipped with a functor f to $\mathbf{G}$, as exhibited in the left half of the following diagram in the (iso-2-)category of smooth etendues:

$$\begin{array}{ccccc} H_1 & \rightrightarrows & H_0 & \xrightarrow{\pi'} & B\mathbf{H} \\ f_1 \downarrow & & f_0 \downarrow & \cong & \downarrow B;f \\ G_1 & \rightrightarrows & G_0 & \xrightarrow[\pi]{} & B\mathbf{G} \end{array} \tag{12}$$

the isomorphism in the right hand square may here be taken to be an identity, at least in so far as the inverse image functors are concerned, by inspecting the explicit construction of the toposes and geometric morphisms in question, and we shall do so, for ease of notation. The isomorphisms making the rows here "short exact" will be denoted $\overline{\theta}$ and θ, respectively.

We now have the following (which, for the localic case, also can be deduced from [16], even just assuming $\mathbf{G}$ etale-complete):

Proposition 1 *Let $f : \mathbf{H} \to \mathbf{G}$ be a discrete fibration with $\mathbf{G}$ a smooth etale groupoid. Then the right hand square in (12) is a bi-pull-back.*

Proof. We have to prove that, for any etendue N, the functor J induced by f_0, q, θ

$$\mathrm{Hom}(N, H_0) \xrightarrow{J} \mathrm{Hom}(N, G_0) \downarrow_{\underline{\mathrm{Hom}}(N, B\mathbf{G})} \underline{\mathrm{Hom}}(N, B\mathbf{H}) \tag{13}$$

is an equivalence of categories (groupoids, in fact). We have used Hom without underline to indicate groupoids which are already known to be discrete. We shall prove that any object in the comma-category on the right, (m, ϕ, z), say, is uniquely isomorphic to an object $J(x)$, and for a unique $x \in \mathrm{Hom}(N, H_0)$. Here, $m \in \mathrm{Hom}(N, G_0)$, z is an object in $\underline{\mathrm{Hom}}(N, B\mathbf{H})$, and ϕ is an arrow $\pi \circ m \to Bf \circ z$ in the groupoid $\underline{\mathrm{Hom}}(N, B\mathbf{G})$. We first consider the case where z may be written in the form $\pi' \circ y$ for some $y \in \mathrm{Hom}(N, H_0)$. Then

$$\pi \circ m \xrightarrow{\phi} Bf \ \circ z = Bf \circ \pi' \circ y = \pi \circ f \circ y,$$

so by effectiveness (etale completeness) of $\mathbf{G}$, there exists a $g \in \mathrm{Hom}(N, G_1)$ with $\theta * g = \phi$. In the groupoid $\mathrm{Hom}(N, G_\bullet) = (\mathrm{Hom}(N, G_1) \rightrightarrows \mathrm{Hom}(N, G_0))$, g is an arrow $g : m \to f(y)$, and since $f_* : \mathrm{Hom}(N, H_\bullet) \to \mathrm{Hom}(N, G_\bullet)$ is a discrete fibration (since $f : \mathbf{H} \to \mathbf{G}$ is), it follows that there is an arrow $h : x \to y$ in $\mathrm{Hom}(N, H_\bullet)$ with $f_*(h) = g$. In particular, its domain x is an element of $\mathrm{Hom}(N, H_0)$; we shall prove that there exists an isomorphism $J(x) \cong (m, \phi, z)$ in the comma category displayed on the right hand side of (13):

The composite $\overline{\theta} * h$ gives an arrow $\pi'(x) \to \pi'(y)$ in $\underline{\mathrm{Hom}}(N, B\mathbf{H})$ whose value under Bf is $\theta * f \circ h = \theta * g = \phi$, (due to $Bf * \overline{\theta} = \theta * f_1$), so that the square commutes. But this implies that

$$(id_m, \overline{\theta} * h) : J(x) = (f(x), id, \pi'(x)) \to (m, \phi, z)$$

is an arrow (isomorphism) in the comma category, as desired.

Also, if for a given (m, ϕ, z), an arrow $J(x) \to (m, \phi, z)$ exists, it is unique, and determines x uniquely: it suffices to prove that any arrow $J(x_1) \to J(x_2)$ in the comma category must be an identity map, and must have $x_1 = x_2$. Such an arrow is given by an arrow β in $\underline{\mathrm{Hom}}(N, B\mathbf{H})$, $\beta : \pi'(x_1) \to \pi'(x_2)$, with $Bf(\beta)$ an identity in $\underline{\mathrm{Hom}}(N, B\mathbf{G})$. But $\mathbf{H}$ is an etale groupoid, hence effective (etale complete), so β is of the form $\overline{\theta} * h$ for some $h \in \mathrm{Hom}(N, H_1)$, and $(Bf)(\beta) = id$ implies that $f_1 \circ h$ is an identity arrow in $\mathrm{Hom}(N, G_\bullet)$. Since $f_* : \mathrm{Hom}(N, H_\bullet) \to \mathrm{Hom}(N, G_\bullet)$ is a discrete fibration, this implies that h is an identity arrow, hence $x_1 = x_2$, and also β, and hence the given arrow $J(x_1) \to J(x_2)$, are identities. This proves the uniqueness.

Now the general case where the given $z : N \to B\mathbf{H}$ is not assumed to lift through H_0: at least it will do so locally, i.e. on an open cover U of N, i.e. on an etale surjection from a manifold U; this makes sense even when N is not a manifold but a general smooth etendue. Now the proof reduces to the case already considered, by speaking some soft words about "uniqueness, and local existence, implies global existence". This proves the proposition.

On *-bundle functors on etendues

We now consider a *-bundle functor F on the category of $\underline{\underline{Et}}$ of smooth etendues (example: cotangent-bundle functor T^*). Being in particular a natural bundle (more precisely: for each n, a natural bundle on $\underline{Mf}_n$), we have already above defined a locally trivial fibre bundle $FE \to E$ for any such E. To extend the *-bundle structure to $\underline{\underline{Et}}$ means in particular, that for any map $H : E \to E'$ of etendues (everything smooth), we should produce a map $\overline{H} : H^*FE' \to FE$ over E, here $H^*FE' = E \times_{E'} FE'$ is defined by means of a bi-pull-back

$$\begin{array}{ccc} H^*FE' & \longrightarrow & FE' \\ \Big\downarrow & \cong & \Big\downarrow \\ E & \xrightarrow[H]{} & E' \end{array} \tag{14}$$

in the iso-2-category of smooth etendues. To get hold of it, i.e. of the etendue H^*FE', we need to construct a smooth etale groupoid to represent it. We may assume that H is represented by $h : \mathbf{G} \to \mathbf{G}'$, (so FE' is represented by the groupoid $F\mathbf{G}'$). We claim that H^*FE' is represented by the *strict* pull-back of groupoids $h^*F\mathbf{G}'$ (This is not too surprising: the strict pull-back is in this case *equivalent* (although not isomorphic) to the pseudo-pull-back (comma-square), since the condition of [6] is satisfied; and bi-pull-backs of etendues are often (always?) formed by forming pseudo-pull-backs of representing groupoids; an example of this is given in [17].) To prove the claim means to prove that the right hand square in the following diagram is a bi-pull-back:

$$\begin{array}{ccccc} h^*FG'_0 & \xrightarrow{\pi} & Bh^*F\mathbf{G}' & \longrightarrow & BF\mathbf{G}' \\ \Big\downarrow & \cong & \Big\downarrow & \cong & \Big\downarrow \\ G_0 & \xrightarrow[\pi]{} & B\mathbf{G} & \xrightarrow[H]{} & B\mathbf{G}' \end{array} . \tag{15}$$

The total square can be rewritten

$$\begin{array}{ccccc} h^*FG'_0 & \longrightarrow & FG'_0 & \longrightarrow & BF\mathbf{G}' \\ \Big\downarrow & = & \Big\downarrow & \cong & \Big\downarrow \\ G_0 & \xrightarrow[h]{} & G'_0 & \xrightarrow[\pi]{} & B\mathbf{G}' \end{array} ,$$

where the right hand square is a bi-pull-back by Proposition 1, and the left hand square is a (bi-)pull-back, being a pull-back of strictly 1-dimensional objects (manifolds); and this pull-back *exists* in the category of manifolds, since

$FG_0' \to G_0'$ is a submersion. Now bi-pull-backs *paste* just like ordinary pull-backs (although it is not so easy to prove; cf. the considerations in [2], where they are called 2-pull-backs), so the total square in (15) is a bi-pull-back. Also, the left hand square in (15) is a bi-pull-back, again by Proposition 1. And the map $\pi : G_0 \to B\mathbf{G}$ there is (stably) a descent coequalizer, being in fact a surjective slice map. To conclude that the right hand square in (15) is a bi-pull-back, we therefore need a 2-dimensional version of the "co-pasting" lemma; for pull-backs in ordinary categories, this says that if in a commutative diagram (ignore for a moment the $\cong$ signs for the moment)

$$\begin{array}{ccccc} C' & \longrightarrow & B' & \longrightarrow & A' \\ \downarrow & \cong & \downarrow & \cong & \downarrow \\ C & \underset{\pi}{\longrightarrow} & B & \longrightarrow & A \end{array} \tag{16}$$

the total diagram and the *left* hand squares are pull-backs, then so is the right hand one, provided π is a stable regular epi.

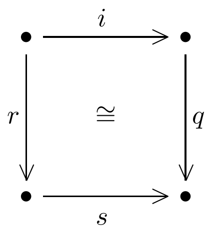
We claim that a similar thing holds for bi-pull-backs in an iso-2-category; the squares in (16) are now not assumed commutative, but have to be supplied with given invertible 2-cells, as displayed; and π has to be stably a descent coequalizer, in an evident sense. I believe this result is more or less well known, but I haven't been able to find a reference. So I shall indicate a proof.

We form a bi-pull-back $B \times_A A'$. Then there is a comparison $B' \to B \times_A A'$ over B (modulo isomorphisms), and we want to prove this comparison to be an equivalence. We bi-pull it back along π. The assumption that the total square is a bi-pull back, and the pasting lemma for bi-pull-backs, then imply that the comparison pulls back to an equivalence. Therefore, by the pasting lemma again, we get a bi-pull-back square of form

$$\begin{array}{ccc} C \times_B B' & \xrightarrow{id} & C \times_B B' \simeq C \times_A A' \\ \downarrow & \cong & \downarrow q \\ B' & \longrightarrow & B \times_A A' \end{array}$$

in which the right hand map is a stable descent coequalizer (it comes about from π by bi-pull-back). We now conclude by appealing to the following Lemma.

Lemma 1 *Consider a bi-pull-back*

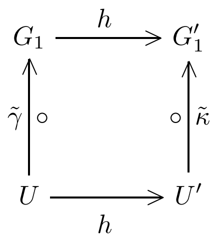


where q is a stable descent coequalizer. If i is and identity[4] *then s is an equivalence.*

Proof. If we (bi-)pull the kernel groupoid of q back along s, we get a kernel groupoid for r, by the pasting lemma. But since i is an identity, it follows that the kernel grouipoid of r may be taken to be equal to the kernel groupoid of q. But since q and r both are descent coequalizers of their kernel pairs, it follows that the comparison s between them is an equivalence.

Now that we therefore know that H^*E' is represented by the groupoid $h^*F\mathbf{G}'$, we get the desired morphism of etendues $H^*E' \to E$ by constructing a functor $h^*F\mathbf{G}' \to F\mathbf{G}$ over $\mathbf{G}$. Both groupoids here are discrete op-fibrations over $\mathbf{G}$, coming about from the $\mathbf{G}$-actions on $h^*FG_0^{'}$ and FG_0, respectively. And as part of the assumed *-bundle functor structure on F, we have a map over G_0, $\overline{h} : h^*FG_0^{'} \to FG_0$, and it gives rise to a functor between the action groupoids, provided that we can show it equivariant with respect to the $\mathbf{G}$-actions; we revert to the subscript notation for the *-bundle functor structure:

To prove that the map $\overline{h} : h^*FG_0' = G_0 \times_{G_0'} FG_0' \to FG_0$ preserves the $\mathbf{G}$-action, let us consider an arrow $\gamma : a \to b$ in $\mathbf{G}$, and an $(a, z) \in G_0 \times_{G_0'} FG_0'$, i.e. $z \in (FG_0')_{h(a)}$. We must prove that $\gamma \cdot h_a = h_b(h(\gamma) \cdot z)$. Using that $\mathbf{G}$ and $\mathbf{G}'$ are etale, it is easy to see that we may find open neighbourhoods U and U' around a and $h(a)$, and local sections $\tilde{\gamma}$ and $\tilde{\kappa}$ of the d_0's of $\mathbf{G}$ and $\mathbf{G}'$, respectively, with $\tilde{\gamma}(a) = \gamma$ and $\tilde{\kappa}(h(a)) = h(\gamma)$, such that the square



commutes.

[4] it should suffice to assume i to be an equivalence

We may assume that $z \in F(i)(z')$ for some (unique) $z' \in F(U')$. The action of γ on $(a, z) \in G_0 \times_{G'_0} FG'_0 \to FG_0$ is described in terms of the action of $h(\gamma)$ on z, and this, in turn, may be described, using the section $\tilde{\kappa}$ through $h(\gamma)$; namely

$$h(\gamma) \cdot z = F(d_1 \circ \tilde{\kappa})(z').$$

On the other hand, the action of γ on $h_a(z)$ is given by

$$F(d_1 \circ \tilde{\gamma})(h_a(z')).$$

Since $d_1 \circ h = h \circ d_1$ (we use d_1 as notation for the codomain formation both in $\mathbf{G}$ and in $\mathbf{G}'$), we get by post-composing the diagram above with d_1 that

$$\begin{array}{ccc} G_0 & \xrightarrow{\;h\;} & G'_0 \\ {\scriptstyle d_1 \circ \tilde{\gamma}}\uparrow{\scriptstyle \circ} & & {\scriptstyle \circ}\uparrow{\scriptstyle d_1 \circ \tilde{\kappa}} \\ U & \xrightarrow[\;h\;]{} & U' \end{array}$$

commutes, so we need to prove that this implies, for $z' \in F(U')$ that

$$F(d_1 \circ \tilde{\gamma})(h_a(z')) = h_b(F(d_1 \circ \tilde{\kappa})(z')).$$

This is a special case of

Lemma 2 *Given a commutative*

$$\begin{array}{ccc} V & \xrightarrow{\;k\;} & V' \\ {\scriptstyle \gamma}\uparrow{\scriptstyle \circ} & & {\scriptstyle \circ}\uparrow{\scriptstyle \kappa} \\ U & \xrightarrow[\;h\;]{} & U' \end{array}$$

*and let $x \in FU'$ and $a \in U$ with $p(x) = h(a)$ (where (F, p) is a *-bundle functor). Assume γ and κ etale. Then, with $b = \gamma(a)$, we have*

$$F(\gamma)(h_a(x)) = k_b(F(\kappa)(x)).$$

Proof. Since $FU' \cong U' \times_{V'} FV'$, we may assume that x is of the form $\kappa_{h(a)}(x')$ with $x' \in FV'$ over $\kappa(h(a)) = k(\gamma(b))$. So

$$k_b(F(\kappa)(x)) = k_b(F(\kappa)(\kappa_{h(a)}(x'))) = k_b(x') = k_{\gamma(a)}(x'),$$

the middle equality by (10). On the other hand

$$F(\gamma)(h_a(x)) = F(\gamma)(h_a(\kappa_{h(a)}(x'))) = F(\gamma)(\kappa \circ h)_a(x')) =$$

$$= F(\gamma)((k \circ \gamma)_a(x')) = F(\gamma)(\gamma_a(k_{\gamma(a)}(x'))) = F(\gamma)(\gamma_a(k_b(x'))) = k_b(x'),$$

the last equality again by (10), and using the associativity law for *-bundle functors twice. This proves the Lemma.

From the lemma, we conclude the $\mathbf{G}$-invariance of the map $\overline{h}$, and hence the existence of a functor $h^*F\mathbf{G}' \to F\mathbf{G}$, which in turn represents the morphism $\overline{H} : H^*FE' \to FE$ we were looking for. This provides F, viewed as a natural bundle defined for all smooth etendues, with structure of *-bundle functor in a suitable sense ("up to coherent isomorphisms" etc.), and this "suitable sense" has of course to be made precise in parallel with the proving of the various compatibilities for associativity etc.. I haven't done so yet -there is probably not much meat in it.

Let me finish by indicating why the structure does what it is supposed to do, namely that cross sections of the bundle T^*E does correspond to differential forms on the etendue E, (and then it easily follows that $\overline{H}$, for $H : E \to E'$ does correspond to pulling forms back). So consider a cross-section $\omega : E \to T^*E$. We represent E by a smooth etale groupoid $\mathbf{G} = (G_1 \rightrightarrows G_0)$, so $E = B\mathbf{G}$ and $T^*E = BT^*\mathbf{G}$. Consider

$$\begin{array}{ccccc}
T^*G_1 & \rightrightarrows & T^*G_0 & \xrightarrow{\pi'} & T^*B\mathbf{G} \\
\downarrow p_1 & & \downarrow p_0 & \cong & p \downarrow \uparrow \omega \\
G_1 & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & G_0 & \xrightarrow[\pi]{} & B\mathbf{G}
\end{array}$$

Because the right hand square is a bi-pull-back by Proposition 1, and $p \circ \omega = id$ (actually, it would suffice to have an isomorphism here), we get a cross-section ω_0 of p_0 with $\pi' \circ \omega_0 \cong \omega \circ \pi$. The isomorphism $\pi \circ d_0 \cong \pi \circ d_1$, by virtue of which the lower row is a bi-pull-back (11), gives the middle isomorphism in

$$\pi' \circ \omega_0 \circ d_0 \cong \omega \circ \pi \circ d_0 \cong \omega \circ \pi \circ d_1 \cong \pi' \circ \omega_0 \circ d_1.$$

Now using that the top row is a bi-pull-back (11), we get from this composite isomorphism a map $\omega_1 : G_1 \to T^*G_1$, making the relevant diagrams on the left commute up to isomorphism; but the codomain object is in this case T^*G_0, which is a manifold, hence the isomorhism is an identity. So things commute, so that the pair ω_1, ω_0 defines a functor $\mathbf{G} \to T^*\mathbf{G}$; and existence of such a functor is tantamount to saying that ω_0 is a $\mathbf{G}$-invariant 1-form on G_0.

Conversely, it is clear that such an invariant 1-form, i.e. a functor $f : \mathbf{G} \to T^*\mathbf{G}$ which is a section of the functor $p : T^*\mathbf{G} \to \mathbf{G}$ gives rise to a section Bf of the bundle $T^*E \to E$.

References

[1] M.Artin, A.Grothendieck, J.-L.Verdier, Théorie des topos et cohomologie étale des schémas, Vol.1, Springer Lecture Notes in Math. 269 (1972)

[2] J.Benabou, Some remarks on 2-categorical algebra (Part I), Bull. Soc. Math. Belg. 41 (1989), 127-194

[3] A.Haefliger, Groupoïdes d'holonomie et classifiant, Astérisque 116 (1984), 70-97

[4] M.Hakim, Topos anneles et schemas relatifs, Ergebnisse der Math. 64, Springer 1972

[5] J. Janyska, Geometric properties of prolongation functors, Casopis Pest. Mat. 108 (1983)

[6] A.Joyal and R. Street, Pseudo pull-backs equivalent to pull-backs, Cahiers de Top. et Geom. Diff. Cat. 1992 ?

[7] A.Joyal and M.Tierney, An Extension of the Galois Theory of Grothendieck, Mem. Amer. Math. Soc. 309 (1984)

[8] A.Kock, Local groupoids and etendues, manuscript Sussex 1994 - available by anonymous ftp from theory.doc.ic.ac.uk

[9] A.Kock and I.Moerdijk, Presentations of étendues, Cahiers de Top. et Géom. Diff. Cat. 32 (1991),

[10] A.Kock and I.Moerdijk, Every étendue comes from a local equivalence relation, Journ. Pure Appl. Algebra 82 (1992), 155-174

[11] A.Kock and I.Moerdijk, Spaces with local equivalences and their monodromy, Aarhus Preprint Series 1994 No. 8. Submitted.

[12] I. Kolar, P. Michor, and J. Slovak, Natural Operations in Differential Geometry, Springer Verlag 1993

[13] I.Moerdijk, The Classifying Topos of a Continuous Groupoid ,I, Trans. Amer. Math. Soc. 310 (1988), 629-668

[14] I.Moerdijk, Classifying Toposes and Foliations, Ann.Inst.Fourier 41 (1991), 189-209

[15] I.Moerdijk, Foliations, Groupoids and Grothendieck Etendues, Lecture Notes, Zaragoza 1993

[16] I.Moerdijk, Descent Theory for Toposes, Bull. Soc. Math .Belg. 41 (1989), 373-391

[17] I. Moerdijk and D. Pronk, Orbifolds, sheaves and groupoids, Utrecht preprint nr. 885 (1994). To appear in K-theory.

[18] R. Palais and C.L. Terng, Natural bundles have finite order, Topology 16 (1977), 271-277

[19] D. Pronk, Etendues and stacks as bicategories of fractions, preprint Utrecht 1993. To appear in Compositio Math.

[20] K.Rosenthal, Etendues and categories with monic maps, J.Pure Appl. Alg. 22 (1981), 193-212

[21] S.Salvioli, On the theory of geometric objects, J. Differential Geometry 7 (1972), 257-278

[22] J. Tapia, Sur la cohomologie de certains espaces d'orbites, Thesis, Univ. Paul Sabatier, Toulouse 1987

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Aarhus September 1995