Extension Theory for Local Groupoids

Anders Kock

October 29, 1997

We intend to provide an algebraic framework in which some of the algebraic theory of connections become identical to some of the (Eilenberg - Mac Lane) theory of extensions of non-abelian groups. In particular, the Bianchi identity for the curvature of a connection is related to one of the crucial equations in extension theory. The Bianchi identity as a purely combinatorial fact was dealt with in [12], where the relationship to differential geometry was substantiated; this relationship will not be an issue here, and we do not presuppose [12].

The algebraic notion underlying our project is that of *local groupoid*; this is a, rather evident, widening of Van Est's and Swierczkowski's notion of *local group* [16], which is the context in which the latter studied extension theory. But by considering local groups only, one fails to bring the *reflexive symmetric graphs* into the scope, and they are the local groupoids that carry the connection theory. Extension theory for (global) groupoids was studied by Brown and Higgins [2], but again, the graphs are not included under groupoids either. Finally, Kirill Mackenzie, in [15] and elsewhere studied extension theory for Lie algebroids, which is explicitly presented as including (differential geometric) connection theory, and is modelled on the classical extension theory. In some sense, Lie algebroids are infinitesimal (a special case of local) groupoids, but I haven't yet been able to get Mackenzie's results (notably [15] Theorem IV.3.20) on Lie Algebroids out as part of the theory to be presented here; its relationship to his theory is therefore at present only an analogy.

I want to thank him for his interest and some correspondence which prompted the present research. I also want to thank Professor Van Est, who in 1991 sent me an inspiring letter and a 1976 manuscript [5], advo-

cating the relationship between combinatorial group theory and differential geometry. Also he called my attention to the work of Swierczkowski ; in fact, Swierczkowski in [16] thanks Van Est for the similar kind of inspiration.

1 Local Groupoids and FDP Complexes

Local groupoids[1] are like groupoids, except that the composition is not always defined, even when source and target match; more precisely, a local groupoid X consists of a set X_0 of objects, or vertices, a set X_1 of morphisms, or arrows, each having specified source and target objects. For each object x, there should be a specified identity arrow id_x, and for each arrow a, there should be a specified "inverse" arrow a^{-1}, and these data satisfying the expected book-keeping conditions, including $(a^{-1})^{-1} = a$; (so the data so far is what we in [10] considered under the name of "oriented graph with identities and inversion"). Finally, there is a partially defined composition of arrows: if target(a) = source(b), a composite arrow $a \cdot b$ is sometimes defined, and has source equal to the source of a, target equal to the target of b, just as for groupoids. Note that we compose in the forward direction. The axioms are, like in [16],

1) If $a \cdot b$ and $b \cdot c$ are defined, then $(a \cdot b) \cdot c$ is defined iff $a \cdot (b \cdot c)$ is, and in that case $(a \cdot b) \cdot c \ = \ a \cdot (b \cdot c)$;
2) Let $a : x \to y$ be any arrow. Then $a \cdot a^{-1}$ and $a^{-1} \cdot a$ are defined, and are equal to the id_x and id_y, respectively; $id_x \cdot a$ and $a \cdot id_y$ are defined, and are equal to a;
3) if $a \cdot b$ is defined, then so is $b^{-1} \cdot a^{-1}$ (it follows then that $b^{-1} \cdot a^{-1} = (a \cdot b)^{-1}$).

Note that as well groupoids as reflexive symmetric relations/graphs are examples. In particular, in the context of Synthetic Differential Geometry (cf. e.g. [8] or [12]), the "first neighbourhood of the diagonal" of a manifold M, $M_{(1)} \subseteq M \times M$, is an example of a local groupoid; in fact the one that led me to consider the notion.

[1] The notion of local groupoid considered here does not agree with a notion with a similar name which has been considered by Ehresmann, cf. also [11]

A *functor* or *homomorphism* $f : X \to Y$ between local groupoids is the expected thing: book-keeping, identities, inversion, are preserved, and also composition is preserved, in the sense that if $a \cdot b$ is defined, then so is $f(a) \cdot f(b)$, and

$$f(a) \cdot f(b) = f(a \cdot b). \tag{1}$$

Stronger notions ("good" functors), as well as weaker ones ("connections"), where (1) is not required, will be considered later.

Just as for groupoids, the data of a local groupoid may be encoded as a simplicial complex, the *nerve* $X_\bullet$ of X; it has $X_0 =$ the set of objects, $X_1 =$ the set of arrows, $X_2 =$ the set of composable pairs, etc. The face operator d^i omits the i'th vertex, and the degeneracy s^i inserts an appropriate identity arrow at the ith vertex.

Note that when we say that an n-tuple $a_1, \ldots, a_n$ is composable, we imply that all adjacent pairs, triples, etc. in the list are likewise composable, so it is not like the "paracategories" of Freyd. The nerve $X_\bullet$ of a local groupoid X carries some further canonical structure, due to the presence of inverses, nameli it is an FDP-complex in the sense of [1]; or, equivalently, it is an object of the Boolean Algebra classifier (cf. [13]). It means that the symmetric group in $n+1$ letters acts on the set of n-simplices, in a way compatible with the face- and degeneracy operators; equivalently, any map $\underline{k} \to \underline{n}$ (where $\underline{k} = \{0, 1, \ldots, k\}$, and similarly for $\underline{n}$) defines a k-dimensional 'face' of any n-simplex. (Note that for an ordinary simplicial set, only an *order preserving* map $\underline{k} \to \underline{n}$ will define a face.) The FDP-viewpoint leads to a more symmetric status of all the faces of a simplex in the nerve of X. For instance, if a 2-simplex e is given by the composable pair

$$x_0 \xrightarrow{a} x_1 \xrightarrow{b} x_2, \tag{2}$$

the permutation $\underline{2} \to \underline{2}$ 'transposing 0 and 1' produces out of e the 2-simplex (composable pair)

$$x_1 \xrightarrow{a^{-1}} x_0 \xrightarrow{a \cdot b} x_2. \tag{3}$$

If e is an n-simplex, and $(i_0, \ldots, i_k)$ is a $k+1$-tuple of numbers $\in \underline{n}$, we may write $e(i_0, \ldots, i_k)$ for the 'face'-k-simplex of e corresponding to the map $\underline{k} \to \underline{n}$ given by the $k+1$-tuple. Thus, if e is the 2-simplex (2), $e(102)$ is the 2-simplex (3), and $e(10)$ is the 1-simplex a^{-1}. Note that we write the (generalized) face operators like (102) or (10) on the right of their argument.

The nerve $X_\bullet$ of a local groupoid X has the property that "any horn $\Lambda^k[n] \to X_\bullet$ has *at most* one filler $\Delta[n] \to X_\bullet$", to use the terminology of [6].

2 Good Homomorphisms, and Connections

We already described the natural notion of functor or homomorphism $X \to Y$ between local groupoids. A functor $f : X \to Y$ will be called *good* if f is the identity map on $X_0 = Y_0$, is surjective $X_1 \to Y_1$, and if it *reflects composability*, meaning that if an n-tuple of arrows in X has the property that the f-images form a composable n-tuple in Y, then the given n-tuple is composable. (In particular, f is a Kan-fibration.)

If $Y \overset{\to}{\to} X_0$ and $J \overset{\to}{\to} X_0$ are two local groupoids on the same set of objects, a *connection* ∇ from Y to J is like a homomorphism, preserving the set of objects, except that we do not require composition of arrows to be preserved. But we do require that ∇ preserves book-keeping, identities, and inversion, as well as composability. (Such things were considered already in [10]. The assumption that inversion is preserved could possibly be circumvented, I don't know at what cost; one would need to circumvent it if one wants to deal with elements of order 2.)

Consider a functor $f : X \to Y$, and assume that f is the identity map on $X_0 = Y_0$. Then a *splitting connection* ∇ of f is just a connection ∇ with $f \circ \nabla =$ identity of Y.

Example. In Synthetic Differential Geometry, Ehresmann's notion [3] of connection in a differentiable groupoid becomes a special case: Let $\Phi \overset{\to}{\to} M$ be a differentiable groupoid on a manifold M. Then an Ehresmann connection in Φ may be construed as a connection $M_{(1)} \to \Phi$. This viewpoint is expounded in [10], [9], [12] (among others). The *curvature* measures the extent to which the connection fails to be a homomorphism (functor).

3 Normal Subgroupoids

It is sometimes useful to use "hom-set notation" for a local groupoid X, writing $a \in X(x_0, x_1)$ to mean that source and target of a is x_0 and x_1, respectively. An *endo-arrow* of a local subgroupoid is an arrow whose source and target agree. A *(local) group bundle* is a (local) groupoid, all of whose arrows are endo-arrows. For instance any "global" groupoid contains a "maximal" group bundle, namely the set of all its endo-arrows. Note that a group bundle is itself a groupoid.

A *normal subgroupoid* N of a local groupoid X is a group bundle in it, with the property that if $n \in N_x$ and $a : x \to y$, then a^{-1}, n, a is a composable triple, and $a^{-1} \cdot n \cdot a \in N_y$. (So in particular, if $n \in N_x$ and the source of a is x, then $n \cdot a$ is composable, and similarly on the target side.) The map $n \mapsto a^{-1} \cdot n \cdot a$ is of course a group isomorphism $N_x \to N_y$, we denote it $\mathrm{ad}a$; we write $n^{\mathrm{ad}a}$ for the value of $\mathrm{ad}a$ on $n \in N_x$.

If $f : X \to Y$ is a good homomorphism of local groupoids over X_0, we get a normal subgroupoid of X, the *kernel* of f; it consists of the arrows a with $f(a)$ an identity arrow in Y. The stability properties come from the fact that f reflects composability. Thus, if a and n are as above, the triple a^{-1}, n, a is composable, because it maps by f to the composable triple $f(a)^{-1}, id_x, f(a)$.

Conversely, given a normal group bundle N in a local groupoid X, we may in the expected way form the quotient $X \to X/N$, which is a good homomorphism of local groupoids, having N for its kernel.

This, as well as the following, are standard notions cf. e.g. [7], and so is the following (at least for the group case) cf. e.g. [15].

For any group bundle $G \to X_0$, we may form a groupoid $Iso(G) \rightrightarrows X_0$, where an arrow from x to y is a group isomorphism from the group G_x to the group G_y. It contains a normal subgroupoid $Inn(G)$, which are the group bundle consisting of the inner automorphisms $G_x \to G_x$. The quotient groupoid $Iso(G)/Inn(G)$ is the groupoid $Out(G) \rightrightarrows X_0$ of *outer isomorphisms* of G (thus an outer isomorphism $x \to y$ is a class of group isomorphisms $G_x \to G_y$).

If N is a normal subgroupoid of a local groupoid X, as above, we get a functor of local groupoids $\mathrm{ad} : X \to Iso(N)$, sending a to $\mathrm{ad}a$.

In particular, consider a good homomorphism of local groupoids $f : X \to Y$ with N as its kernel (an "*extension of* Y *by* N"). Then the composite homomorphism

$$X \stackrel{ad}{\to} Iso(N) \to Out(N)$$

factors across the quotient map $X \to X/N = Y$ to provide a homomorphism of local groupoids $Y \to Out(N)$. Generally, if Y is a local groupoid on X_0 and N a group bundle on X_0, we call a homomorphism (over X_0) $\Theta : Y \to Out(N)$ an *abstract kernel from* Y *to* N. So in particular, an extension of Y

by N,

$$N \subseteq X \xrightarrow{f} Y, \tag{4}$$

gives rise to such an abstract kernel Θ, called the abstract kernel *of* the extension.

4 The Extension problem

We shall deal here with only one aspect of the extension problem for local groupoids, well known for extensions of groups [15] Chapter 4, and for local groups [16]; for groupoids, the first of them is dealt with in [2] in "crossed complex" terms.

It consists in reconstructing an extension, together with a splitting connection ∇ in it, from the data of the curvature $R = R_\nabla$ of ∇ and the adjoint connection $\alpha = ad(\nabla)$.

We begin by an analysis. Let an extension (4) be given, together with a splitting connection $\nabla : Y \to X$. Its *curvature* R_∇ is the law which to any composable pair of arrows a, b in Y measures the defect of ∇ in preserving their composite,

$$R_\nabla(a, b) := \nabla(a)\nabla(b)\nabla((ab)^{-1}). \tag{5}$$

(Note that we omit the dot denoting composition of arrows, for brevity, where it does not cause confusion.) The fact that the right hand side here is indeed composable in X follows because f reflects composability and preserves composition. The expression for curvature will become more symmetric and amenable to calculation if we re-express it in terms of the FDP structure of the nerve of Y, using the notation introduced above. If the 2-simplex given by the composable pair a, b is called e, the defining equation for R_∇ may be written

$$R_\nabla(012) = \nabla(01)\nabla(12)\nabla(20)$$

where (012) denotes the 2-simplex $e = e(012)$ and, similarly, (01) denotes the 1-simplex $e(01)(= a)$, and likewise for the other 1-simplices occurring. Using that ∇ is a splitting of f, which does commute with compositions, it follows that the arrow $R_\nabla(a, b)$ in (5) actually belongs to the kernel N of f, in fact to N_x where x is the domain of a.

Also, the adjoint connection $ad\,\nabla$ of ∇ is the composite

$$Y \xrightarrow{\nabla} X \xrightarrow{ad} Iso(N),$$

which is a connection on Y with values in the groupoid $Iso(N)$.

We deduce some equations for $R = R_\nabla$ and $\alpha = ad\nabla$. First, for any 2-simplex $e = e(012) = (012)$, we have the "normalization" equations

$$R(012) = R(021)^{-1} \text{ and } R(010) = id_x \tag{6}$$

(where x is the vertex corresponding to 0 in $e = (012)$).

The second equation refers to a 2-simplex $e = e(012) = (012)$, as above, and to $n \in N_x$. Recall that we write actions of elements γ in $Iso(N)$ on elements $n \in N_x$ as exponents. Then, writing $\overline{R}$ for the curvature R_α of α,

$$(n)^{\overline{R}(012)} = R(012)^{-1} n R(012). \tag{7}$$

Finally, for any 3-simplex $t = t(0123) = (0123)$, (with x corresponding to 0, as above), we have the 'Bianchi Identity'

$$R(123)^{\alpha(10)} R(013) R(032) R(021) = id_x. \tag{8}$$

To prove (6), just observe that

$$R(021)^{-1} = (\nabla(02)\nabla(21)\nabla(10))^{-1} = \nabla(10)^{-1}\nabla(21)^{-1}\nabla(02)^{-1},$$

but since ∇ commutes with inversion, $\nabla(10)^{-1} = \nabla(01)$ and similarly for the other two factors, so that we get the defining expression for $R(012)$. The second normalization equation is proved similarly.

The equation (7) expresses the curvature of the adjoint connection α in terms of R, more precisely, says that "formation of *ad* and formation of curvature commute". This follows because

$$\overline{R}(012) = \alpha(01)\alpha(12)\alpha(20)$$

which is the isomorphism consisting in consecutively conjugating by $\nabla(01)$, $\nabla(12)$, and $\nabla(20)$, thus the left hand side of (7) is

$$\nabla(20)^{-1}\nabla(12)^{-1}(\nabla(01))^{-1} n \nabla(01)\nabla(12)\nabla(20),$$

and again using $(\nabla(20))^{-1} = \nabla(02)$ etc, and the defining equation for $R_\nabla(012)$ and $R_\nabla(021)$, we immediately get the right hand side of (7) (for

$R = R_\nabla$). - Note that (7) implies that the curvature of α is expressed by an *inner* automorphism, thus the composite connection

$$Y \xrightarrow{\alpha} Iso(N) \to Out(N)$$

is curvature free.

Finally, the main equation (8) is, according to [12], the "Bianchi Identity". The proof is as in [12], but we give it for completeness, and for uniformity of conventions. The four factors on the left hand sides are, by the definition of curvature $R = R_\nabla$, and of α as $ad\nabla$, equal to the four bracketed expressions in the product

$$[\nabla(01)\nabla(12)\nabla(23)\nabla(31)\nabla(10)]\ [\nabla(01)\nabla(13)\nabla(30)]$$
$$[\nabla(03)\nabla(32)\nabla(20)]\ [\nabla(02)\nabla(21)\nabla(10)];$$

now remove the brackets and cancel out succesively suitable adjacent terms, using $\nabla(30)\nabla(03) = id$ and similarly for other pairs of indices. Then "nothing" is left, meaning that we arrive at id_x.

Note that (6) and (8) together imply that $R(120)^{\alpha(10)} = R(012)$; simply apply (8) to a degenerate simplex (0120) and then use (6).

There are twenty-three further versions of Bianchi identity for the given 3-simplex t, with id_x as their value (see Remark below), and also, the equation (8) can be rearranged, to get some of the factors on the other side of the equality sign. Thus, for instace if we multiply on the right by $R(021)^{-1} = R(012)$, we get the following version of (8)

$$R(123)^{\alpha(10)}R(013)R(032) = R(012), \tag{9}$$

which is the form we shall need later on.

Thus, given an extension $N \to X \to Y$ equipped with a splitting connection ∇, we get $R = R_\nabla$ (taking 2-simplices of Y into elements of N) as well as the adjoint connection $\alpha = ad\nabla$ taking values in $Iso(N)$, and satisfying the equations (6), (7), (8).

We now have a converse result, generalizing [15] IV,8.1, and [16], Section 7, namely

Theorem 1 *Given a local groupoid Y, and a group bundle N on Y_0, and given R and α satisfying the equations (6), (7), (8). Then there exists an extension $N \to X \to Y$ with a splitting connection ∇ with curvature R and adjoint connection α.*

Proof/Construction. This is a standard construction. We let the arrows of X be pairs (a, ξ), where $\xi : x \to y$ in Y and $a \in N_x$. More generally, an n-simplex of (the nerve of) X is a pair (a, t) where $t = t(0, \ldots, n)$ is an n-simplex of Y, and $a \in N_x$ (where x is the vertex of t corresponding to 0). To define the composition in X, let ξ, η be a composable pair in Y,

$$x_0 \xrightarrow{\xi} x_1 \xrightarrow{\eta} x_2,$$

and let $a \in N_{x_0}$, $b \in N_{x_1}$. Let the 2-simplex in Y determined by ξ, η be denoted $t = t(012) = (012)$. Then

$$(a, \xi) \cdot (b, \eta) := (a \cdot b^{\alpha(10)} \cdot R(012), \xi \cdot \eta);$$

and

$$\nabla(\xi) := (id_x, \xi).$$

The maps $N \to X$ and $X \to Y$ are obvious, as are the facts that $N \to X$ preserves composition, and $X \to Y$ preserves composition and reflects composability. To prove associativity, consider a composable 3-tuple $(a; \xi), (b, \eta), (c, \zeta)$. thus we have a composable 3-tuple ξ, η, ζ in Y. Let $t = t(0123) = (0123)$ denote the 3-simplex given by ξ, η, ζ, so with our standard abuse $\xi = (01)$, $\eta^{-1}\xi^{-1} = (20)$, etc. In this notation,

$$(a, \xi) \cdot (b, \eta) = (a, 01) \cdot (b, 12) = (a \cdot b^{\alpha(10)} \cdot R(012), 02)$$

so $((a, \xi) \cdot (b, \eta)) \cdot (c, \zeta)$ has for its first component

$$a \cdot b^{\alpha(10)} \cdot R(012) \cdot c^{\alpha(20)} \cdot R(023), \tag{10}$$

and similarly $(a, \xi) \cdot ((b, \eta) \cdot (c, \zeta))$ has for its first component

$$a \cdot (b \cdot c^{\alpha(21)} \cdot R(123))^{\alpha(10)} \cdot R(013),$$

and using the fact that $\alpha(10)$ is a group homomorphism, this may be written

$$a \cdot b^{\alpha(10)} \cdot c^{\alpha(21)\alpha(10)} \cdot R(123)^{\alpha(10)} \cdot R(013). \tag{11}$$

Comparing (10) with (11) and cancelling $a \cdot b^{\alpha(10)}$, we see that we need to prove, for arbitrary c in the relevant N-fibre, that

$$R(012) \cdot c^{\alpha(20)} \cdot R(023) = c^{\alpha(21)\alpha(10)} R(123)^{\alpha(10)} \cdot R(013),$$

or, multiplying on the right by $R(023)^{-1} = R(032)$, that

$$R(012) \cdot c^{\alpha(20)} = c^{\alpha(21)\alpha(10)} \cdot R(123)^{\alpha(10)} \cdot R(013) \cdot R(032). \qquad (12)$$

The three last factors here may, by the version (9) of Bianchi identity, be replaced by $R(012)$, so that (12) reads

$$R(012) \cdot c^{\alpha(20)} = c^{\alpha(21)\alpha(10)} \cdot R(012),$$

or, multiplying on the right by $R(021) = R(012)^{-1}$,

$$R(012) \cdot c^{\alpha(20)} \cdot R(021) = c^{\alpha(21)\alpha(10)}. \qquad (13)$$

Now since $\alpha(02)$ is an isomorphism, we may assume that c is of form $d^{\alpha(02)}$ for some d in the relevant N-fibre, and substituting this, we see that to prove (13) for all c is equivalent to proving, for all d,

$$R(012) \cdot d \cdot R(021) = d^{\alpha(02)\alpha(21)\alpha(10)}. \qquad (14)$$

The "exponent" on the right is $\overline{R}$, the curvature of α, applied to (021), so that (14) holds because $\overline{R}(021)$ acts as conjugation by $R(021)$, by equation (7). This proves the associative law.

To prove that the connection $\nabla : Y \to X$ constructed has the given R as its curvature, let ξ, η be a composable pair in Y, defining the 2-simplex

$$e = e(012) = (012).$$

Then we calculate the curvature of ∇ (writing id indiscriminately for the identity elements in the various N-fibres):

$$\begin{aligned}
\nabla(01) \cdot \nabla(12) \cdot \nabla(20) &= \\
&= (id, 01) \cdot (id, 12) \cdot (id, 20) \\
&= (id \cdot id^{\alpha(10)} \cdot R(012), 02) \cdot (id, 20) \\
&= (id \cdot id^{\alpha(10)} \cdot R(012) id^{\alpha(20)} \cdot R(020), 00);
\end{aligned}$$

but α takes identities to identities; and $R(020)$ is an identity by virtue of the normalization (6), so we are left with $(R(012), 00)$. Under the identification of the kernel of $X \to Y$ with N, this is just $R(012)$, as desired. The proof that ad$\nabla = \alpha$ is similar, and omitted. This proves the Theorem.

It is worthwhile to record that

$$(a, \xi)^{-1} = ((a^{-1})^{\alpha(\xi)}, \xi^{-1});$$

in particular, therefore,

$$(a, \xi) \cdot \nabla(\xi)^{-1} = (a, id_x),$$

noting that $R(t) = id_x$where t is the 2-simplex given by the composable pair (ξ, i_y).

Remark. For what it is worth, we may count the possible Bianchi identities. For a given 3-simplex, there are four vertices 0,1,2,3 among which we may choose to have the Bianchi expression appear as an endo-arrow. Choose one of the, say 0. There are four factors in the Bianchi expression, one of which appears as conjugated along a 1-simplex, using the connection. We may choose to have the conjugated factor appear in any of the four positions. Also, we may choose freely among the three 1-simplices emanating from 0 which of them we want to use for the conjugation purpose. And finally, the sense (clockwise or counterclockwise) of going round the conjugated factor then determines both the order in which the remaining three factors are to be taken, and in which sense we must go round them. This makes 24 Bianchi identities "centered" at 0.

Example. Here is an example of a naturally arising extension of local groupoids $N \to X \to Y$. We take $X_0 = Y_0 =$ set of (unoriented) lines in the Euclidean plane; Y_1 is the reflexive symmetric relation "lines l_1 and l_2 are not perpendicular". The set of arrows in X from l_1 to l_2 is the set of isometries $l_1 \to l_2$ *which preserve orientation*; this makes sense even though the lines are unoriented, since the condition that they are not perpendicular implies that it makes sense to talk about *sameness* of orientations on them. Finally, we let $\nabla(l_1, l_2)$ be reflection in the bisector of the *acute* angle the lines form with each other (for parallel lines, reflection in their midline.) The curvature R_∇ is non-trivial, as the reader may verify by drawing some geometric triangles, but the adjoint connection α has trivial curvature, because the vertex groups are commutative.

References

[1] M. Barr, Oriented singular homology, Theory and Appl. of Categories, Vol. 1, No. 1 (1995), 1-9.

[2] R. Brown and P.J. Higgins, Crossed complexes and non-abelian extensions, in Category Theory, Proceedings Gummersbach 1981, Springer Lecture Notes in Math. 962 (1982), 39-50.

[3] C. Ehresmann, Sur les connexions d'ordre superieur, Atti V. Cong. Un. Mat. Italiana, Pavia-Torino (1956), 326-328.

[4] S. Eilenberg and S. Mac Lane, Cohomology theory in abstract groups, Ann. of Math. 48 (1947), 326-341.

[5] W.T. Van Est, Peiffer-Smith-Whitehead non-commutative homology and cohomology, unpublished notes, Amsterdam 1976.

[6] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory, Springer Verlag 1967.

[7] P.J. Higgins, Categories and Groupoids, Van Nostrand Reinhold, 1971.

[8] A. Kock, Synthetic Differential Geometry, Cambridge Univ. Press 1981.

[9] A. Kock, Ehresmann and the fundamental structures of differential geometry, seen from a synthetic viewpoint, in C. Ehresmann, Collected Works Vol. 1, 549-554, Amiens 1984.

[10] A. Kock, The Integration Theorem for Lie Groupoids, Czechoslovak Math. J. 39 (114) (1989), 423-431.

[11] A. Kock, Local Groupoids and Etendues, Manuscript Sussex 1994.

[12] A. Kock, Combinatorics of curvature and the Bianchi identity, Theory and Appl. of Categories, Vol.2 , No.7 (1996), 69-89.

[13] F.W. Lawvere, Toposes generated by codiscrete objects in combinatorial topology and functional analysis, unpublished manuscript, Buffalo 1989.

[14] K. Mackenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, Cambridge Univ. Press 1987.

[15] S. Mac Lane, Homology, Springer Verlag 1963.

[16] S. Swierczkowski, Cohomology of local group extensions, Trans. Amer. Math. Soc. 128 (1967), 291-320.