Affine connections, midpoint formation, and point reflection*

Anders Kock
University of Aarhus

Preface

It is a striking fact that differential calculus exists not only in analysis (based on the real numbers $\mathbb{R}$ and limits therein), but also in algebraic geometry, where no limit processes are available. In algebraic geometry, one rather uses the idea of *nilpotent elements* in the "affine line" R; they act as infinitesimals. (Recall that an element x in a ring R is called *nilpotent* if $x^k = 0$ for suitable non-negative integer k.)

Synthetic differential geometry (SDG) is an axiomatic theory, based on such nilpotent infinitesimals. It can be proved, via topos theory, that the axiomatics covers both the differential-geometric notions of algebraic geometry and those of calculus.

I shall illustrate this synthetic method, by presenting its application to three particular types of differential-geometric structure, namely that of *affine connection*, *midpoint formation*, and *point reflection* (geodesic symmetry).

I shall not go much into the foundations of SDG, whose core is the so-called KL[1] axiom scheme. This is a very strong kind of axiomatics; in fact, a salient feature of it is: *it is inconsistent* – if you allow yourself the luxury of reasoning with so-called classical logic, i.e. use the "law of excluded middle", "proof by contradiction", etc. Rather, in SDG, one uses a weaker kind of logic, often called "constructive" or "intuitionist". Note the evident logical fact that there is a trade-off: with a *weaker* logic, *stronger* axiom systems become consistent. For the SDG axiomatics, it follows for instance that any function from the number line to itself is infinitely often differentiable (smooth); a very useful simplifying feature in differential geometry – but incompatible with the law of excluded

*Expanded version of [8]. The notion of midpoint formation considered in loc.cit. has been generalized, so that a notion of point reflection can be considered, together with their interdependence, cf. Theorem 4.2 and Figure (4.1) below. Also, some proofs have been supplied, using Christoffel symbols for affine connections.

[1] for "Kock-Lawvere"

middle, which allows you to construct the non-smooth function

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if not} \end{cases}.$$

1 Nilpotents, and neighbours

Nilpotent elements on the number line serve as *infinitesimals*[2], in a sense which is "forbidden" when the number line is $\mathbb{R}$. Nilpotent infinitesimals come in a precise hierachy, since

$$x^k = 0 \quad \text{implies} \quad x^{k+1} = 0.$$

The method of SDG combines the "nilpotency" ideas from algebraic geometry with category theory, and categorical logic: category theory has provided a sense by which reasoning in (constructive) naive set theory is *sound* for geometric reasoning. So the following is formulated in such naive set theoretic terms.

We plunge directly into the geometry of infinitesimals (in the "nilpotency" sense): let us denote by $D \subseteq R$ the set of $x \in R$ with $x^2 = 0$ (the "first order infinitesimals"), more generally, let $D_k \subseteq R$ be the set of kth order infinitesimals, meaning the set of $x \in R$ with $x^{k+1} = 0$. (So $D = D_1$.) The basic instance of the KL axiom scheme says that any map $D_k \to R$ extends uniquely to a polynomial map $R \to R$ of degree $\leq k$. Thus, given any map $f : R \to R$, the restriction of f to D_k extends uniquely to a polynomial map of degree $\leq k$, the kth *Taylor polynomial* of f at 0.

For x and y in R, we say that x and y are kth order *neighbours* if $x - y \in D_k$, and we write $x \sim_k y$. It is clear that $\sim_k$ is a reflexive and symmetric relation. It is not transitive. For instance, if $x \in D$ and $y \in D$, then $x + y \in D_2$, by binomial expansion of $(x + y)^3$; but we cannot conclude $x + y \in D$. So $x \sim_1 y$ and $y \sim_1 z$ imply $x \sim_2 z$, and similarly for higher k.

We now turn to the (first order) neighbour relations in the coordinate plane R^2. It is, in analogy with the 1-dimensional case, defined in terms of a subset $D(2) \subseteq R^2$; we put

$$D(2) = \{(x_1, x_2) \in R \times R \mid x_1^2 = 0, x_2^2 = 0, x_1 \cdot x_2 = 0\}.$$

So $D(2) \subseteq D \times D$. We define the "first order neighbour relation" $\sim$ (or $\sim_1$) by putting $\underline{x} \sim \underline{y}$ if $\underline{x} - \underline{y} \in D(2)$, where $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$. Similarly for $D(n) \subseteq R^n$, and the resulting first order neighbour relation on the higher "coordinate vector spaces" R^n.

If $\underline{x} \in D(n)$, we have $B(x, x) = 0$ for any bilinear $B : R^n \times R^n \to R^m$, and if B is furthermore symmetric, we therefore also have the useful

$$B(x + y, x + y) = 2B(x, y) \tag{1.1}$$

[2]they are not to be compared to the infinitesimals of non-standard analysis

for x and y in $D(n)$.

There is also a kth order neighbour relation $\sim_k$ on R^n, defined in a completely analogous manner from the set

$$D_k(n) := \{(x_1, \ldots, x_n) \in R^n \mid \text{ the product of any } k+1 \text{ of the } x_i\text{s is } 0\},$$

namely $x \sim_k y$ if $\underline{x} - \underline{y} \in D_k(n)$.

A higher dimensional version of the KL axiom scheme says that any map $D(n) \to R$ extends uniquely to an affine map. More generally, any map $D_k(n) \to R$ extends uniquely to a poynomial map $R^n \to R$ of degree $\leq k$. The codomain R here may be replaced by any other finite dimensional vector space. In particular, if a map $D(n) \to R^m$ takes 0 to 0, it extends uniquely to a linear map $R^n \to R^m$. It is then easy to prove that if a map $\Gamma : D(n) \times D(n) \to R^m$ "vanishes on the two axes", i.e. if $\Gamma(\underline{d}, 0) = 0 = \Gamma(0, \underline{d})$ for all $\underline{d} \in D(n)$, then Γ extends uniquely to a bilinear map $R^n \times R^n \to R^m$; and this bilinear map is symmetric iff Γ itself is so.

The following (cf. [7], Proposition 1.5.1) is another consequence of the KL axiom scheme:

Theorem 1.1 *Any map $f : R^n \to R^m$ preserves the kth order neighbour relation,*

$$\underline{x} \sim_k \underline{y} \quad \textit{implies} \quad f(\underline{x}) \sim_k f(\underline{y}).$$

Proof sketch for $n = 2$, $m = 1$, for the first order neighbour relation $\sim_1$. It suffices to see that $\underline{x} \sim_1 \underline{0}$ implies $f(\underline{x}) \sim_1 f(\underline{0})$, i.e to prove that $\underline{x} \in D(2)$ implies $f(\underline{x}) - f(\underline{0}) \in D$. Now from a suitable version of the KL axiom scheme follows that on $D(2)$, f agrees with a unique affine function $T_1 f : R^2 \to R$, so for $\underline{x} \in D(2)$,

$$f(\underline{x}) - f(\underline{0}) = a_1 x_1 + a_2 x_2.$$

Squaring the right hand side here yields 0, since not only $x_1 \in D$ and $x_2 \in D$, but also $x_1 \cdot x_2 = 0$. So $f(\underline{x}) - f(\underline{0}) \in D$.

From the Theorem follows that the relation $\sim_k$ on R^n is *coordinate free*, i.e. is a truly geometric notion: any re-coordinatization of R^n (by *any* map, not just by a linear or affine one) preserves the relation $\sim_k$.

For suitable definition of what an *open* subsets of R^n is, and for a suitable definition of "*n-dimensional manifold*" (= something that locally can be coordinatized with open subsets of R^n), one concludes that on any manifold, there are canonical reflexive symmetric relations $\sim_k$: they may be defined in terms of a local coordinatization, but, by the Theorem, are independent of the coordinatization chosen.

Any map between manifolds preserves the relations $\sim_k$.

We shall mainly be interested in the *first* order neighbour relation $\sim_1$, (also written just $\sim$). In Sections 3 and 4, we study aspects of the second order neighbour relation $\sim_2$.

So for a manifold M, we have a subset $M_{(1)} \subseteq M \times M$, the "first neighbourhood of the diagonal", consisting of $(x, y) \in M \times M$ with $x \sim y$. It was in terms of this "scheme" $M_{(1)}$ that algebraic geometers in the 1950s gave nilpotent infinitesimals a rigourous role in geometry. Note that for $M = R^n$, we have $M_{(1)} \cong M \times D(n)$, by the map $(\underline{x}, \underline{y}) \mapsto (\underline{x}, \underline{x} - \underline{y})$.

Let us consider some notions from "infinitesimal geometry" which can be expressed in terms of the first order neighbour relation $\sim$ on an arbitrary manifold M. Given three points x, y, z in M. If $x \sim y$ and $x \sim z$ we call the triple (x, y, z) a *2-whisker at* x (sometimes: an *infinitesimal* 2-whisker, for emphasis); since $\sim$ is not transitive, we cannot in general conclude that $y \sim z$; if y happens to be $\sim z$, we call the triple (x, y, z) a *2-simplex* (sometimes an *infinitesimal* 2-simplex). Similarly for k-whiskers and k-simplices. A k-simplex is thus a $k+1$-tuple of mutual neighbour points. The k-simplices form, as k ranges, a simplicial complex, which in fact contains the information of differential forms, and the de Rham complex of M, see [2], [6], [1], [7].

(When we say that $(x_0, x_1, \ldots, x_k)$ is a k-whisker, we mean to say that it is a k-whisker *at* x_0, i.e. that $x_0 \sim x_i$ for all $i = 1, \ldots, k$. On the other hand, in a simplex, none of the points have a special status.)

Given a k-whisker $(x_0, \ldots, x_k)$ in M. If U is an open subset of M containing x_0, it will also contain the other x_is, and if U is coordinatized by R^n, we may use coordinates to define the affine combination

$$\sum_{i=0}^{k} t_i \cdot x_i, \tag{1.2}$$

(where $\sum t_i = 1$; recall that this is the condition that a linear combination deserves the name of *affine* combination). The affine combination (1.2) can again be proved to belong to U, and thus it defines a point in M. The point thus obtained has in general *not* a good geometric significance, since it will depend on the coordinatization chosen. However (cf. [5], [7] 2.1), it does have geometric significance, if the whisker is a simplex:

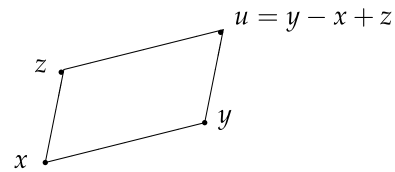
Theorem 1.2 *Let $(x_0, \ldots, x_k)$ be a k-simplex in M. Then the affine combination (1.2) is independent of the coordinatization used to define it. All the points that arise in this way are mutual neigbours. And any map to another manifold M' preserves such combinations.*

Proof sketch. This is much in the spirit of the proof of Theorem 1.1: it suffices to see that any map $R^n \to R^m$ (not just a linear or affine one) preserves affine combinations of mutual neighbour points. This follows by considering a suitable first Taylor polynomial of f (expand from x_0), and using the following purely algebraic fact: If $\underline{x}_1, \ldots, \underline{x}_k$ are in $D(n)$, then any linear combination of them will again be in $D(n)$ *provided* the $\underline{x}_i$s are *mutual* neighbours.

Examples. If $x \sim y$ in a manifold (so they form a 1-simplex), we have the affine combinations "midpoint of x and y", and "reflection of y in x",

$$\tfrac{1}{2}x + \tfrac{1}{2}y \quad \text{and} \quad 2x - y,$$

respectively. If x, y, z form a 2-simplex, we may form the affine combination $u := y - x + z$; geometrically, it means completing the simplex into a parallelogram by adjoining the point u. Here is the relevant picture:



(1.3)

(All four points here are neighbours, not just those that are connected by lines in the figure.) The u thus constructed will be a neighbour of each of the three given points.

Remark. If x, y, z and u are as above, and if x, y, and z belong to a subset $S \subseteq M$ given as the zero set of a function $f : M \to R$, then so does $u = y - x + z$; for, f preserves this affine combination.

2 Affine connections

If x, y, z form a 2-whisker at x (so $x \sim y$ and $x \sim z$), we cannot canonically form a parallelogram as in (1.3); rather, parallelogram formation is an added *structure*:

Definition 2.1 *An* affine connection *on a manifold M is a law λ which to any 2-whisker x, y, z in M associates a point $u = \lambda(x, y, z) \in M$, subject to the conditions*

$$\lambda(x, x, z) = z, \quad \lambda(x, y, x) = y. \tag{2.1}$$

(Cf. [3].) It can be verified (cf. [7] 2.3) that several other laws follow; in a more abstract combinatorial situation than manifolds, these laws should probably be postulated. Some of the laws are: for any 2-whisker (x, y, z)

$$\lambda(x, y, z) \sim y \text{ and } \lambda(x, y, z) \sim z \tag{2.2}$$

$$\lambda(y, x, \lambda(x, y, z)) = z \tag{2.3}$$

One will not in general have or require the "symmetry" condition

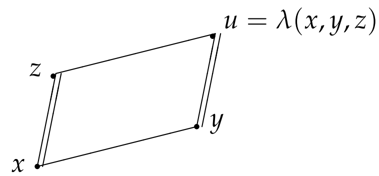
$$\lambda(x, y, z) = \lambda(x, z, y); \tag{2.4}$$

nor do we in general have, for 2-simplices x, y, z, that

$$\lambda(x, y, z) = y - x + z. \tag{2.5}$$

The laws (2.4) and (2.5) are in fact equivalent, and affine connections satisfying either are called *symmetric* or *torsion free*. We return to the torsion of an affine connection below.

If x, y, z, u are four points in M such that (x, y, z) is a 2-whisker at x, the statement $u = \lambda(x, y, z)$ can be rendered by a diagram



(2.6)

The figure[3] is meant to indicate that the data of λ provides a way of closing a whisker (x, y, z) into a *parallellogram* (one may say that λ provides a notion of *infinitesimal parallelogram*); but note that λ is not required to be symmetric in y and z, which is why we in the figure use different signatures for the line segments connecting x to y and to z, respectively.

Here, a line segment (whether single or double) indicates that the points connected by the line segment are neighbours.

If x, y, z, u are four points in M that come about in the way described, we say that the 4-tuple form a *λ-parallelogram* . The fact that we in the picture did not make the four line segments *oriented* contains some symmetry assertions, which can be proved by working in a coordinatized situation; namely that the 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the set of λ-parallelograms; so for instance (u, z, y, x) is a λ-parallelogram, equivalently

$$\lambda(\lambda(x, y, z), z, y) = x.$$

On the other hand

$$\lambda(\lambda(x, y, z), y, z) \sim x, \tag{2.7}$$

but it will not in general be equal to x; its discrepancy from being x is an expression of the *torsion* of λ. Even when $y \sim z$ (so x, y, z form a simplex), the left hand side of (2.7) need not be x. Rather, we may define the *torsion* of λ to be the law b which to any 2-simplex x, y, z associates $\lambda(\lambda(x, y, z), y, z)$. Then $b(x, y, z) = x$ for all simplices iff λ is symmetric, cf. [7] Proposition 2.3.1.

There is also a notion of *curvature* of λ: Let M be a manifold equipped with an affine connection λ. Note that $\lambda(x, y, -)$ takes any neighbour v of x into a neighbour of y ("*λ-parallel* transport of v from x to y). If now x, y, z form an infinitesimal 2-simplex, we may successively make three transports,

[3]Note the difference between this figure and the figure (1.3), in which y and z are assumed to be neighbours, and where the parallelogram *canonically* may be formed.

from x to y, then from y to z, and finally from z back to x. Thus the 2-simplex x, y, z gives rise to a permutation of the set of neighbour points v of x, and the connection is *flat* (curvature free) if all permutations arising this way are the identity permutation. More generally, the curvature r of λ is defined as the law, which to a infinitesimal 2-simplex x, y, z provides the permutation of the neighbours of x just described. (In the terminology of [7], r is a group-bundle valued combinatorial 2-form.)

We give two examples of affine connections on the unit sphere S.

Example 1. The unit sphere S sits inside Euclidean 3-space, $S \subseteq E$. Since E is in particular an affine space, we may for any three points x, y, z in it form $y - x + z \in E$. For x, y, z in S, the point $y - x + z$ will in general be outside S; if x, y, z are mutual neighbours, however, $y - x + z$ will be in S, cf. Remark at the end of Section 1. What if x, y, z only form an infinitesimal 2-whisker? Then we cannot expect $y - x + z$ to be in S, but we may "project it down" to S and define $\lambda(x, y, z) \in S$ to be the point, where the half line from the center of S to $y - x + z$ meets S. If S is the surface of the earth, this just means that $\lambda(x, y, z)$ which is vertically below $y - x + z$.

This affine connection λ is evidently symmetric in y and z, so is torsion free; it does, however, have curvature, which one can see from the "integrated version" (holonomy) of the connection, i.e. the parallel transport (according to λ) along curves on the sphere: for instance, transporting along a spherical triangle, whose sides each are 90^o, will provide a permutation of the neigbour points of any vertex, namely a rotation by 90^o (make a picture!). This connection is the Riemann- or Levi-Civita connection on sphere.

Example 2. (This example does not work on the whole sphere, only away from the two poles.) Given x, y and z with $x \sim z$ ($x \sim y$ is presently not relevant). Since x and z are quite close, we can uniquely describe z in a rectangular two-dimensional coordinate system at x with coordinate axes pointing East and North. Now take $\lambda(x, y, z)$ to be that point near y, which in the East-North coordinate system at y has same coordinates as the ones obtained for z in the coordinate system that we considered at x.

The description of this affine connection is asymmetric in y and z, and it is indeed easy to calculate that it has torsion ([7], Section 2.4). It has no curvature.

Connections constructed in a similar way also occur in materials science: for a crystalline substance, one may attach a coordinate system at each point, by using the crystalline structure to define directions (call them "East" and "North" and "Up", say). The torsion for a connection λ constructed from such coordinate systems is a measure for the imperfection of the crystal lattice (dislocations), – see [10], [4] and the references therein.

For calculations, and even for communication, one usually needs coordinates. Coordinate expressions for an affine connection are the "Christoffel symbols". Let λ be an affine connection on an n-dimensional manifold M; assume that we have identified some open subset of M with some open subset of R^n.

If $x \sim y$ in this subset, $y = x + \underline{d}$ for some $\underline{d} \in D(n)$, by definition of the neighbour relation. So a whisker (x, y, z) at x may be written $(x, x + \underline{d}_1, x + \underline{d}_2)$ with $(\underline{d}_1, \underline{d}_2) \in D(n) \times D(n)$. Define, for fixed x, the function $\Gamma : D(n) \times D(n) \to R^n$ by the rule

$$\Gamma(\underline{d}_1, \underline{d}_2) = \lambda(x, x + \underline{d}_1, x + \underline{d}_2) - \big(x + \underline{d}_1 + \underline{d}_2\big).$$

(To record the dependence of Γ on the point x, we may write $\Gamma(x; \underline{d}_1, \underline{d}_2)$.) Thus, Γ measures the discrepancy between λ and the canonical affine connection λ_0 in the affine space R^n. From the law $\lambda(x, x, z) = z$ follows $\Gamma(0, \underline{d}_2) = 0$, and similarly $\lambda(x, y, x) = y$ gives $\Gamma(\underline{d}_1, 0) = 0$. From the discussion prior to Theorem 1.1 follows that Γ extends uniquely to a bilinear map $\Gamma : R^n \times R^n \to R^n$, which is the *Christoffel symbol* for λ at the point x – relative to the coordinatization assumed around x. It is clear that λ is a *symmetric* (= torsion free) affine connection iff the Christoffel symbols $\Gamma(x; -, -)$ are symmetric bilinear maps, for all $x \in M$.

We can rephrase the relation between λ and the Christoffel symbols Γ (in a coordinatized situation) by

$$\lambda(x, x + \underline{d}_1, x + \underline{d}_2) = x + \underline{d}_1 + \underline{d}_2 + \Gamma(x; \underline{d}_1, \underline{d}_2) \tag{2.8}$$

where $\underline{d}_1$ and $\underline{d}_2$ are in $D(n)$.

3 Second order notions; midpoint formation

We describe a geometric notion of *midpoint formation structure* which can be used to construct torsion free affine connections.

Let $M_{(2)} \subseteq M \times M$ denote the set of pairs (x, u) of second order neighbours; $M_{(2)}$ is the "second neighbourhood of the diagonal", in analogy with the first neighbourhood $M_{(1)}$ described in Section 1. We have $M_{(1)} \subseteq M_{(2)}$.

Recall that for $x \sim_1 y$ in M, we have canonically the affine combination $\frac{1}{2}x + \frac{1}{2}y$, the midpoint formation for first order neighbours; it defines a map $M_{(1)} \to M$.

Definition 3.1 *A* midpoint formation structure *on M is a map $\mu : M_{(2)} \to M$ which extends the midpoint formation $M_{(1)} \to M$ for pairs of first order neighbour points.*

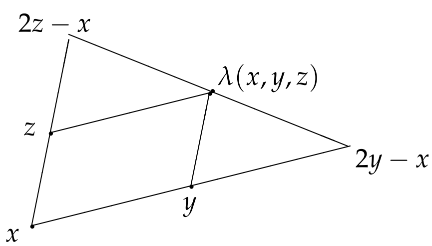
Thus, $\mu(x, u)$ is defined whenever $x \sim_2 u$; and $\mu(x, u) = \frac{1}{2}x + \frac{1}{2}u$ whenever $x \sim_1 u$. It can be proved that such μ is automatically symmetric, $\mu(x, u) = \mu(u, x)$. (The proof follows the same lines as the symmetry of λ in the proof of Theorem 4.2 below.) It can also be proved that $\mu(x, u) \sim_2 x$ and $\sim_2 u$. Beware that "$\mu(x, u)$ is midpoint of x and u, (where $x \sim_2 u$)" does not imply $x \sim_1 \mu(x, u) \sim_1 u$; in fact either of these will hold *only* if x is already $\sim_1 u$.

Theorem 3.2 *Any midpoint formation structure μ on M gives canonically rise to a symmetric affine connection λ on M.*

Proof. Given μ, and given an infinitesimal 2-whisker (x, y, z). Since $x \sim_1 y$, we may form the affine combination $2y - x$ (reflection of x in y), and it is still a first order neigbour of x. Similarly for $2z - x$. So $(2y - x) \sim_2 (2z - x)$, and so we may form $\mu(2y - x, 2z - x)$, and we define

$$\lambda(x, y, z) := \mu(2y - x, 2z - x).$$

The relevant picture is here:



It is symmetric in y and z, by the symmetry of μ. Also, if $y = x$, we get

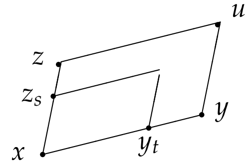
$$\lambda(x, x, z) = \mu(x, 2z - x) = \tfrac{1}{2}x + \tfrac{1}{2}(2z - x),$$

since $x \sim_1 2z - x$ and μ extends the canonical midpoint formation for first order neighbours. But this equals z, by evident equations for affine combinations. This proves the first equation in (2.1), and the second one then follows by symmetry.

Theorem 3.3 *Any symmetric affine connection λ on M gives canonically rise to a midpoint formation μ.*

This is less evident. First it requires an "interpolation" axiom, consistent with the KL axiomatics, namely that any $\underline{\delta} \in D_2(n)$ may be written $\underline{d}_1 + \underline{d}_2$ for suitable $\underline{d}_1$ and $\underline{d}_2$ in $D(n)$. This implies that for any manifold M, and any $x \sim_2 u$ in M, we may interpolate a y, in the sense that $x \sim_1 y \sim_1 u$. If M is equipped with an affine connection λ, we may therefore also for $x \sim_2 u$ find a λ-parallelogram (x, y, z, u) (take $z = \lambda(y, x, u)$).

Now given a λ-parallelogram (x, y, z, u), and given a scalar $t \in R$. Since $x \sim y$, we may form the affine combination $x_t := (1 - t)x + ty$, and similarly, given $s \in R$, we may form $z_s := (1 - s)x + sz$; both these points are $\sim x$, and so we may form the point $\lambda(x, (1 - t)x + ty, (1 - s)x + sz)$. See picture:



(3.1)

The picture suggests that for $s = t$, we could define $(1-t)x + tu$ as $\lambda(x, y_t, z_t)$ (this will certainly be correct in an affine space). Thus we have described a candidate for this affine combination of x and u, even though x and u are in general only *second* order neighbours. In particular, taking $t = \frac{1}{2}$, we would get a candidate for the midpoint $\mu(x, u)$. The problem with this definition is of course that it depends not only on x and u (and on t, of course), but also (seemingly) on the "interpolating" y and z. By working in coordinates, using the Christoffel symbol $\Gamma = \Gamma(x; -, -)$ for λ at x, we shall prove that this dependence is only apparent for symmetric λ. We may write $y = x = \underline{d}_1$ and $u = y + \underline{d}_2 = x + \underline{d}_1 + \underline{d}_2$. It is not in general true that $z = x + \underline{d}_2$, but rather

$$z = x + \underline{d}_2 - \Gamma(\underline{d}_1, \underline{d}_2). \tag{3.2}$$

For, since $\lambda(x, y, -)$ maps the the set of neighbours of x bijectively to the set of neighbours of y, it suffices to see that

$$\lambda(x, y, x + \underline{d}_2 - \Gamma(\underline{d}_1, \underline{d}_2)) = x + \underline{d}_1 + \underline{d}_2.$$

Calculating the left hand side with Γ yields (since $y - x = \underline{d}_1$)

$$x + \underline{d}_1 + \underline{d}_2 - \Gamma(\underline{d}_1, \underline{d}_2) + \Gamma(\underline{d}_1, \underline{d}_2 - \Gamma(\underline{d}_1, \underline{d}_2))$$
$$= x + \underline{d}_1 + \underline{d}_2 - \Gamma(\underline{d}_1, \underline{d}_2) + \Gamma(\underline{d}_1, \underline{d}_2) - \Gamma(\underline{d}_1, \Gamma(\underline{d}_1, \underline{d}_2)),$$

using bilinearity of Γ, so we end up with $x + \underline{d}_1 + \underline{d}_2 - \Gamma(\underline{d}_1, \Gamma(\underline{d}_1, \underline{d}_2))$. Now the last term vanishes: $\underline{d}_1$ here occurs in a quadratic fashion, and $\underline{d}_1 \in D(n)$; more precisely, since $\Gamma(-, \Gamma(-, r))$ is bilinear, it vanishes if a vector from $D(n)$ is put in both the empty slots. Thus we finally end up with $x + \underline{d}_1 + \underline{d}_2$, which is u.

Substituting the expression (3.2) for z gives

$$\lambda(x, (1-t)x + ty, (1-t)x - tz) = \lambda(x, x + t\underline{d}_1, x + t(\underline{d}_2 - \Gamma(\underline{d}_1, \underline{d}_2));$$

let us calculate this using Γ; we get that it equals

$$x + t\underline{d}_1 + t\underline{d}_2 - t\Gamma(\underline{d}_1, \underline{d}_2) + \Gamma(t\underline{d}_1, t\underline{d}_2 - t\Gamma(\underline{d}_1, \underline{d}_2)).$$

As before, the "nested" appearance of Γ vanishes since $\underline{d}_1 \in D(n)$, and we are left with

$$x + t\underline{d}_1 + t\underline{d}_2 - t\Gamma(\underline{d}_1, \underline{d}_2) + \Gamma(t\underline{d}_1, t\underline{d}_2). \tag{3.3}$$

Now we use that λ was assumed symmetric, so that Γ is a symmetric bilinear form. Then by (1.1), $\Gamma(\underline{d}_1, \underline{d}_2) = \frac{1}{2}\Gamma(\underline{d}_1 + \underline{d}_2)$, and similarly for $\Gamma(t\underline{d}_1, t\underline{d}_2)$. Thus the expression (3.3) only depends on x and $\underline{d}_1 + \underline{d}_2$, that is, on x and u only. This proves the desired independence of $\underline{d}_1$ and $\underline{d}_2$ individually.

Let us show that one gets the symmetric affine connection λ back from the midpoint formation μ to which it gives rise. Let $\tilde{\lambda}$ be the affine connection constructed from μ, so for a whisker x, y, z at x, use x as interpolating point between $2y - x$ and $2z - x$; so

$$\tilde{\lambda}(x, y, z) = \mu(2y - x, 2z - x) = \lambda(x, \tfrac{1}{2}x + \tfrac{1}{2}(2y - x), \tfrac{1}{2}x + \tfrac{1}{2}(2z - x)),$$

but $\frac{1}{2}x + \frac{1}{2}(2y - x) = y$ and $\frac{1}{2}x + \frac{1}{2}(2z - x) = z$, by purely affine calculations; so we get $\lambda(x, y, z)$ back.

In [5], it is shown how a Riemannian metric geometrically gives rise to a midpoint formation (out of which, in turn, the Levi-Civita affine connection may be constructed, by the process given by the Theorem).

4 Point reflection (geodesic symmetry)

For a pair of first order neighbours, $x \sim_1 y$, on a manifold M, one has canonically the point reflection of y in x, namely the affine combination $2x - y$; it thus defines a map $M_{(1)} \to M$.

Definition 4.1 *A* point reflection *on a manifold M is a map $* : M_{(2)} \to M$, which extends the point reflection $M_{(1)} \to M$ for pairs of first order neighbour points.*

We write $x * y$ for the values of this map, "$x * y$ is the reflection of y in x". It should be thought of as an infinitesimal geodesic symmetry.

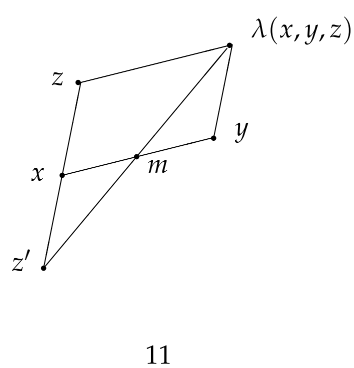
Problem: One would like to investigate the conditions when a point reflection structure satisfies the equation,

$$(x * y) * (x * z) = x * (y * z)$$

for "symmetric spaces in the sense of Loos" (cf. [9]), whenever x, y, z are mutual 2-neighbours. The equation does not immediately make sense in our context, since we cannot assert that $x \sim_2 (y * z)$. However, it can be be proved that $x \sim_3 (y * z)$, and one can prove that $* : M_{(2)} \to M$ extends uniquely to a map $* : M_{(3)} \to M$ satisfying $x * (x * u) = u$ for all $u \sim_3 x$, and with this extended $*$, the expression $x * (y * z)$ makes sense.

Theorem 4.2 *Any point reflection structure $*$ on M gives canonically rise to a symmetric affine connection λ on M.*

Proof. Given $*$, and given an infinitesimal 2-whisker (x, y, z). Since $x \sim_1 z$, we may form the affine combination $z' := 2x - z$ (reflection of z in x), and it is still a first order neigbour of x. Also, we may form $m := \frac{1}{2}x + \frac{1}{2}y$, likewise a first order neighbour of x, so $m \sim_2 z'$, and therefore we may form $m * z'$. We define $\lambda(x, y, z)$ to be this point. The relevant picture is here:



(4.1)

It is easy to see that $\lambda(x, x, z) = z$ and $\lambda(x, y, x) = y$, so λ is indeed an affine connection. The fact that is symmetric is not immediate from the definition; we prove it by considering an coordinatized situation, so identify an open neighbourhood of x, y, z with an open subset of R^n, so $y = x + \underline{d}_1$, $z = x + \underline{d}_2$ for $\underline{d}_1$ and $\underline{d}_2$ in $D(n)$. Now, since $*$ extends the canonical point reflection for first order neighbours, it is by KL of the form $x * u = 2x - u + Q_x(u - x)$ where $Q_x : R^n \to R^n$ is a quadratic map, i.e. $Q_x(v) = \Gamma(x; v, v)$ for some (unique) symmetric bilinear map $\Gamma(x; -, -) : R^n \times R^n \to R^n$. The symmetry of the constructed λ will now follow by proving that the Christoffel symbols for λ are the Γs. Here is the calculation. Note that $m = x + \frac{1}{2}\underline{d}_1$, and that $z' = x - \underline{d}_2$- Also, $z' - m = (x - \underline{d}_2) - m = -\frac{1}{2}\underline{d}_1 - \underline{d}_2$, so

$$\lambda(x, y, z) = m * (2x - z) = 2m - (x - \underline{d}_2) + \Gamma(x + \tfrac{1}{2}\underline{d}_1; -\tfrac{1}{2}\underline{d}_1 - \underline{d}_2, -\tfrac{1}{2}\underline{d}_1 - \underline{d}_2).$$

Now the terms before the Γ term yield by simple additive calculation $x + \underline{d}_1 + \underline{d}_2 = y - x + z$. The Γ term may be rewritten by (1.1) using symmetry and bilinearity of $\Gamma(x + \frac{1}{2}\underline{d}_1; -, -)$; it gives $\Gamma(x + \frac{1}{2}\underline{d}_1; \underline{d}_1, \underline{d}_2)$. Here, $\underline{d}_1$ appears as an argument in a linear position (after the semicolon), and then a Taylor expansion argument gives that the $x + \frac{1}{2}\underline{d}_1$ in front of the semicolon may be replaced by x. (This is because $\underline{d}_1$ is in $D(n)$; for a precise formulation of this "Taylor principle", see [7] (1.4.2).) Putting things together, we thus have

$$\lambda(x, y, z) = y - x + z + \Gamma(x; \underline{d}_1, \underline{d}_2),$$

proving that $\Gamma(x; -, -)$ is indeed the Christoffel symbol at x for λ. This proves the symmetry.

Theorem 4.3 *Any symmetric affine connection λ on M gives canonically rise to a point reflection structure (geodesic symmetry) $*$.*

Just as Theorem 4.3, the construction depends on interpolating a λ-parallelogram x, y, z, u between x and u for $x \sim_2 u$ (see figure (3.1): one then puts

$$x * u := \lambda(x, 2x - y, 2x - z),$$

and the proof that this is independent of the choice of the interpolation is as for Theorem 4.3. Again, the constructions are inverse of each other. We may summarize the results of the last two sections in

Theorem 4.4 *On any manifold M, there are canonical bijective correspondences between the following three kinds of geometric structure:*

- *symmetric affine connections λ*
- *midpoint formations $\mu : M_{(2)} \to M$*
- *point reflection structures $* : M_{(2)} \to M$.*

Any map $f : M' \to M$ between manifolds preserves the neighbour relations $\sim_1$ and $\sim_2$ (Theorem 1.1); therefore if M' and M are equipped with affine connections λ' and λ, respectively, it makes sense to ask whether f is connection preserving

$$f(\lambda'(x,y,z)) = \lambda(f(x), f(y), f(z)),$$

for any $x \sim_1 y, x \sim_1 z$ in M'. If M' and M are equipped with midpoint formation structures μ' and μ, respectively, it makes sense to ask whether f preserves midpoint formation,

$$f(\mu'(x,u)) = \mu(f(x), f(u))$$

for pairs of second order neighbours $x \sim_2 u$ in M'. Similarly if M' and M are equipped with point reflection structures.

Symmetric affine connections, midpoint formation structures, and point reflection structures correspond, by Theorem 4.4; the correspondences are constructed using affine combinations of first order neighbour points, preserved by any f by Theorem 1.2. Therefore it follows that the assertions "f is connection preserving ", "f is midpoint preserving", and "f is point-reflection preserving" are equivalent (for symmetric affine connections, and the corresponding μ and $*$). Such maps $f : M' \to M$ deserve the name *geodesic* maps.

In particular, the number line R (being an affine space) carries canonical structures λ', μ', $*$ (which correspond to each other), namely

$$\lambda'(x,y,z) = y - x + z, \quad \mu'(x,u) = \tfrac{1}{2}x + \tfrac{1}{2}u, \quad x * u = 2x - u,$$

(in fact without any restrictions like $x \sim_2 u$). A map $f : R \to M$ into a manifold M equipped with a symmetric affine connection λ deserves the name *(parametrized) geodesic curve* if f is geodesic in the general sense just defined. This is equivalent to f preserving midpoint formation or point reflection. In particular, f is geodesic if $f(x+2d) = f(x+d) * f(x)$ for $d \in D_2$. A subset $C \subseteq M$ deserves the name *unparametrized curve* if there is an embedding $f : R \to M$ mapping bijectively R onto C. In this case, C deserves the name *geodesic* if $x \sim_2 u$ in C implies $x * u \in C$, in other words, if C is stable under point reflection.

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