Monads for which Structures are Adjoint to Units*

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Introduction

We present here the equational two-dimensional categorical algebra which describes the process of freely completing a category under some class of limits or colimits. It is crystallized out of the authors 1967 dissertation [6] (revised form [7]). I presented a purely equational aspect of that already in 1973 [9], [10] , and the present note is in some sense identical to that, but with some further equational consequences added. The kind of structure introduced in [9], [10] has in the meantime been applied and improved by various authors, notably Street [12] [13], who used the term "monads with the Kock property" and "KZ doctrine" ("Kock-Zöberlein"). Some of Street's improvements are incorporated in our results below. We shall use the term 2-doctrine, for the reason given in Section 2 below.

Thus, a 2-doctrine **T** is an endofunctor T on the 2-category of categories, which is equipped with $y : I \to T$ and $m : TT \to T$, just as monads; but the monad laws hold only up to isomorphisms, and these isomorphisms, as well as the further two-dimensional structure, required for the adjointness alluded to in the title, arise out of a single natural transformation

$$\lambda : yT \Rightarrow Ty : T \to TT,$$

assumed to satisfy certain equations. There is also an equational notion of 'algebra' or 'module', for **T**; such a thing turns out to be equivalent to a map

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$AT \to A$ having Ay as a right adjoint section (a right adjoint left inverse - we are composing from left to right), and also equivalent to a lax version of algebra in the monad theoretic sense. These equivalences are summarized and made explicit in the main Theorem, Theorem 7 below.

An important example of such 2-doctrine is the construction Fam which to a category C associates the category Fam(C) of families of objects in C; it is the completion of C under coproducts. It has been applied in [3], the study of which gave me the impetus to have [9] completed. (The finitary case of Fam was given as example in [9].) A module structure $a : AT \to A$ in this example amounts to a category A equipped with a law a which to a family of objects of A associates a coproduct for it. Also the ind-completion of categories, as considered in [1], and in [4] is an example. We comment on the examples in the last section.

We shall work, as in [9], in the generality that we consider a strictly monoidal 2-category $\mathcal{C}$. If $\mathcal{C}$ is the category [Cat,Cat] of 2-functors from the category of categories to itself, with composition as monoidal structure, we have the set-up appropriate for the above examples (monoids in $\mathcal{C}$ being the same thing as (2-) monads on Cat). This generality has, in the present paper, just the purpose of providing a notational and conceptual simplification; but I am convinced that the generality, when suitably generalized to bicategories, will have mathematical applications as well; for instance, the structure considered in Proposition 8 of [12] cries out to be understood as a 2-doctrine in our sense.

We shall use standard notation, terminology, and notions from 2-category theory, as in [5], except that we compose from left to right, both for the vertical composition . and for the horizontal composition $*$. The monoidal structure is denoted by $\otimes$, and we agree that $\otimes$ binds more strongly than $*$, thus $f \otimes g * h$ denotes $(f \otimes g) * h$. If A is an object, A also denotes the identity 1-cell on A, as well as the identity 2-cell of that again, and similarly for the identity 2-cell of a 1-cell. Units and counits for adjoint 1-cells (or arrows) are also called front- and back- adjunctions, respectively. We remind the reader that a 2-cell between two parallel right adjoint arrows, has a *mate* which is a 2-cell, going in the opposite direction, between the left adjoints, and it is constructed explicitely out of the given front- and back adjunctions. The mate of an invertible 2-cell is again invertible.

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1 The equational theory

We consider a monoidal 2-category $\mathcal{C}$ with strictly associative $\otimes$, and strictly unitary I; the latter will be omitted from notation when possible.

Definition 1 *A* **2-doctrine T** *on $\mathcal{C}$ consists of an object T of $\mathcal{C}$, arrows $y : I \to T$, $m : T \otimes T \to T$, and a 2-cell $\lambda : y \otimes T \Rightarrow T \otimes y$, satisfying*

T0 *y is a two-sided unit for m*
T1 $y * \lambda = y * T \otimes y$ $(= y * y \otimes T)$;
$\lambda * m = T$
T2 $T \otimes \lambda * m \otimes T * m = m.$

Neither T1 nor T2 introduces any equations between 1-cells. For instance for T2, the domain 1-cell of the left hand side is

$$T \otimes y \otimes T * m \otimes T * m = ((T \otimes y * m) \otimes T) * m = m$$

since $T \otimes y * m = T$; and the codomain is

$$T \otimes T \otimes y * m \otimes T * m = m * T \otimes y * m = m,$$

using bifunctorality of $\otimes$ for the first equality.

A condition T2* which is a kind of mirror image of T2 will be considered in Section 2, but will not be part of the axiomatics.

Let **T** be a 2-doctrine, as above.

Definition 2 *A* module *A for* **T** *consists of an object A of $\mathcal{C}$ and an arrow $a : A \otimes T \to A$ satisfying*

M0 *$A \otimes y$ is a left inverse for a, $A \otimes y * a = A$*
M1 $A \otimes \lambda * a \otimes T * a = a.$

The same calculation as above shows that M1 does not introduce any equation between 1-cells.

Proposition 1 *Suppose $a : A \otimes T \to A$ satisfies M0. Then M1 holds if and only if a is left adjoint to $A \otimes y$ by virtue of $A \otimes \lambda * a \otimes T$ as front adjunction (and A as back adjunction).*

Note that the domain of $A \otimes \lambda * a \otimes T$ is $A \otimes T$ by virtue of M0, and the codomain is $a * A \otimes y$ by virtue of bifunctorality of $\otimes$.

Proof. One easily sees that M1 is exactly the one of the two triangle equations for the adjointness; the other triangle equation follows from T1, so holds in any case.

Corollary 2 *For any object B, the 1-cell $B \otimes m : B \otimes T \otimes T \to B \otimes T$ is left adjoint for $B \otimes T \otimes y$.*

Proof. It makes $B \otimes T$ into a module, by virtue of T0 and T2.

Let $(A, a), (B, b)$ be modules for a 2-doctrine $\mathbf{T}$ as above, and let $f : A \to B$ be an arbitrary 1-cell. We construct a 2-cell

$$\phi : f \otimes T * b \Rightarrow a * f : A \otimes T \to B$$

by the formula

$$\phi := A \otimes \lambda * a \otimes T * f \otimes T * b.$$

We call it the *canonical* 2-cell associated with f. To see that its codomain is really $a * f$,, one utilizes

$$A \otimes y * f \otimes T * b = f * B \otimes y * b = f,$$

using bifunctorality of $\otimes$ and $B \otimes y * b = B$.

Let $(A, a), (B, b), f$ and ϕ be as above. Recalling that a and b typically could be assignment of colimit diagrams of a certain type, it is not surprising that a left adjoint arrow f should preserve these assignments, up to canonical isomorphism:

Theorem 3 *If f is a left adjoint arrow, ϕ is invertible. More precisely, let g be a right adjoint of f. Then ϕ is mate of the invertible identity 2-cell $g * A \otimes y \Rightarrow B \otimes y * g \otimes T$.*

Proof. Let $f \dashv g$ by virtue of front- and back adjunctions η and ϵ. Let $g_1 = g * A \otimes y, g_2 = B \otimes y * g \otimes T$, with left adjoints $f_1 = a * f, f_2 = f \otimes T * b$, respectively. The formula for mating requires us to utilize the front adjunction η_1 for f_1, g_1, and the back adjunction ϵ_2 for f_2, g_2. The standard recipe for constructing the front adjunction for a composite adjoint gives us

$\eta_1 = \overline{\eta}.(a * \eta * A \otimes y)$, where $\overline{\eta}$ is the front adjunction for $a \dashv A \otimes y$, thus $\overline{\eta} = A \otimes \lambda * a \otimes T$; ϵ_2 is similarly constructed, but easier since the back adjunction for $b \dashv B \otimes y$ is an identity, so $\epsilon_2 = B \otimes y * \epsilon \otimes T * b$. The general mating formula constructs the mate for $\alpha : g_1 \Rightarrow g_2$ as

$$(\eta_1 * f_2).(f_1 * \alpha * f_2).(f_1 * \epsilon_2)$$

but in the present case the middle dot-factor disappears since the α is now an identity 2-cell. Inserting the formulae for $\eta_1, \epsilon_2, f_1, f_2, g_1, g_2$ in the mating formula then yields

$$(A \otimes \lambda * a \otimes T * f \otimes T * b).(a * \eta * A \otimes y * f \otimes T * b).(a * f * B \otimes y * \epsilon \otimes T * b).$$

To see that this is our ϕ, we just have to see that the two last dot-factors compose to an identity 2-cell, since the first dot-factor already equals ϕ. By the interchange law of the $*$- and the dot-composition, we can collect the two $a * -$ in the front, and the two $- * b$ in the end, and then it suffices to see that $(\eta * A \otimes y * f \otimes T).(f * B \otimes y * \epsilon \otimes T)$ is an identity 2-cell. But rewriting each of the two dot-factors here using bifunctorality of $\otimes$ gives $(\eta * f * B \otimes y).(f * \epsilon * B \otimes y)$; if we move the $B \otimes y$ outside on the right using the interchange law for the $*$ and dot composites, we see that we have an identity 2-cell, by virtue of the triangular equation for η and ϵ. This proves the Theorem.

Note that $A \otimes T$ carries a distinguished module structure, namely $A \otimes m$. We leave to the reader to make explicit in which sense this is a *free* module on A. If now a is a module structure on A, it is a left adjoint arrow, by Proposition 1, and so the the Theorem gives the following

Corollary 4 *Let a provide A with module structure. Then the canonical 2-cell associated to the arrow $a : A \otimes T \to A$ is invertible.*

Canonical 2-cells are recognizable:

Proposition 5 *Let $(A, a), (B, b)$ be modules, and let $f : A \to B$ be an arrow. Then a 2-cell*

$$\phi : f \otimes T * b \Rightarrow a * f$$

is annihilated by $A \otimes y * -$, (i.e. $A \otimes y * \phi$ is an identity 2-cell), if and only if ϕ is the canonical 2-cell associated with f.

Proof. That canonical 2-cells are thus annihilated is immediate from axiom T1. Conversely, assume the annihilation condition. We calculate $A \otimes \lambda * \phi \otimes T * b$ in two ways, using bifunctorality of $\otimes$. On the one hand

$$A \otimes \lambda * \phi \otimes T * b = (A \otimes y \otimes T * \phi \otimes T * b).(A \otimes \lambda * a \otimes T * f \otimes T * b);$$

the first dot-factor here is an identity, by the annihilation assumption, so we are left with the second factor, which is the canonical 2-cell associated with f. On the other hand

$$\begin{aligned} & A \otimes \lambda * \phi \otimes T * b \\ & \quad = (A \otimes \lambda * f \otimes T \otimes T * b \otimes T * b).(A \otimes T \otimes y * \phi \otimes T * b) \\ & \quad = (A \otimes \lambda * f \otimes T \otimes T * b \otimes T * b).(\phi * B \otimes y * b) \\ & \quad = (A \otimes \lambda * f \otimes T \otimes T * b \otimes T * b).\phi \\ & \quad = (f \otimes T * B \otimes \lambda * b \otimes T * b).\phi, \end{aligned}$$

the last by naturality of λ . But the first dot-factor here is an identity 2-cell by virtue of M1, so we are left with ϕ, and this proves the Proposition.

For a 2-doctrine,we know by Proposition 1 that strucures are adjoint to units, in fact are reflection left adjoints in the sense that the back adjunction is an identity 2-cell. The following result is a converse:

Proposition 6 *Let $a : A \otimes T \to A$ be a reflection left adjoint for $A \otimes y$, with front adjunction η, say, so $\eta * a = a$. Then $\eta = A \otimes \lambda * a \otimes T$, and a provides A with module structure.*

Proof. We calculate $\eta * A \otimes \lambda * a \otimes T$ in two ways, using bifunctorality of $\otimes$ .On the one hand,

$$\begin{aligned} & \eta * A \otimes \lambda * a \otimes T \\ & \quad = (A \otimes \lambda * a \otimes T).(\eta * A \otimes T \otimes y * a \otimes T \\ & \quad = (A \otimes \lambda * a \otimes T).(\eta * a * A \otimes y) \\ & \quad = A \otimes \lambda * a \otimes T, \end{aligned}$$

since $\eta * a$ is an identity 2-cell by assumption. On the other hand,

$$\eta * A \otimes \lambda * a \otimes T$$

$$= (\eta * A \otimes y \otimes T * a \otimes T).(a * A \otimes y * A \otimes \lambda * a \otimes T)$$
$$= \eta.(a * A \otimes y * A \otimes \lambda * a \otimes T)$$
$$= \eta,$$

since $A \otimes y * a = A$, and since $A \otimes y * A \otimes \lambda$ is an identity 2-cell by T1.This proves $\eta = A \otimes \lambda * a \otimes T$; applying $*a$ to this equation, and using $\eta * a = a$, we get $A \otimes \lambda * a \otimes T * a = a$,which is M1. This proves the Propostion.

We now consider 2-doctrines and their modules from the aspect of monads and their algebras, or rather, in the present setting, from the viewpoint of monoids and their actions. The 'multiplication' m on T, and the action a of T on a module (A, a) are not assumed associative, but they are associative up to isomorphisms (invertible 2-cells), namely the canonical ones; this follows immediately from Corollary 4. Furthermore, these isomorphism satisfy a number of coherence equations; these are proved by observing that the isomorphisms in questions are mates of identity 2-cells, which evidently are coherent. We refer to [12]. There is also an independent notion of 'action-of-T which is associative and unitary up to coherent isomorphisms', cf. loc.cit., where they are called *pseudo-algebras* for the doctrine. There is also, cf. loc.cit., an even weaker notion of *lax algebras* where the 2-cells in question are not even assumed invertible.

We shall consider here a seemingly weaker notion of pseudo- and lax 'algebra' ('module', in our termiology). It follows, however, from the Theorem below, and the coherence results for modules in the sense of Definition 2 that it is not really weaker than Street's notion (it is a little more special, since we consider what he calls the *normalized* case).

Definition 3 *Let* $\mathbf{T} = (T, y, m, \lambda)$ *be a 2-doctrine. A* lax module *for it consists of* A, a, α*, where* $a : A \otimes T \to A$ *and* $\alpha : a \otimes T * a \Rightarrow A \otimes m * a$*, such that* $A \otimes y * a$ *is an identy 2-cell, and* α *satisfies the coherence conditions that* $A \otimes T \otimes y * \alpha$ *and* $A \otimes y \otimes T * \alpha$ *are identity 2-cells. If* α *is invertible, we say* pseudo module *instead of lax module.*

We can now summarize most of our results in the following

Theorem 7 *Let* $\mathbf{T} = (T, y, m, \lambda)$ *be a 2-doctrine and* A *an object equipped with* $a : A \otimes T \to A$*, with* $A \otimes y * a$ *an identity 2-cell. Then the following conditions are equivalent:*

1. a makes A into a lax module, for suitable α
2. a makes A into a pseudo module, for suitable α
3. a is a reflection left adjoint for $A \otimes y$, for suitable η
4. a makes A into a module (in the sense of Definition 2).

In case the conditions hold, the α assumed to exist in 1. and 2. is unique, in fact can be expressed in terms of λ,

$$\alpha = A \otimes T \otimes \lambda * A \otimes m \otimes T * a \otimes T * a,$$

and the front adjunction η assumed to exist in 3. is unique, in fact can be expressed in terms of λ,

$$\eta = A \otimes \lambda * a \otimes T.$$

Proof. The equivalence of 3. and 4., and the uniqueness of (and expression for) η is immediate from Propositions 1 and 6 above. Assume 4. From the explicit formula for α and Axiom T1 it immediately follows that α is annihilated by $A \otimes T \otimes y$. For the other coherence condition, we calculate

$$\begin{aligned} &A \otimes y \otimes T * \alpha \\ &\quad = A \otimes y \otimes T * A \otimes T \otimes \lambda * A \otimes m \otimes T * a \otimes T * a \\ &\quad = A \otimes \lambda' A \otimes y \otimes T \otimes T * A \otimes m \otimes T * a \otimes T * a \\ &\quad = A \otimes \lambda * a \otimes T * a, \end{aligned}$$

which is an identity 2-cell by M1. Thus (a, α) provides a lax algebra structure on A. Utilizing that the explicit α is in fact the canonical 2-cell associated to a,we get from Corollary 4 that it is indeed an invertible 2-cell, so provides not only lax, but pseudo algebra structure. This proves 1. and 2. Conversely assume 1. or 2. We prove that α is in fact given by the explicit formula (which at the same time proves the uniqueness assertion). This we do by calculating $A \otimes \lambda * \alpha$ in two ways (cf also the calculation in [12] p.111). On the one hand, it equals

$$(A \otimes \lambda * a \otimes T \otimes a).(A \otimes T \otimes y * \alpha) = A \otimes \lambda * a \otimes T \otimes a,$$

since the second dot-factor is an identity, by one of the equations for lax modules. On the other hand, it equals

$$(A \otimes y \otimes T * \alpha).(A \otimes \lambda * A \otimes m * a),$$

and both dot-factors here are identity 2-cells, the first by a lax-module law, the second by T1. So we conclude that $A \otimes \lambda * a \otimes T * a$ is an identity 2-cell, but this is the module law M1. So (A, a) is a module for $\mathbf{T}$. This proves the Theorem.

2 Other aspects of 2-doctrines

Besides the axiom T2

$$T \otimes \lambda * m \otimes T * m = m$$

and the immediate consequences of axiom T1

$$T \otimes \lambda * T \otimes m * m = m \tag{1}$$

$$\lambda \otimes T * m \otimes T * m = m, \tag{2}$$

one may consider the following "mirror image" T2* of T2

$$\lambda \otimes T * T \otimes m * m = m. \tag{3}$$

Proposition 8 *If m is a strictly associative multiplication on T, then T2* holds.*

Proof. In (2) (which holds), just replace $m \otimes T * m$ by $T \otimes m * m$, by associativity, and we have (3), ie. T2*.

Proposition 9 *Assume that* $\mathbf{T}$ *is a 2-doctrine for which furthermore T2* holds. Then $y \otimes T \vdash m$ by virtue of $\lambda \otimes T * T \otimes m$ as back adjunction ϵ and the identity 2-cell on T as front adjunction.*

Proof. The triangular equations for adjointness reduce to $\epsilon * m = m$ and $y \otimes T * \epsilon = y \otimes T$. With the explicit ϵ given, the first of these conditions is T2*, and the second follows from axiom T1.

I have not been able to prove T2* without he assumption of strict associativity of m. But since m is in any case associative up to isomorphism, by Corollary 4, one can prove that the left hand side of T2* is an invertible 2-cell.

The cocompletion 2-doctrines **T** considered in [6] and [7] (and reported on in [11]) are, with hook and crook, made strictly associative. (For instance, for the 2-doctrine Fam, as considered in the introduction, this is achieved by letting the objects of Fam(C) of families in the category C consist of such families of objects in C, whose index set is an ordinal number; and the strict associativity of ordinal coproducts ("ordinal sums") leads to the strict associativity of the doctrine.)

Since T2* always holds up to isomorphism, it is clear that if the monoidal 2-category $\mathcal{C}$ in which the 2-doctrine **T** lives has partially ordered sets for its hom-categories, then T2* holds, so that again Proposition 9 applies. In particular, let $\mathcal{C}$ be the category of endo-(2-)functors on the category Ord of partially ordered sets (posets). All (co-)completion constructions on posets known to he author are 2-doctrines in this $\mathcal{C}$. In particular, this applies to the construction Idl which to a poset A associates the ordered set Idl(A) of ideals in A (=lower sets which are upward directed, cf. e.g. [2], VII.2, or [8]); Idl(A) is the free completion of A under directed joins, and a left adjoint a: Idl(A) $\rightarrow A$ for the natural embedding $\downarrow (-) : A \rightarrow$Idl($A$) assigns to a directed lower set its join (and exists iff A has all directed joins). Now a well known succinct way of stating the notion of *continuous poset* is to say that it is a poset with directed joins, in which formation of directed joins a: Idl(A) $\rightarrow A$ in turn has a left adjoint.

From Proposition 9, we therefore derive the following (well known, cf [4]) fact as a Corollary:

Corollary 10 *Any poset of the form* Idl(A) *(A any poset) is a continuous poset.*

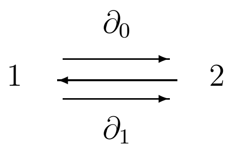
We shall finally consider the "simplicial" aspect of 2-doctrines. This is based on viewpoints of Lawvere and Street, and will justify the term "2-doctrine". Consider the category $\mathbf{\Delta}$ of finite ordinals 0,1,2,... and their order preserving maps. It is a (strict) monoidal category, using ordinal sum as $\otimes$. Lawvere observed in [11] that a monad on a category C can be considered as a strict homomorphism of monoidal categories $\mathbf{\Delta} \rightarrow [C, C]$, (the codomain category having composition as monoidal structure); and he defined an (equational) *doctrine* to be such monad in the case where C =Cat, and also analyzed the algebras for monads or doctrines in terms of the monad 1+(-) on $\mathbf{\Delta}$.

This was extended, or specialized, by Street in [13]. He first observes that $\mathbf{\Delta}$ is in fact a 2-category (being a category of posets and order preserving maps), and that its basic 1-cells, the famous face and degeneracy operators ∂_i and σ_j are connected by a string of adjointness relations

$$\partial_0 \vdash \sigma_0 \vdash \partial_1 \vdash \cdots \vdash \partial_n. \tag{4}$$

He then proceeds to analyze doctrines $\mathbf{T}$: $\mathbf{\Delta}\rightarrow [C, C]$, (where C is a 2-category) which take the 2-dimensional structure into account. It is reasonable, then, to call such a *2-doctrine*. Street then further essentially observes that these are the "KZ doctrines" except that his description only involves $\mathbf{\Delta}^+$, the last-element preserving maps between non-zero ordinals, presumably because he does not include T2*. (So to justify our terminology of 2-doctrines completely, we should have included T2* among the axioms.)

From this perspective, the basic data of a 2-doctrine $\mathbf{T}$ is the image under $\mathbf{T}$ of the arrows in $\mathbf{\Delta}$



where ∂_0 corresponds to $y \otimes T$, ∂_1 to $T \otimes y$, the arrow coming back (which is σ_0) to m, and the inequality $\partial_0 \leq \partial_1$ to λ. The adjointnesses of (4) (for n=1) correspond to those proved in Propositions 9 and 1.

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