Monads for which Structures are Adjoint to Units

Anders Kock

We present here the purely equational 2-dimensional categorical algebra underlying the process of freely completing categories under a given suitable class of colimits. Examples are the Ind-completion (completion under filtered colimits) studied in SGA 4 [2], or the Fam-construction, cf e.g.[16] or [9], which completes a category under coproducts. Also, the exact completion of a left exact category studied in [5], and the completion procedures for partially ordered sets with respect to "subset systems" (cf [25] and the references therein) fall under our theory. General information about free cocompletion of categories may, at various levels of generality, be found in [14], [28], [11], [3], and [1].

A common feature is that the category $F(\mathbf{C})$, which freely completes $\mathbf{C}$, for its objects has diagrams (functors) $\mathbf{I} \to \mathbf{C}$ in $\mathbf{C}$ with index category $\mathbf{I}$ any category of the prescribed class, thus $\mathbf{I}$ filtered, for the Ind-completion case, and discrete for the Fam-case. The morphisms in $F(\mathbf{C})$ are usually more complicated to describe. In any case, to say that a category $\mathbf{C}$ has colimits of the appropriate kind can be expressed by saying that there is a functor $\lim_{\rightarrow} : F(\mathbf{C}) \to \mathbf{C}$ which to an object of $F(\mathbf{C})$ - i.e. to a diagram $D : \mathbf{I} \to \mathbf{C}$ - associates a colimit $\lim_{\rightarrow}(D)$ for it. The universal property defining the notion of colimit can be expressed by saying that $\lim_{\rightarrow}$ is a left adjoint to the canonical functor $y_{\mathbf{C}} : \mathbf{C} \to F(\mathbf{C})$ (the one assigning to an object $C \in \mathbf{C}$ the singleton diagram $C : 1 \to \mathbf{C}$).

On the other hand, the statement that $F(\mathbf{C})$ is the *free* (co-)completion expresses a universal property pointing in the direction of adjoint functors and monads; thus, F is "up to isomorphism" a left adjoint for the forgetful functor $U : S\underline{Cat} \to \underline{Cat}$ (where $\underline{Cat}$ is the (2-)category of (locally small) categories, and $S\underline{Cat}$ is the category of categories admitting (and functors preserving, up to isomorphism) the appropriate kind of colimits. (Similarly, the exact completion of [5] is a left adjoint, up to isomorphism, for the the forgetful functor from the category of exact categories to that of left exact categories.) The composite $T = U \circ F$ becomes a monad-up-to-isomorphism, with the canonical "singleton" functors $y_{\mathbf{C}}$ as units; and, as we shall see (Proposition 3.3 below), the category of (Eilenberg-Moore-) algebras-up-to-isomorphism is equivalent to $S\underline{Cat}$, with $c : T(\mathbf{C}) \to \mathbf{C}$ being a T-structure (up to isomorphism) precisely when it is a (retraction-) left adjoint for $y_{\mathbf{C}} : \mathbf{C} \to T(\mathbf{C})$.

From this fact stems the title of the paper. Our thesis is, that all the special properties and data for the "monad" and its "algebras" (e.g. the various isomorphisms and their coherence) all stem from one single piece of 2-dimensional data, λ (and three equations for it). This data λ is a 2-cell (natural in $\mathbf{C}$):

$$T(\mathbf{C}) \underset{y_{T(\mathbf{C})}}{\overset{T(y_{\mathbf{C}})}{\underset{\Downarrow \lambda_{\mathbf{C}}}{\rightrightarrows}}} T^2(\mathbf{C})$$

constructed out of the front adjunction for the adjointness which defines the colimit assignment $m_{\mathbf{C}}$ for $T(\mathbf{C})$.

The four pieces of data T, y, m, λ, and the equations, lead to a purely formal calculus, which makes sense in the generality that T could be any endo(2-)functor on any 2-category $\mathcal{C}$ ($\underline{Cat}$, $\underline{Lex}$ or $\underline{Posets}$, say); one could even view T as an object of the strict monoidal 2-category $(\mathcal{M}, \otimes)$ of endo-2-functors $\mathcal{C} \to \mathcal{C}$ (with composition of functors as $\otimes$) and develop the whole calculus for objects T of such a category (as we did in some of the preliminary versions [16, 19]). However, we feel that we stick closer to the intuition and applications of the theory if we phrase it in terms of endofunctors T.

It should be mentioned that if the hom-categories of the 2-category $\mathcal{C}$ are partially ordered sets, so that the various structural isomorphisms are actually equalities, a simpler axiomatics is available, with the 1-dimensional equation that m is strictly associative, instead of the three 2-dimensional equations T1-T3. Also, if m is strictly associative, the axiomatics simplifies, and it turns out [20] that this simpler axiomatics in fact describes precisely the 2-category Δ of finite ordinal numbers.

However, the notion considered in the present paper does not include strict associativity for the monad multiplication, but rather *derives* associativity-up-to-isomorphism. Note, though, that the cocompletion procedures described in [13] and [14] were actually, by hook and by crook, constructed so as to be monads in the *strict* sense; likewise, it was, in certain cases, argued that, by replacing the category in question by an equivalent one, colimits could be chosen so as to be strict algebra structures for the monad. But, for instance, the skeletal category of sets does not admit a strict structure for the (strict) finite-coproduct completion Fam, as proved in Proposition 7.5 of [13]).

On the other hand, a different chain of reasoning was employed in [6] to prove that there does exist a strict (KZ-) monad on $\underline{Cat}$ whose strict algebras are categories equipped with finite colimits. Max Kelly has indicated to me that the same reasoning should apply to more general classes of colimits, say with size of the indexing categories bounded by some cardinal.

Due to these strictness properties, [4] were able to apply their 2-dimensional monad theory, with its flexibility notions, to such cocompletion monads. At present, no general 2-dimensional theory, in the spirit of [4], seems to be available explaining to what extent KZ-doctrines may be strictified into monads, or whether (pseudo-) algebras may be strictified. Such theory would have to extend the [1] theory of classes of colimits into a theory of Albert-Kelly equivalence of KZ-doctrines.

This work has developed and crystallized slowly, since my Ph.D. thesis [13] in 1967; stages in the development are marked by [14, 16, 17, 19] (with [20] and [18] as sideline developments). The development comprises contributions and insights by several other people, e.g. Zöberlein [30], and, notably, Street [26, 27], who called these "monads" "*monads with the Kock property*" or (in a slightly more general context) *Kock-Zöberlein*- or, for short, *KZ-doctrines*; we adopt the latter name.

The crystallization process has also been spurred by conversations and correspondence with many friends and colleagues during the years: Lawvere, Ulmer, Kelly, Street, Wood, Johnstone, Jacobs, Carboni, and Rosebrugh (in approximately chronological order). I want to thank them all.

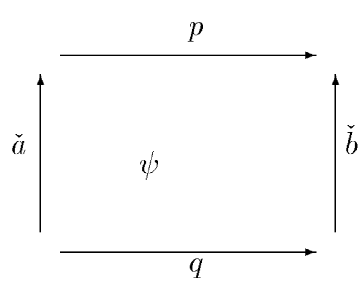
Special reference should be made to the influence of Street, who in [27] gave a very conceptual global account of the notion of KZ-doctrine, in terms of the 2-category Δ of finite ordinals (which in turn led to [20]). In some sense, the present paper puts Street's theory back in a form which is directly applicable for understanding cocompletion processes. But it also completes Street's formulation, in the sense that I prove the existence of one adjoint more than Street has; essentially, he explicitly (in [27] 2.27) excluded the "leftmost" adjoint ("$T(y) \dashv m$" of our Theorem 3.2) from his notion, and did not provide any indication whether it could be put back in.

Prerequsites from 2-dimensional categorical algebra

We shall employ standard conventions and notation from 2-dimensional category theory. The horizontal ("Godement") composition of 1-cells and 2-cells is denoted $*$, the vertical composition of 2-cells is denoted by a dot $\cdot$. We compose from right to left (unlike [19] and its predecessors). Identity 2-cells are often just denoted id, so that the context will have to make it clear what 1-cell it is the identity 2-cell of. Adjointness relation between 1-cells is denoted $f \dashv g$ (f left adjoint to g). Of course, one must specify the 2-cells $\eta : id \Rightarrow g*f$ and $\epsilon : f * g \Rightarrow id$ (front- and back-adjunctions) by virtue of which one has the adjointness. We shall write $f \dashv_r g$ for the special case where the back adjunction ϵ is an identity 2-cell ("f is a reflection(-left)-adjoint to g), and $f \dashv_{co} g$ if the front adjunction is ("g is coreflection(-right)-adjoint of f"); the terminology of [7] would be that f is a lali, respectively that g is a rali. If an arrow m is at the same time a reflection and coreflection adjoint, it is what Lawvere [22] calls a unity-and-identity-of-opposites ("UAIO"). Our main "extra" left adjoint (Theorem 3.2) will in fact show that the colimit formation for freely cocompleted categories will always be a UAIO.

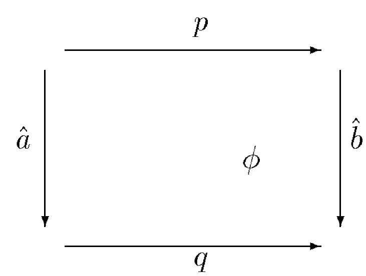
We shall employ a convenient "macro-package" for simplifying certain 2-dimensional calculations, the so called *mate calculus*, as it has been developed mainly by the Sydney category theorists, cf. [12]. We shall briefly recall what we need.

Given a square with a 2-cell ψ



(1)

and assume there are given adjointnesses $\hat{a} \dashv \check{a}$, $\hat{b} \dashv \check{b}$. The *mate* ϕ of ψ (with respect to these adjointnesses) is the 2-cell



(2)

obtained by pasting the front adjunction for $\hat{a} \dashv \check{a}$ and the back adjunction for $\hat{b} \dashv \check{b}$ on the left and right side of (1), respectively. There is a similar process, likewise called mating, leading from data (2) to data (1), and these processes are mutually inverse.

If one has a further square which can be pasted on the right of (1), i.e. a 2-cell $\psi' : p' * \check{b} \Rightarrow \check{c} * q'$, where $\check{c}$ has a left adjoint $\hat{c}$, then the mate of the composite (=pasted) square equals the composite (=paste) of the mates. (There is also a statement of this kind about vertical pasting, which we shall not consider.)

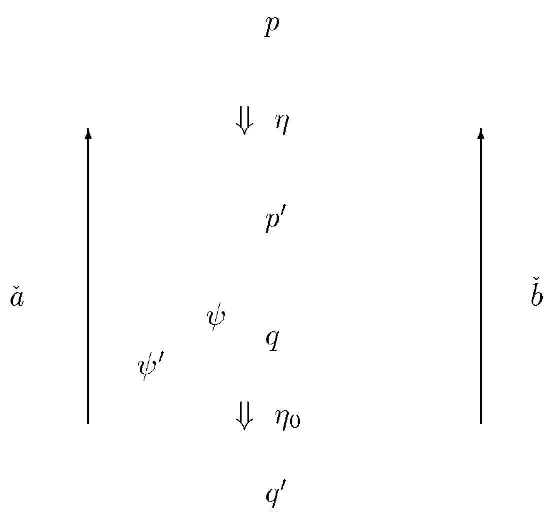
The mate of an invertible (or even an identity-) 2-cell need not be invertible. But if $\check{a} = \check{b}$, and p and q are the respective identity 1-cells, and ψ is the identity 2-cell of $\check{a}$, then the mate of ψ is the identity 2-cell of $\hat{a}$.

For the special case where p and q are identity 1-cells, we get the familiar bijective correspondence between 2-cells

$$\frac{\check{a} \Rightarrow \check{b}}{\hat{b} \Rightarrow \hat{a}}$$

which we shall call *simple mating.*

We shall finally need the following fact about naturality of the mating process for fixed $\check{a}, \hat{a}, \check{b}, \hat{b}$. Consider

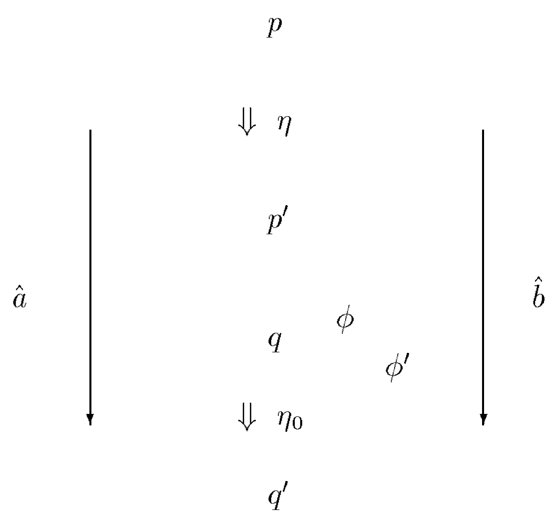


(3)

and assume there are given 2-cells ψ and ψ' in the unprimed and primed square, respectively, each ready to be mated. If the total diagram commutes on the 2-dimensional level, meaning

$$(\check{b} * \eta_0) \cdot \psi = \psi' \cdot (\eta * \check{a}),$$

as 2-cells $p * \check{a} \Rightarrow b * q'$, and if ϕ and ϕ' are the mates of ψ and ψ', respectively, then the diagram



commutes on the 2-dimensional level,

$$(\eta_0 * \hat{a}) \cdot \phi = \phi' \cdot (\hat{b} * \eta)$$

as 2-cells $\hat{b} * p \to q' * \hat{a}$.

1 The axioms and the motivation

Let $\mathcal{C}$ be a 2-category. Let $I : \mathcal{C} \to \mathcal{C}$ denote the identity functor.

Definition 1.1 *A* KZ-doctrine *on $\mathcal{C}$ consists of an endo-2-functor $T : \mathcal{C} \to \mathcal{C}$, and two 2-natural transformations $y : I \to T$, $m : T \circ T \to T$, and, for each $C \in \mathcal{C}$, a 2-cell $\lambda_C : T(y_C) \Rightarrow y_{T(C)}$, (natural in C)*

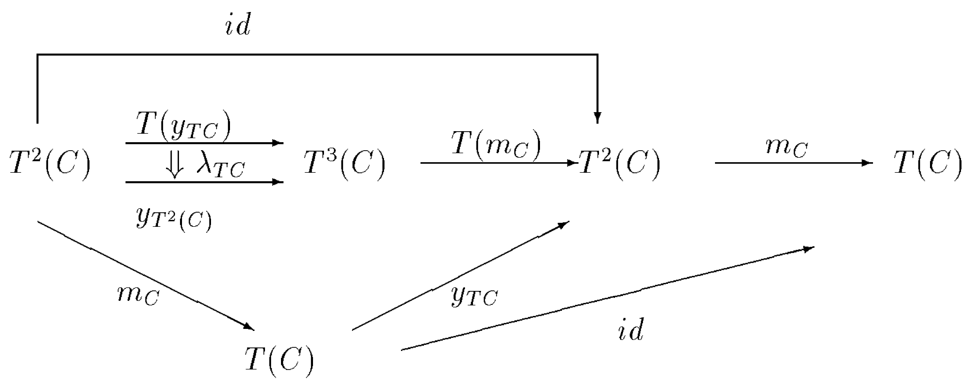
$$T(C) \underset{y_{T(C)}}{\overset{T(y_C)}{\underset{\Downarrow \lambda_C}{\rightrightarrows}}} T^2(C)$$

satisfying the following axioms

- **T0** *y is a strict two-sided unit for m, $m_C * T(y_C) = m_C * y_{T(C)} = id_{T(C)}$.*
- **T1** *$\lambda_C * y_C$ is an identity 2-cell*
- **T2** *$m_C * \lambda_C$ is an identity 2-cell*

- **T3** *$m_C * T(m_C) * \lambda_{TC}$ is an identity 2-cell.*

(The naturality of λ in C is in the evident ("modification-") sense, cf e.g. [12] p.82.) We note that the equations T1-T3 do not impose any equations on 1-cells since the domain- and codomain- 1-cells of the 2-cells in question are already equal, by virtue of T0. Thus, for example, both the domain- and the codomain 1-cell of the 2-cell in T3 equal m_C, as the reader may see by contemplating the diagram



(4)

where the square commutes by naturality of y with respect to m_C and the unmarked regions commute by T0.

Note that if m were associative, $m_C * T(m_C) = m_C * m_{TC}$, so then the middle 1-cell in the diagram could be replaced by m_{TC}, which then would immediately "kill" λ_{TC} (i.e. convert it into an identity 2-cell), by T2. So for strictly associative m, T3 follows from T2.

Proposition 1.2 *We have the reflection adjointness $m_C \dashv_r y_{TC}$, with $T(m_C) * \lambda_{TC}$ as front adjunction.*

Proof. The left two thirds of (4) reveals that $T(m_C) * \lambda_{TC}$ indeed is a 2-cell $id \Rightarrow y_{TC} * m_C$, and axiom T3, i.e the assumption that (4) is an identity 2-cell, is one of the triangular equations for the front- and back-adjunction; the other follows from $\lambda_{TC} * y_{TC} = id$ which is a consequence of T1.

There is a "mirror image" version T3* of T3, which we shall prove (on basis of T0-T3) below, Proposition 3.1 below; for associative m, T3* follows from T2.

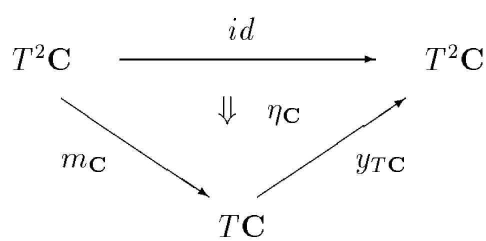
We now motivate the axiomatics. The "freely adjoining colimits"- constructions mentioned in the introduction consists in a functor $F : \underline{Cat} \to S\underline{Cat}$ which is adjoint-up-to-isomorphism for the forgetful functor $U : S\underline{Cat} \to \underline{Cat}$, in the precise sense that there is, for each $\mathbf{C} \in \underline{Cat}$, a 2-cell

$$y_{\mathbf{C}} : \mathbf{C} \to UF\mathbf{C} = T\mathbf{C}$$

such that composing with $y_{\mathbf{C}}$ provides an equivalence of hom-categories (for any $\mathbf{D} \in S\underline{Cat}$)

$$hom_{S\underline{Cat}}(T\mathbf{C}, \mathbf{D}) \to hom_{\underline{Cat}}(\mathbf{C}, \mathbf{D});$$

furthermore, for $\mathbf{C}$ to be in $S - \underline{Cat}$ (i.e. to have the appropriate kind of colimits), it is necessary and sufficient that $y_{\mathbf{C}}$ admits a left adjoint (=colimit formation) $\xi_{\mathbf{C}}$ (which may be taken in such a way as to be split by $y_{\mathbf{C}}$, i.e. a retraction left adjoint for $y_{\mathbf{C}}$). In particular, any freely cocompleted category $T\mathbf{C}$ admits a reflection left adjoint $m_{\mathbf{C}} : T^2\mathbf{C} \to T\mathbf{C}$ for $y_{T\mathbf{C}}$; $m_{\mathbf{C}}$ may at the same time be taken to be split by $T(y_{\mathbf{C}})$. (In fact, the general theory, Theorem 3.2 below, will prove that $m_{\mathbf{C}}$ will then be a coreflection right adjoint for $T(y_{\mathbf{C}})$.) The adjointness $m_{\mathbf{C}} \dashv y_{T\mathbf{C}}$ is expressed by means of a front adjunction $\eta_{\mathbf{C}}$,



satisfying the usual triangular equations, which here simplify (by $m_{\mathbf{C}}$ being a retraction) to

$$\eta_{\mathbf{C}} * y_{T\mathbf{C}} = id_{y_{T\mathbf{C}}}, \tag{5}$$

$$m_{\mathbf{C}} * \eta_{\mathbf{C}} = id_{m_{\mathbf{C}}}. \tag{6}$$

Let us put $\lambda_{\mathbf{C}} = \eta_{\mathbf{C}} * T(y_{\mathbf{C}})$. Then T1 easily follows from (5) and naturality of y; T2 follows immediately from (6). Finally, for T3, the 2-cell to be proved an identity 2-cell is, as we observed contemplating (4), a 2-cell $m_{\mathbf{C}} \Rightarrow m_{\mathbf{C}}$. But $m_{\mathbf{C}}$ is an arrow in $\underline{SCat}$, so is an object of $hom_{\underline{SCat}}(T^2\mathbf{C}, T\mathbf{C})$, and since $y_{T\mathbf{C}} : hom_{\underline{SCat}}(T^2\mathbf{C}, T\mathbf{C}) \to hom_{\underline{Cat}}(T\mathbf{C}, T\mathbf{C})$ is faithful (in fact an equivalence), it suffices to see that the 2-cell in question (ie. the left hand side of T3) is killed by pre-multiplication by $y_{T\mathbf{C}}$. But we may after the premultiplication by $y_{T\mathbf{C}}$ replace the resulting $T(y_{T\mathbf{C}}) * y_{T\mathbf{C}}$ by $y_{T^2\mathbf{C}} * y_{T\mathbf{C}}$ (by naturality of y), and this will kill $\eta_{\mathbf{C}}$, by (5) (with $\mathbf{C}$ replaced by $T\mathbf{C}$.)

This proves that a cocompletion process $T = UF$ does in fact carry structure of a KZ doctrine. -The reader may wonder why $m_{\mathbf{C}} : T^2\mathbf{C} \to T\mathbf{C}$ should be natural in $\mathbf{C}$ - after all, the back adjunctions for the adjointness-up-to-isomorphism $F \dashv U$ are colimit assignments, and such cannot be formed *strictly* naturally. But $m_{\mathbf{C}} : T^2\mathbf{C} \to T\mathbf{C}$ is colimit formation for the *free* cocompletion, and the construction of colimits (of the appropriate kind) in $T\mathbf{C}$ will by inspection reveal itself to be strictly natural (uniform) in $\mathbf{C}$ - as is to be expected of a construction of "syntactic" or "term model" type as $Ind\mathbf{C}$ or $Fam\mathbf{C}$. For the same reason, $\lambda_{\mathbf{C}}$ is natural in $\mathbf{C}$. Secondly, the reader may wonder why $m_{\mathbf{C}}$ may be a retraction of both $y_{T\mathbf{C}}$ and $T(y_{\mathbf{C}})$. This is (for the *Fam*- case) essentially the assertion that we may choose coproducts in $\underline{Sets}$ in such a way that $\coprod_X 1 = X$ and $\coprod_1 X = X$.

2 Algebras and homomorphisms

Let $\mathbf{T}= (T, y, m, \lambda)$ be a KZ-doctrine on a 2-category $\mathcal{C}$. It will turn out (Section 3) that (T, y, m) is automatically a monad-up-to-coherent-isomorphisms. Also the notion of *algebra* and *homomorphism* of such, which we now introduce, turn out to be equivalent to the monad-theoretic ones, up to isomorphism, thus justifying the terminology.

Definition 2.1 *An* algebra *for* $\mathbf{T}$ *(or a* $\mathbf{T}$-algebra*) consists of an object* $A \in \mathcal{C}$ *and a map* $a : TA \to A$, *which is a reflection left adjoint for* $y_A : A \to TA$. *We call* a *the* structure *of the algebra.*

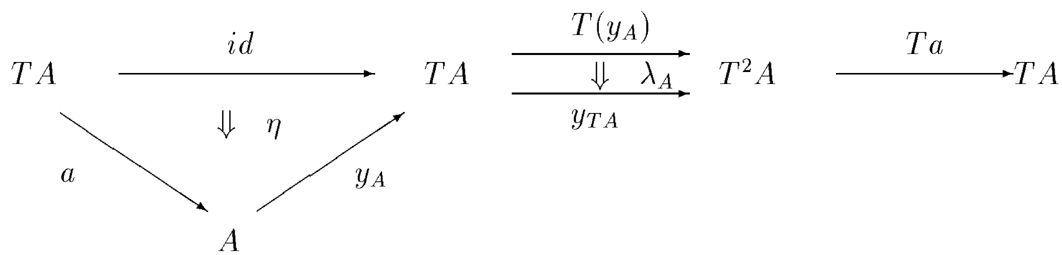
Thus, structures a are adjoint to units, $a \dashv_r y_A$, whence the title of the paper. Therefore, a structure on an object A is unique, up to isomorphism, if it exists. It furthermore turns out that the front adjunction η for $a \dashv_r y_A$ is unique, in fact given by an explicit expression involving λ :

Proposition 2.2 *Let $a : TA \to A$ be an arrow such that $a \dashv_r y_A$ by virtue of a 2-cell $\eta : id \Rightarrow y_A {*} a$. Then $\eta = Ta {*} \lambda_A$. In order that a 1-cell $a : T(A) \to A$ with $a * y_A = id$ is a structure, it is necessary and sufficient that*

$$a * T(a) * \lambda_A = id. \tag{7}$$

(The calculation that the domain- and codomain 1-cells of $Ta * \lambda_A$ are in fact id and $y_A * a$, respectively, are similar to the one contemplated in the left two-third of (4).)

Proof. Consider the horizontal composite



We calculate it in two ways as a $\cdot$-composite, by the fundamental exchange law for $*$ and $\cdot$. On the one hand, we may calculate it as

$$\begin{aligned} & (Ta * y_{TA} * \eta) \cdot (Ta * \lambda_A) \\ = \ & (y_A * a * \eta) \cdot (Ta * \lambda_A) \\ = \ & Ta * \lambda_A, \end{aligned}$$

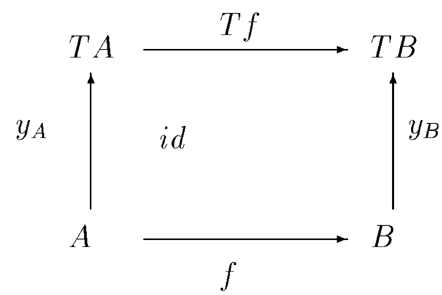
using naturality of y with respect to a, and the assumed triangular equation $a * \eta = id$ for $a \dashv_r y_A$. On the other hand, we may calculate it as

$$\begin{aligned} & (Ta * \lambda_A * y_A * a) \cdot (T(a) * T(y_A) * \eta) \\ = \ & id \cdot T(a) * T(y_A) * \eta \\ = \ & \eta, \end{aligned}$$

using that the left hand $\cdot$-factor is an identity 2-cell by Axiom T1, and that $T(a) * T(y_A) = T(a * y_A) = id_A$. For the last assertion, if a is reflection left adjoint for y_A by virtue of a front adjunction η, one of the triangular equations for the adjointness says $a * \eta = id$. But since η must be $T(a) * \lambda_A$, this equation is (7). Conversely $T(a) * \lambda_A$ will serve as front adjunction η, with (7) being one of the required triangular equations; the other one, $\eta * y_A = id$ follows from T1. This proves the Proposition.

Consider now two algebras (A, a) and (B, b), and an arbitrary map $f : A \to B$. By the *canonical 2-cell* for f, we shall understand the 2-cell

which is the mate (under $a \dashv y_A, b \dashv y_B$) of the identity 2-cell



(this diagram commutes, by naturality of y).

Remark 2.3 Consider the case where TA is a free colimit completion of a category A, so that $a : TA \to A$ is colimit formation, (assuming A has the relevant kind of colimits). If $D : \mathbf{I} \to A$ is a diagram in A, the value ϕ_D of ϕ at this diagram $D \in TA$ is the well known comparison

$$\lim_{\to}(f(D_i)) \to f(\lim_{\to}(D_i)).$$

To say that ϕ is invertible is thus to say that f commutes with colimits. This also motivates the following definition.

Definition 2.4 *Let (A, a) and (B, b) be* **T**-*algebras, and $f : A \to B$ an arbitrary map. We say that f is a* **T**-homomorphism *if the canonical 2-cell associated to f is invertible.*

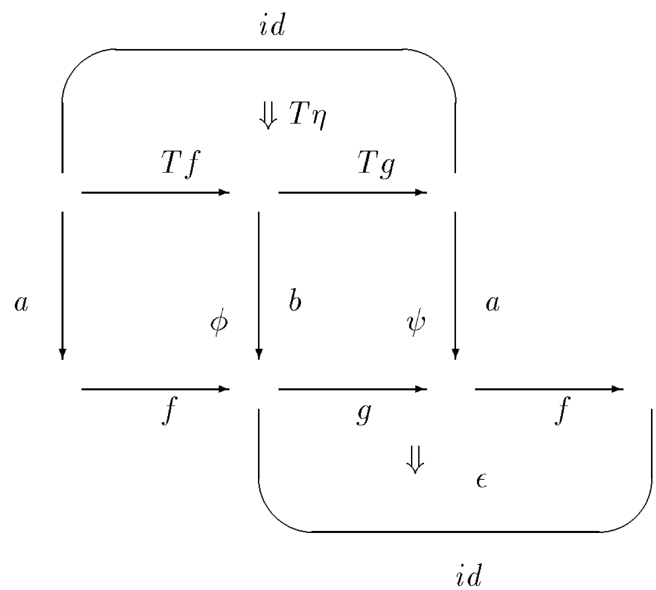
From the mate calculus ("paste of mates is the mate of paste") immediately follows that the composite of two **T**-homomorphisms is a **T**-homomorphism. Also identity maps are **T**-homomorphisms; and then it also follows from the naturality of the mating process, as described above, that any isomorphism $A \to B$ is a **T**-homomorphism (in fact an invertible such).

We thus get a category $\mathbf{T}$-$\underline{Alg}$ of $\mathbf{T}$-algebras and their homomorphisms, and a forgetful functor $U : \mathbf{T}\text{-}\underline{Alg} \to \mathcal{C}$. We may make $\mathbf{T}$-$\underline{Alg}$ into a 2-category by counting any 2-cell in $\mathcal{C}$ as a 2-cell in $\mathbf{T}$-$\underline{Alg}$. Thus U is faithful, and locally full and faithful.

In view of Remark 2.3, and the known fact that left adjoint functors preserve colimits of any kind that may exist, the following useful result is not surprising.

Proposition 2.5 *Let (A, a) and (B, b) be $\mathbf{T}$-algebras, and let $f : A \to B$ be a map. If f is a left adjoint arrow, it is a $\mathbf{T}$-homomorphism.*

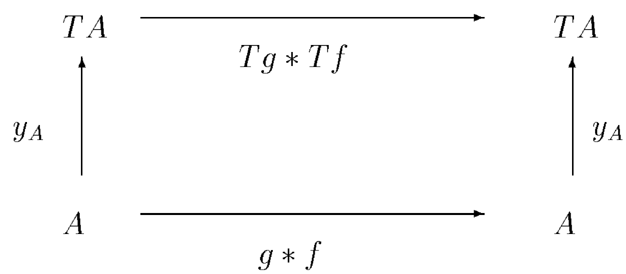
Proof. Let $g : B \to A$ be some right adjoint for f, by virtue of $\eta : id \Rightarrow g*f$ and $\epsilon : f*g \Rightarrow id$. In the following diagram, the two 2-cells in the squares are the canonical 2-cells for f and g, respectively. Also, ϵ is displayed. And because T is a 2-functor, $Tf \dashv Tg$ by virtue of $T\eta$, $T\epsilon$, and $T\eta$ is displayed also:



(8)

The three 2-cells $T\eta$, ψ and ϵ in this diagram paste to a 2-cell $\psi' : f*a \to b*Tf$ (which is in fact the mate of ψ, but now with respect to $f \dashv g$, $Tf \dashv Tg$). We claim that this 2-cell is an inverse of $\phi : b * Tf \Rightarrow f * a$. We prove that $\phi \cdot \psi' = id_{f*a}$ (the argument that $\psi' \cdot \phi = id_{b*Tf}$ is similar, but requires consideration of the diagram with ϕ and ψ pasted together in the opposite order). To prove the desired equation $\phi \cdot \psi' = id$ is, by the construction of ψ', equivalent to proving that the total paste in (8) is an identity 2-cell.

By naturality of y with respect to η, we have a commutative diagram



The mate of the front (straight) square under $a \dashv y_A$ is the canonical 2-cell for $g * f$, hence the paste of ϕ and ψ appearing in (8). The mate of the rear (curved) square is an identity 2-cell. The naturality principle for mating thus says that pasting $T\eta$ on the paste of ϕ and ψ gives the same result as pasting an identity 2-cell on η. In other words, the paste of the three cells $T\eta$, ϕ and ψ in (8) yields as result just $\eta * a$. Using this, the further pasting of the ϵ of (8) yields

$$(\epsilon * f * a) \cdot (f * \eta * a) = (\epsilon * f \cdot f * \eta) * a = f * a,$$

the last equality by one of the triangular equations for η, ϵ. This proves the desired equation, and thus the Proposition.

Another, completely austere equational proof, using no diagrams or mate calculus, may be found in the version [19].

Since front adjunctions $id \Rightarrow y_A * a$ for $\mathbf{T}$-algebra structures $a : TA \to A$ by Proposition 2.2 have an explicit expression in terms of λ, and since canonical 2-cells are mates of identity 2-cells under adjunctions $a \dashv y_A$, $b \dashv y_B$, it follows that the canonical 2-cell ϕ for $f : A \to B$, where (A, a) and (B, b) are $\mathbf{T}$-algebras, has a canonical expression in terms of λ, namely given by

$$\phi = b * Tf * Ta * \lambda_A. \tag{9}$$

The following useful result is due to Street ([26], Prop.4):

Lemma 2.6 *(Recognition Lemma) Given* **T***-algebras* (A, a) *and* (B, b)*, and a map* $f : A \to B$*. In order that a 2-cell* $\phi : b * Tf \Rightarrow f * a$ *is the canonical 2-cell for* f*, it is necessary and sufficient that it is annihilated by* y_A*, i.e. that* $\phi * y_A = id$*.*

Proof. If ϕ is the canonical 2-cell for f, it is given by (9), and this expression is annihilated by y_A, because of T1

On the other hand, assume $\phi * y_A = id$. We calculate the mate of ϕ by pasting the front adjunction η for $b \dashv y_B$ and the (trivial) back adjunction for $a \dashv y_A$, obtaining

$$(y_B * \phi * y_A) \cdot (\eta * Tf * y_A)$$

$$= (y_B * \phi * y_A) \cdot (\eta * y_B * f).$$

The right hand factor here is an identity 2-cell by the triangular equation for $b \dashv_r y_B$, and the left hand one is an identity 2-cell by assumption. So the mate of ϕ is (the appropriate) identity 2-cell, so ϕ is the mate of that identity 2-cell, hence is the canonical 2-cell for f.

3 Monad theoretic aspects

Up till now, we have not been utilizing the data $m : T^2 \Rightarrow T$ or the equations T2, T3 for it. But we did note that T3 (together with T0, T1) implies that, for any C, $m_C : T^2C \to TC$ is a reflection left adjoint for y_{TC}, thus provides TC with structure of **T**-algebra.

We now prove the 'dual' T3* of Axiom T3.

Proposition 3.1 *We have, for any* $C \in \mathcal{C}$,

$$m_C * m_{TC} * T\lambda_C = id. \tag{10}$$

Proof. Similarly to the diagram (4) for T3, one sees that the domain and the codomain 1-cells of the 2-cell in (10) is m_C. Let us view it as a 2-cell

$$\begin{array}{ccc} T^2C & \xrightarrow{T(id)} & T^2C \\ {\scriptstyle m_C}\downarrow & \swarrow & \downarrow{\scriptstyle m_C} \\ TC & \xrightarrow[id]{} & TC. \end{array}$$

By the recognition Lemma, to see that it is an identity 2-cell (which is the canonical 2-cell for the identity map of TC), it suffices that its precomposition with y_{TC} is an identity 2-cell. But

$$\begin{aligned} & m_C * m_{TC} * T\lambda_C * y_{TC} \\ = \; & m_C * m_{TC} * y_{T^2C} * \lambda_C \\ = \; & m_C * \lambda_C \; = \; id, \end{aligned}$$

the first equation by 2-naturality of y with respect to λ_C, and the other two equations by T1 and T2. This proves the Proposition.

As a Corollary, we have

Theorem 3.2 *For any KZ-doctrine* $\mathbf{T} = (T, y, m, \lambda)$, *we have* $T(y_C) \dashv_{co} m_C \dashv_r y_{TC}$.

Proof. We observed in Proposition 1.2 that $m_C \dashv_r y_{TC}$ follows from T0, T1, and T3. And $T(y_C) \dashv_{co} m_C$ follows in exactly the same way from T0, T1, and T3* (=(10)). (In particular, the back adjunction for $T(y_C) \dashv m_C$ is $m_{TC} * T(\lambda_C)$.)

Since m_C is a left adjoint, it follows from Proposition 2.5, applied with

$f = m_C$, that the canonical 2-cell for m_C

$$\begin{array}{ccc} T^3C & \xrightarrow{Tm_C} & T^2C \\ {\scriptstyle m_{TC}}\downarrow & \mu_C & \downarrow{\scriptstyle m_C} \\ T^2C & \xrightarrow[m_C]{} & TC \end{array} \tag{11}$$

is invertible. More generally, if $a : TA \to A$ is a **T**-algebra structure, a is a left adjoint, so that the canonical 2-cell for a

$$\begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle m_A}\downarrow & \alpha & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array} \tag{12}$$

is invertible. These canonical isomorphisms μ_C and α satisfy the coherence conditions $\mu_C * y_{T^2C} = id$, $\alpha * y_{TA} = id$, which in fact characterize them, by the recognition Lemma 2.6 for canonical 2-cells. Also, $\alpha * T(y_A) = id$ holds. For, (9) implies that

$$\alpha = a * Ta * Tm_A * \lambda_{TA}, \tag{13}$$

so that

$$\begin{aligned} \alpha * T(y_A) &= a * Ta * Tm_A * \lambda_{TA} * T(y_A) \\ &= a * Ta * Tm_A * T^2(y_A) * \lambda_A \\ &= a * Ta * \lambda = id, \end{aligned}$$

using naturality of λ with respect to y_A, T0, and (7).

Finally, if (A, a), (B, b) are **T**-algebras, and $f : A \to B$ a homomorphism, the relevant square from TA to B commutes up to isomorphism, namely up to the canonical 2-cell for f, by the very definition of the homomorphism notion.

These isomorphisms are all coherent with each other; this easily follows because they are all constructed as mates of identity 2-cells (and because "paste of mates is mate of paste"). In particular T, y, m, μ is a pseudo-monad (monad-up-to-coherent-isomorphisms μ). (In fact, by our normalization concerning y, the isomorphisms involving y are identities.) Also, **T**-algebras in our sense are pseudo-algebras for this pseudo-monad, meaning that $a : TA \to A$ is associative-up-to-isomorphism α, coherent with the associativity isomorphism μ for m. Finally, homomorphisms f in our sense are pseudo-homomorphisms in the pseudo-monad theoretic sense, meaning that they commute with structures up to coherent isomorphisms. Thus we have a functor (commuting with the evident forgetful functors)

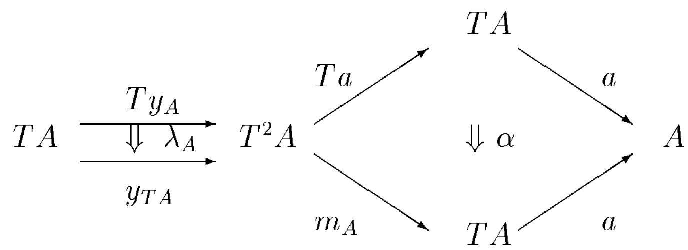
$$\mathbf{T}\text{-}\underline{Alg} \to (T, y, m, \mu)\text{-}\underline{Alg}, \tag{14}$$

sending (A, a) to (A, a), where $\mathbf{T}$-$\underline{Alg}$ denotes the category of **T**-algebras and homomorphisms in our sense (Definitions 2.1 and 2.4), whereas (T, y, m, μ)-$\underline{Alg}$ is the category of pseudo-algebras and pseudo-homomorphisms in the pseudo-monad theoretic sense.

We have been a little vague about what the coherence conditions are for the structure elements in the latter category; the reason being that we shall only need a few of them in order to conclude that (14) is actually an *isomorphism of categories*:

Proposition 3.3 *Let (A, a, α) be a pseudo-algebra (meaning that $a : TA \to A$ and that α is a 2-cell as in (12), satisfying the coherence conditions $\alpha * T(y_A) = \alpha * y_{TA} = id$). Then (A, a) is a* **T***-algebra, and α is the canonical 2-cell as constructed in (13). If (A, a, α), (B, b, β) are pseudo-algebras, and (f, ϕ) is a pseudo-homomorphism (meaning that $f : A \to B$ and that ϕ is an invertible 2-cell $b * Tf \Rightarrow f * a$, satisfying the coherence condition $\phi * y_A = id$), then f is a* **T***-algebra homomorphism, and ϕ is the canonical 2-cell for it.*

Proof. The statement about homomorphisms is immediate from the Recognition Lemma. To prove that a pseudo-algebra (A, a, α) has $a \dashv_r y_A$, we prove that the 2-cell $Ta * \lambda_A$ will serve as a front adjunction. The one triangle equation is $Ta * \lambda_A * y_A = id$, which follows from T1. The other one, namely $a * Ta * \lambda_A = id$, we prove by calculating the horizontal composite



in two ways, by the fundamental exchange law for $*$- and $\cdot$-composites in a 2-category. On the one hand, we may calculate it as

$$(\alpha * y_{TA}) \cdot (a * Ta * \lambda_A) = a * Ta * \lambda_A \tag{15}$$

by the coherence condition for α. On the other hand, we may calculate it as

$$(a * m_A * \lambda_A) \cdot (\alpha * Ty_A) = \alpha * Ty_A = id, \tag{16}$$

by T2 and the coherence condition for α. This proves $a * Ta * \lambda_A = id$, and thus the Proposition.

We summarize part of the conclusion more succinctly in

Corollary 3.4 *Let A be an object of $\mathcal{C}$. The following data are equivalent: 1) a reflection left adjoint $a : TA \to A$ for y_A; 2) a (T, y, m, μ)- pseudo algebra structure (a, α) on A. (In particular, α is uniquely determined by a.)*

We shall now address the problem: which objects A can carry an algebra structure for a given KZ-doctrine? The question is meaningful since such a structure is unique up to isomorphism. The answer is partly modelled on the list of equivalent conditions in [9]; I am indebted to Peter Johnstone for indicating to me the proof of the implication from (2) to (1). This proof depends on splitting idempotent 2-cells in the ambient 2-category. Such splittings are only determined up to isomorphism, and we have to make an assumption about how such splittings may be chosen; this is a price we have to pay for our insisting on the *normalized* algebra structures a, i.e. for insisting that $a * y_A$ is *equal* to the identity on A, (rather than just *isomorphic* to it, with an isomorphism satisfying certain coherence conditions, cf. [26]).

So consider a choice of splitting af idempotent 2-cells in a 2-category (idempotency and splitting with respect to the vertical composition $\cdot$; let us denote the chosen splitting of an idempotent 2-cell α by $\alpha' \cdot \alpha''$. We say that the choice is *natural* if $(\alpha * f)' = \alpha' * f$ for any 1-cell f and idempotent 2-cell α (composable with f), (and similarly for $f * \alpha$, although we shall not need this), and if $\alpha' = \alpha$ for any identity 2-cell α . (The similar equations for the α'' then follow). We can now state the

Theorem 3.5 *Let (T, y, m, λ) be a KZ-doctrine on a 2-category $\mathcal{C}$, and A an object of $\mathcal{C}$. Assume that $\mathcal{C}$ admits a natural splitting of idempotent 2-cells. Then the following conditions are equivalent:*

1. *There exists a reflection left adjoint for y_A*
2. *There exists a retraction for y_A.*
3. *A is a retract of some object $T(D)$*
4. *A is a retract of some object satisfying (1)-(3).*

Proof. The implications going down are trivial. Also, a retract of a retract is a retract, so (4) implies (3) is clear. To see that (3) implies (2), let A be a retract of $T(D)$ by virtue of i, p with $p * i = id_A$. Then

$$p * m_D * Ti * y_A = p * m_D * y_{TD} * i = p * i = id,$$

the second equality sign by T0. This shows there exists a retraction for y_A, proving (2). Finally, assume (2). Let a be a retraction for y_A and define the 2-cell $\eta : id_{T(A)} \Rightarrow y_A * a$ as $T(a) * \lambda_A$. In other words, η is constructed as in Proposition 2.2. As in there, one of the triangle equations for adjointness holds, the other (i.e. (7)) one may not. However, by an observation of Paré, quoted in [23] Chapter IV.1, Exercise 4, the fact that one triangle equation does hold implies that the other triangle 2-cell, in our case $a * T(a) * \lambda_A$, is at least an idempotent, which, when split, yields an adjoint pair, with the same right adjoint, and with left adjoint that 1-cell through which the splitting of the idempotent 2-cell passes. In this way y_A is proved to have a left adjoint. The fact that it is a reflection left adjoint comes from the naturality assumption for splitting of idempotents, together with the fact that the idempotent 2-cell in question $a * T(a) * \lambda_A$ becomes an identity 2-cell when $*$-composed on the right with y_A, by T1. This proves that y_A has a reflection left adjoint, so the condition (1) holds. This proves the Theorem.

4 Comonad theoretic aspects

A recurrent theme in relation to the various cocompletion processes that occur in the literature is the question: when is it the case that the colimit- (or sup-) formation $a : TA \to A$ (which is left adjoint to $y_A : A \to TA$) itself has a left adjoint ?

For instance, if TA is the ideal completion of the poset A (so TA = set of lower, upwards filtering, subsets of A, and a supremum formation for such, cf.e.g. [15]), a left adjoint for a, $\twoheaddownarrow : A \to TA$ exists iff A is a *continuous* poset (in the sense of continuous lattice theory, cf e.g. [24]), with $\twoheaddownarrow(x)$ being the set of elements way-below $x \in A$. See also [10].

Our Theorem 3.2 gives a general sufficient condition for existence of such a left adjoint.

For the case of a KZ-doctrine $\mathbf{T}$=(T, y, m, λ) on the category of posets, (T, y, m) is of course not just a pseudo monad, but a genuine monad, and the isomorphism of categories following from Proposition 3.3 in this case becomes a genuine monadicity theorem: the category of $\mathbf{T}$-cocomplete posets is monadic over posets, by the monad (T, y, m).

Whenever we have a monad $\mathbf{T}$ on a category $\mathcal{C}$, we get a comonad $\mathbf{T}$' on the category $\mathcal{C}^{\mathbf{T}}$ of $\mathbf{T}$-algebras, simply as the composition $F \circ U$, where $U : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ is the forgetful functor and $F : \mathcal{C} \to \mathcal{C}^{\mathbf{T}}$ its left adjoint.

Bart Jacobs raised the question of the coalgebras for the comonads arising from the various cocompletion monads on Posets, and found (private communication) that they are exactly those cocomplete posets (for the relevant notion of cocomplete) where the supremum formation has itself a left adjoint. We shall in the following prove and extend this result.

Let $\mathbf{T}$= (T, y, m, λ) be a KZ-doctrine on a 2-category $\mathcal{C}$. Let $\mathbf{T}$-$\underline{Alg}$ be the 2-category of $\mathbf{T}$-algebras and their homomorphisms, as defined in Section 2. There is a faithful, locally full-and-faithful forgetful 2-functor U : $\mathbf{T}$-$\underline{Alg}$ $\to \mathcal{C}$. There is also a 2-functor $F : \mathcal{C} \to$$\mathbf{T}$-$\underline{Alg}$ which is pseudo left adjoint to U (in a sense which we shall not need to make precise here; for the case where $\mathcal{C}$ is Posets, it is an actual left adjoint), given by

$$F(C) = (TC, m_C)$$

$$F(f) = T(f).$$

We know already by Proposition 1.2 that $m_C \dashv y_{TC}$ so that (TC, m_C) is indeed a $\mathbf{T}$-algebra in our sense. The fact that Tf is a homomorphism follows since $m_D * T^2 f = Tf * m_C$, by naturality of m.

Let $G = F \circ U$. We construct $e_X : GX \to X$ for any $X = (A, a) \in$**T**-$\underline{Alg}$; it is simply $a : TA \to A$, which, by being a left adjoint, is indeed a **T**-homomorphism, by Proposition 2.5. Note that e is not natural except up to an isomorphism (which can be specified in terms of λ). (In the Poset case, the naturality is strict, of course.)

Also, we have a $d_X : GX \to G^2X$; if $X = (A, a)$, this is $T(y_A) : TA \to T^2A$, which is a homomorphism, being of form $T(f)$; (it is even a left adjoint arrow, by Theorem 3.2); and d_X is strictly natural since y is. And e is a strict two-sided unit for d. Finally, we shall provide a 2-cell ρ_X for any $X = (A, a) \in$**T**-$\underline{Alg}$,

$$G^2X \underset{e_{GX}}{\overset{G(e_X)}{\underset{\Uparrow \rho_X}{\rightrightarrows}}} GX$$

This is the same as a 2-cell

$$T^2A \underset{m_A}{\overset{T(a)}{\underset{\Uparrow \rho_X}{\rightrightarrows}}} TA$$

Since we have $T(a) \dashv T(y_A)$, $m_A \dashv y_{TA}$, we may construct such ρ_X by taking the simple mate of the 2-cell $\lambda_A : T(y_A) \Rightarrow y_{TA}$.

On **T**-$\underline{Alg}$, we therefore have data **T**' = **G** = (G, e, d, ρ), which is like the data for a KZ-monad, except for the reversion of 1- and 2-cells, and for the fact that e is not natural except up to isomorphism. For the case where the base category $\mathcal{C}$ has all its hom-categories posets instead of genuine categories, e.g. when $\mathcal{C}$=Posets, e will be strictly (2-) natural, and also, in this case, the equations T1-T3 (suitably dualized) vacuously hold. If we let $(\mathbf{T}\text{-}\underline{Alg})^{co-op}$ denote **T**-$\underline{Alg}$ with both 1-cells and 2-cells reversed, we therefore have

Theorem 4.1 *Let the hom categories of the 2-category $\mathcal{C}$ be posets. Then for any KZ-doctrine* **T** *on $\mathcal{C}$, the data* **G**=(G, e, d, ρ) *described above, is a KZ-doctrine on* $(\mathbf{T}\text{-}\underline{Alg})^{co-op}$.

If we from the outset had developed the theory of KZ-doctrines with the naturality assumptions on y and m replaced by suitable pseudo-naturality (i.e. naturality-up-to-specified-coherent isomorphisms), we would presumably have had the theorem even without the restriction on the hom-categories of $\mathcal{C}$.

An interesting feature of the theorem is that it immediately admits iteration. If we thus apply it to the KZ-doctrine **T**'=**G** on $(\mathbf{T}\text{-Alg})^{co-op}$, we get a KZ-monad **T**" on the 2-category of coalgebras for **T**'=**G**, etc. This iterative feature should provide a conceptual frame for the question of iterated adjoints in the context of Yoneda structures, as studied by [29].

Note that, under the restriction posed on the hom categories, the notion of **T**-algebra, **G**-coalgebra, etc., are synonymous with the (strict) monad theoretic notions with the same names.

As a Corollary of Theorem 4.1, we derive Bart Jacobs' result, namely the equivalence of the 1. and 2. in

Theorem 4.2 *Let the hom categories of $\mathcal{C}$ be posets. Let* $\mathbf{T}= (T, y, m)$ *be a KZ-doctrine on $\mathcal{C}$ (i.e. a monad with $T(y_C) \leq y_{TC}$ for all $C \in \mathcal{C}$.) Then the following conditions on a **T**-algebra (A, a) are equivalent*

1. *there exists a left adjoint* $\downarrow$ *for* $a : TA \to A$
2. *there exists a **G**-costructure* $\downarrow$ *on* $(A, a) \in \mathbf{T}\text{-}Alg$
3. *there exists a **T**-homomorphism* $\downarrow\ : A \to TA$ *with* $a \circ \downarrow\ = id$.

Proof. By Theorem 4.1, **G** is a KZ-doctrine on $(\mathbf{T}\text{-}\underline{Alg})^{co-op}$, so Corollary 3.4 applies. But with $X = (A, a)$, a left adjoint for $X \to GX$ in $(\mathbf{T}\text{-}\underline{Alg}^{co-op}$ is the same as a left adjoint for $a : TA \to A$ in $\mathbf{T}\text{-}\underline{Alg}$. Thus 1. and 2. are equivalent. Clearly 2. implies 3., since this means just that we give up the requirement of co-associativity for the costructure $\downarrow$. To prove 3. $\Rightarrow$ 1., note that from standard monad theory, precomposition with y_A yields an isomorphism (for $B = (B, b)$ any algebra)

$$Hom_{\mathbf{T}\text{-}\underline{Alg}}(TA, B) \to Hom_{\mathcal{C}}(A, B),$$

which, since **T** is a 2-monad, in the present case is an order isomorphism. To prove $\downarrow\ \dashv_{co} a$, it suffices to prove $\downarrow * a \leq id$. Since both things to be compared here are **T**-homomorphisms $TA \to TA$, it suffices to prove the inequalities after precomposition with y_A, i.e. to prove $\downarrow\ \leq y_A$. This follows by precomposing the adjunction inequality $id_{TA} \leq y_A * a$ by $\downarrow$ and utilizing $a * \downarrow\ = id$. This proves the adjointness and hence the Theorem.

References

[1] M.H.Albert and G.M.Kelly, The closure of a class of colimits, J. Pure Appl. Alg. 51 (1988),1-17

[2] M.Artin, A.Grothendieck and J.-L.Verdier, Theorie des Topos et Cohomologie etale des Schemas, Springer Lecture Notes in Math.269 (1972)

[3] R.Betti, Cocompleteness over coverings, J. Austral.Math.Soc. Ser.A 39 (1985),169-177

[4] R.Blackwell, G.M.Kelly and A.J.Power, Two-dimensional monad theory, Journ. Pure Appl. Alg. 59 (1989), 1-41

[5] A.Carboni and R.Celia Magno, The free exact category on a left exact one, Journ. Austral. Math. Soc. (A) 33 (1982), 295-301

[6] E.Dubuc and G.M.Kelly, A presentation of topoi as algebraic relative to categories or graphs, J. Alg. 81 (1983), 420-433

[7] J.Gray, Fibred and cofibred categories, in Proceedings of the Conference on Categorical Algebra, La Jolla 1965, 21-83

[8] P.T.Johnstone, Stone Spaces, Cambridge University Press 1982

[9] P.T.Johnstone, Partial products, bagdomains, and hyperlocal toposes, Preprint Cambridge 1991

[10] P.T.Johnstone and A.Joyal, Continuous categories and exponentiable toposes, Journ.Pure Appl.Alg. 25(1982), 255-296

[11] G.M.Kelly, Basic Concepts of Enriched Category Theory, London Math.Soc. Lecture Notes Series No.64, Cambridge University Press 1982

[12] G.M.Kelly and R.Street, Review of the elements of 2-categories, in Category Seminar Sydney, Springer Lecture Notes in Math. 420 (1974), 75-103

[13] A.Kock,Limit monads in categories, Dissertation, Univ. of Chicago 1967

[14] A.Kock, Limit monads in categories, Aarhus Preprint Series 1967/68 No.6

[15] A.Kock, Bilinearity and cartesian closed monads, Math.Scand. 29 (1971), 161-174

[16] A.Kock, Monads for which structures are adjoint to units, (Version 1), Aarhus Preprint Series 1972/73 No.35

[17] A.Kock, Formal theory of free-completion monads, Oberwolfach Tagungsbericht 30 (1973), 33-34

[18] A.Kock, Constructive theory of the lift monad on posets, Aarhus Preprint Series 1992 No.22

[19] A.Kock, Monads for which structures are adjoint to units, (Version 2), Aarhus Preprint Series 1992 No.3

[20] A.Kock, Generators and relations for Δ as a monoidal 2-category, Beiträge zur Algebra und Geometrie 34 (1993), 201-208

[21] F.W.Lawvere, Ordinal sums and equational doctrines, in Seminar on Triples and Categorical Homology Theory,Springer Lecture Notes in Math. 80 (1969), 141-155

[22] F.W. Lawvere, Tools for the advancement of objective logic: closed categories and toposes, in The Logical Foundations of Cognition (ed. J. Macnamara and G.E. Reyes); Oxford University Press 1993

[23] S.Mac Lane, Categories for the Working Mathematician, Springer Verlag 1971

[24] G.Markowsky, A motivation and generalization of Scott's notion of continuous lattices, in Springer Lecture Notes in Mathematics 871 (1982)

[25] J.Meseguer, Order completion monads, Alg. Universalis 16 (1983), 63-82

[26] R.Street, Fibrations and Yonedas Lemma in a 2-category, in Category Seminar Sydney, Springer Lecture Notes in Math. 420 (1974), 75-103

[27] R.Street, Fibrations in bicategories, Cahiers de Top. et Geom. Diff. 21 (1980), 111-160

[28] R.Street, Conspectus of variable categories, J.Pure Appl.Alg. 21 (1981), 307-338

[29] R.Rosebrugh and R.Wood, Constructive complete distributivity, Constructive complete distributivity 2, Math. Proc. Camb. Phil. Soc. 110 (1991), 245-249

[30] V.Zöberlein, Doctrines on 2-categories, Math.Zeitschrift 148 (1976), 267-279