Aarhus Preprint 72/73 No. 35

MONADS FOR WHICH STRUCTURES ARE ADJOINT TO UNITS

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1. Introduction

The purpose of the present note is to present the formal 2-dimensional category theory relevant to a precise formulation of properties of free completion procedures. More specifically, structures of the kind considered here came up in connection with limit monads, [2] and [3]. It, however, seems to have other applications also, e.g. in [1], Example 3.3.

What we consider here is, in a slightly more general set up, endofunctors T one the (illigitimate) category Cat of legitimate categories. The functors considered are equipped with $y: I \rightarrow T$ and $m: T^2 \rightarrow T$, just as monads, but the monad laws only hold up to specified isomorphisms (invertible natural transformations). However, the whole 2-dimensional structure, including these invertible 2-cells, arise out of one single natural transformation

$$T \underset{T\,y}{\overset{y\,T}{\rightrightarrows}} T^2 \qquad \Downarrow \lambda$$

We call these structures $(T,y,m;\lambda)$ by the name λ-monads. There is an immediate notion of "algebra" for such a monad: an $A \in$ Cat equipped with structure $x: AT \rightarrow A$ which again is not strictly associative, but only up to invertible 2-cells, coherent with those already present. It turns out, however, that such a notion of "weak algebra" in case of a λ-monad is equivalent to the notion of an "action $x: AT \to A$ which is left adjoint to the unit $Ay: A \to AT$". (The adjunction should arise out of the given λ in a certain way). This is the content of the Theorem 2.3.

It is understandable that such situations arise in connection with free-completion procedures of categories, or of ordered sets. For instance, the limit monads T of [2] assign to a category $\mathcal{A}$ the category $\mathcal{A}T$ of colimit data in $\mathcal{A}$; an "algebra structure" on $\mathcal{A}$ is a colimit assignment $\mathcal{A}T \to \mathcal{A}$ (to each colimit data, associate its colimit); and colimit formation is, not surprisingly, left adjoint to the "diagonal" functor $\mathcal{A} \to \mathcal{A}T$. In section 3 we give a simple example of a (co-) limit monad, in order to illustrate this adjointness as well as exhibiting the crucial λ.

The most logical framework of this article would be the 3-dimensional category of 2-dimensional categories. However, we can, for the things we are interested in, reduce dimension by one, by considering a "one-object part" of this, namely the 2-dimensional category $\mathcal{C}$ of 2-functors $Cat \to Cat$. It is a monoidal 2-category with composition of functors as $\otimes$; this is a 2-functor, is strictly associative, and with a strict unit I. Both $\otimes$ and I are omitted from notation. Of course, a 2-monad on Cat becomes a monoid object in $\mathcal{C}$; and an object on which it acts is again an object in $\mathcal{C}$, that is, a 2-endofunctor on Cat. To interpret an algebra $(\mathcal{A},x)$ for a 2-monad on Cat, one has to view it as a <u>constant</u> endofunctor on Cat. This is the slight artifice which is the cost of having <u>functors</u> appearing as 0-cells, <u>transformations</u> as 1-cells, and "transformations between transformations" as 2-cells.

The notational conventions for Section 2 are: 0-cells (objects) are denoted by capital Latin letters T, A, etc.; 1-cells by small Latin letters m, y, x, etc. except when they are identity maps on 0-cells T,..., in which case they are again denoted T,... ; 2-cells are denoted by small Greek letters, except when they are identity 2-cells of 1-cells x,... (or T,...) in which case they are again denoted x,... (or T,...). Composition is from left to right, both for the horizontal (Godement) composition, denoted *, and for the vertical composition of 2-cells, denoted by a dot . . In diagrams, identity 1-cells or 2-cells are denoted by the digit 1.

We make a remark on a well known piece of 2-dimensional category theory, the

<u>Technique of adjoint (or conjugate) 2-cells</u>. If

$$A \underset{g_i}{\overset{f_i}{\rightleftarrows}} B$$

are adjoints, $f_i \dashv g_i$ by means of $\eta_i : A \to f_i * g_i$ and $\varepsilon_i : g_i * f_i \to B$ for $i = 1,2$, and if $\alpha : f_1 \Rightarrow f_2$ is a 2-cell, then one gets an "adjoint" 2-cell $\gamma : g_2 \Rightarrow g_1$ as the composite

(1.1) $$g_2 \underset{g_2 * \eta_1}{\Longrightarrow} g_2 * f_1 * g_1 \underset{g_2 * \alpha * g_1}{\Longrightarrow} g_2 * f_2 * g_1 \underset{\varepsilon_2 * g_1}{\Longrightarrow} g_1 .$$

This process takes invertible 2-cells to invertible 2-cells (but not necessarily identity 2-cells to identity 2-cells).

Let us finally recall that if

$$A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} A_2 = B_1 \underset{g_2}{\overset{f_2}{\rightleftarrows}} B_2$$

with $f_i \dashv g_i$ by means of $\eta_i : A_i \Rightarrow f_i * g_i$ and $\varepsilon_i : g_i * f_i \Rightarrow B$, then

$$f_1 * f_2 \dashv g_2 * g_1$$

by means of

(1.2) $$\eta = \eta_1 \cdot (f_1 * \eta_2 * g_1)$$

and

$$\varepsilon = (g_2 * \varepsilon_1 * f_2) \cdot \varepsilon_2 \, .$$

For the construction of this type in this paper, all ε's are identities.

2. The Formalism

We consider a 2-dimensional monoidal category $\mathcal{C}$ with strictly associative $\otimes$ and I, omitted from notation.

We consider in it an object T equipped with three items of structure

(i) $y : I \longrightarrow T$

(ii) $m : TT \longrightarrow T$

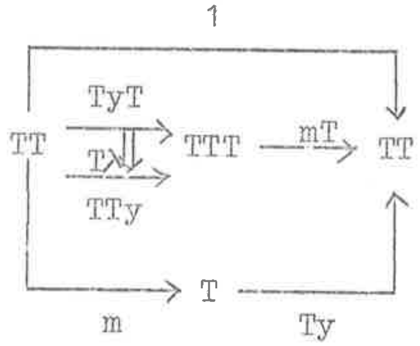
(iii) a 2-cell λ: $$T \overset{yT}{\underset{Ty}{\rightrightarrows}} T^2 \quad (\lambda : yT \Downarrow Ty) .$$

We require the following axioms to hold

<u>A0</u> y is a 2-sided unit for m, that is

$$yT * m = Ty * m = T$$

<u>A1</u> m is left adjoint to Ty with the identity 2-cell on $Ty*m = T$ as end-adjunction, and the 2-cell $T\lambda*mT$ as front adjunction:



<u>A2</u> $\lambda*m = T$ (this of course implies <u>A0</u>).

Clearly, <u>A1</u> can be reformulated into two equations between the adjunctions, namely

(2.1) $$T\lambda*mT*m = m$$

(2.2) $$Ty*T\lambda*mT = Ty$$

If we multiply the left hand side of (2.2) by y on the left, we get

$$y*Ty*T\lambda*mT$$
$$= y*yT*T\lambda*mT \qquad \text{(bifunctorality of } \otimes\text{)}$$
$$= y*\lambda*yT^2*mT \qquad \text{(2-bifunctorality of } \otimes\text{)}$$
$$= y*\lambda$$

whence from (2.2) conclude

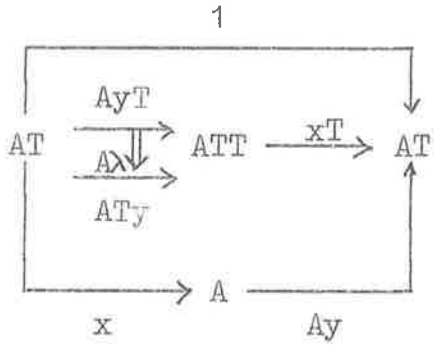
(2.3) $$y*\lambda = y*Ty = y*yT.$$

A (right) <u>λ-module</u> is a pair (A,x), where $x\colon AT \to A$, and satisfying

M0 y is a right unit for x, that is

$$Ay*x = A$$

M1 x is left adjoint to Ay, with the identity 2-cell on $Ay*x$ as end adjunction, and the 2-cell $A\lambda*xT$ as front adjunction:

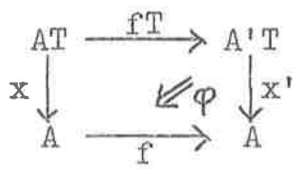


Because of (2.3), M1 is equivalent to just one equational condition

(2.4) $$A\lambda*xT*x = x.$$

Clearly (T,m) is a λ-module. Also if B is an arbitrary object and (A,x) is a λ-module, (BA,Bx) is a λ-module. A λ-module (BT,Bm) is called <u>free</u>.

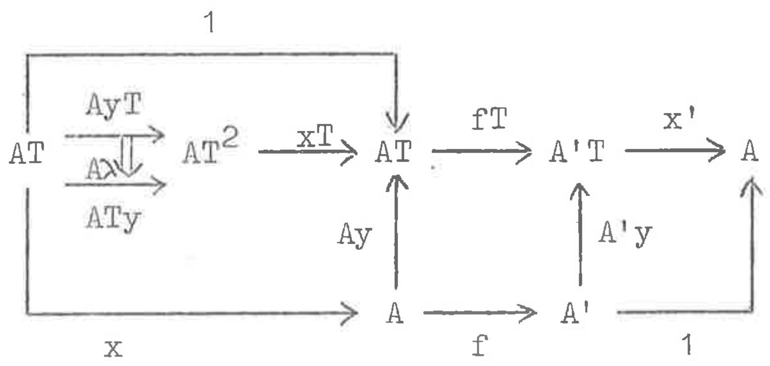
We consider a map $f: A \longrightarrow A'$, where (A,x) and (A',x') are λ-modules. To that, we associate a 2-cell φ, called the <u>canonical</u> one:



It is given as

$$A\lambda*xT*fT*x',$$

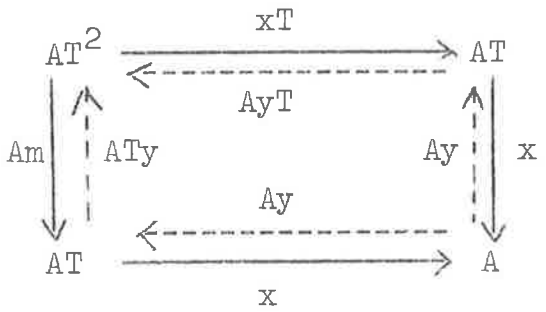
or, diagrammatically



We call f a λ-homomorphism provided the canonical φ associated to it is invertible.

Proposition 2.1. If (A, x) is a λ-module, then x: AT ⟶ A is a λ-homomorphism (considered as a map from the free -module (AT, m) to (A, x)).

Proof. In the diagram of bold arrows



all arrows have right adjoints (dotted arrows), and the diagram of dotted arrows commutes by bifunctorality of ⊗.

We shall show that the canonical 2-cell ξ associated to the map x: AT ⟶ A (where AT and A have structures Am and x, respectively) has the identity 2-cell of Ay*ATy = Ay*AyT corresponding to it under the technique of adjoint 2-cells. From this, it follows that ξ is an invertible 2-cell. Let us first make ξ explicit by specializing the above construction of the canonical

2-cell φ; we get

$$\xi = AT\lambda * AmT * xT * x.$$

Now we analyze the 2-cell arising out of ξ by the technique; by (1.1), it equals the conposite

(2.5) $$(Ay*ATy*\eta_1) \cdot (Ay*ATy*\xi*Ay*AyT)$$

where η_1 is the front adjunction for the adjointness

$$xT*x \dashv Ay*AyT;$$

by (1.2), we see that η_1 is

$$(A\lambda*xT)T \cdot (xT*A\lambda*xT*AyT).$$

We prove that each of the factors in (2.5) is an identity-2-cell; the first one:

(2.6)
$$\begin{aligned} Ay*ATy*\eta_1 &= (Ay*ATy*A\lambda T*xT^2) \cdot (Ay*ATy*xT*A\lambda*xT*AyT) \\ &= (Ay*A\lambda*AT^2y*xT^2) \cdot (Ay*ATy*AT\lambda*xT^2*xT*AyT) \end{aligned}$$

where we in both factors have used bifunctorality of $\otimes$ (which, when we have the concrete interpretation in mind is rather the naturality of y and λ, respectively). Now both factors in (2.6) are identity 2-cells, by (2.3). - Next, the second factor in (2.5) is, (using the explicit form of ξ):

$$Ay*ATy*AT\lambda*(\text{six other 1-cells})$$

which again is an identity 2-cell, by (2.3). Thus, (2.5) is an identity 2-cell, thus ξ is an invertible 2-cell.

Remark. We note in passing that the only property of ξ we use in this proof is that $ATy*\xi$ is an identity 2-cell.

Corollary 2.2. The "multiplication" $m: T^2 \to T$ is associative up to a canonical invertible 2-cell α. For any λ-module (A,x), the "action" $x: AT \to A$ is associative up to a canonical invertible 2-cell ξ.

Many coherence statements between ξ and α can be deduced by observing that the canonical invertible 2-cells are constructed by the technique of adjoint 2-cells from identity 2-cells, which clearly are coherent. We shall only need and prove the coherence statement:

$$AyT*\xi = x = ATy*\xi \tag{2.7}$$

So, if (A,x) is a λ-module, $AyT*\xi$ and $ATy*\xi$ are, respectively,

$$ATy*AT\lambda*AmT*xT*x$$

and

$$AyT*AT\lambda*AmT*xT*x.$$

The former is an identity 2-cell by (2.3). The latter is rewritten

$$\begin{aligned} & A\lambda*AyT^2*AmT*xT*x \\ = \; & A\lambda*(AyT*Am)T*xT*x \\ = \; & A\lambda*xT*x \\ = \; & x, \end{aligned}$$

by AO and (2.4). This proves (2.7).

We now view T, y, m as a "monoid, associative only up to a (coherent) invertible 2-cell α". We want to prove a converse of the above Corollary, namely that a "weak module" over T, y, m is actually a λ-module. By a <u>weak module</u>, we understand a pair (A, x) where $x\colon AT \to A$, is an action with Ay as a unit, and which is associative modulo some 2-cell ξ:

$$AT^2 \ \underset{m^*x}{\overset{xT^*x}{\rightrightarrows}}\!\!\Downarrow\ A$$

This ξ is supposed to be coherent with units, in the sense that the unit law for x holds strictly, and coherent with ξ, meaning that (2.7) holds.

To prove that (A, x) is a λ-module, we just have to verify (2.4).

Consider the diagram

$$AT \ \underset{ATy}{\overset{AyT}{\rightrightarrows}}\!\!{\scriptstyle A\lambda}\Downarrow\ AT^2 \ \underset{Am^*x}{\overset{xT^*x}{\rightrightarrows}}\!\!{\scriptstyle \xi}\Downarrow\ A$$

Rewriting the composite 2-cell $A\lambda^*\xi$, we get

(2.8) $$\begin{aligned} A\lambda^*\xi &= (AyT^*\xi)\cdot(A\lambda^*Am^*x) \\ &= x, \quad \text{by (2.7) and A2.} \end{aligned}$$

On the other hand, rewriting the other way round:

(2.9) $$\begin{aligned} A\lambda^*\xi &= (A\lambda^*xT^*x)\cdot(ATy^*\xi) \\ &= A\lambda^*xT^*x, \quad \text{by (2.7).} \end{aligned}$$

Putting (2.8) and (2.9) together yields

$$x = A\lambda{*}xT{*}x$$

which is the desired adjunction equation (2.4).

We note that since x now has been proved to make A into a λ-module, thus having Ay as right adjoint, we can use the technique of adjoint 2-cells to find the adjoint 2-cell of the ξ: $xT{*}x \Longrightarrow m{*}x$, assumed to exist for the weak module (A,x). The proof of Proposition 2.1 (cf. the Remark following that proof) now gives that because $ATy{*}\xi$ is an identity, the adjoint 2-cell of ξ is the identity 2-cell of $Ay{*}ATy = Ay{*}ATy$, thus ξ is in fact invertible.

We have thus proved:

Theorem 2.3. Let $(T,y,m;\lambda)$ be a λ-monoid. Then a pair (A,x) (where $x\colon AT \to A$ has Ay as a left inverse) is a λ-module if and only if the action x is associative up to a 2-cell ξ coherent with units (in the sense of (2.7) holding); such a 2-cell is automatically invertible, and unique.

Or briefly:

Summary. The notion of "action $AT \to A$, left adjoint to the unit $A \to AT$" is equivalent to the notion "action $AT \to A$, associative up to coherent 2-cell (which is then automatically invertible).

3. Example

All the (co-) limit monads considered by the author in [2], [3] are, when viewed as monoids in $\mathcal{C} = Cat^{Cat}$, examples of

λ-monads. We illustrate what λ is by means of the simplest of these monads, namely the one which completes categories under finite coproducts. So, we consider the functor $T\colon \mathrm{Cat} \longrightarrow \mathrm{Cat}$ which to a legitimate category $\mathcal{A}$ associates the "category $\mathcal{A}T$ of finite coproduct data"; the objects of this category are functors $\Phi : \underline{n} \to \mathcal{A}$ where $\underline{n}$ is the discrete category with the finite integer $n = \{0, 1, \ldots, n-1\}$ as its set of objects. A morphism in $\mathcal{A}T$ is a not necessarily commutative triangle of functors

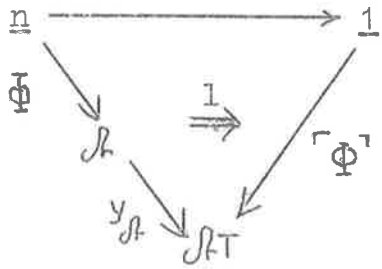
$$(3.1) \qquad \begin{array}{ccc} \underline{n} & \xrightarrow{\quad f \quad} & \underline{m} \\ & {\scriptstyle \Phi}\searrow \overset{\alpha}{\Rightarrow} \swarrow{\scriptstyle \Psi} & \\ & \mathcal{A} & \end{array}$$

together with a natural transformation $f \cdot \Psi \Rightarrow \Phi$. Composition of such morphisms is obvious.

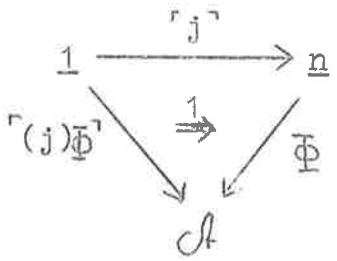
It is clear how to make T into a functor (in fact, a 2-functor): given a functor $\mathcal{A} \xrightarrow{\Gamma} \mathcal{B}$, we get a functor $(\Gamma)T\colon (\mathcal{A})T \longrightarrow (\mathcal{B})T$ by Godement composition with Γ . Thus the morphism (3.1) in $\mathcal{A}T$ goes by $(\Gamma)T$ to $\alpha * \Gamma$.

Let us describe $y_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}T$; it associates to an object $A \in |\mathcal{A}|$ the functor $\underline{1} \xrightarrow{\ulcorner A \urcorner} \mathcal{A}$ which picks out that object. The definition on maps is obvious. So $(y_{\mathcal{A}})T\colon \mathcal{A}T \to \mathcal{A}T^2$ associates to $\underline{n} \xrightarrow{\Phi} \mathcal{A}$ in $\mathcal{A}T$ the object $\underline{n} \xrightarrow{\Phi} \mathcal{A} \xrightarrow{y_{\mathcal{A}}} \mathcal{A}T$ in $\mathcal{A}T^2$. And $y_{\mathcal{A}T}$ associates to $\underline{n} \xrightarrow{\Phi} \mathcal{A}$ the object $\underline{1} \xrightarrow{\ulcorner \Phi \urcorner} \mathcal{A}T$. Consider the triangle

(3.2)



Then 1 is described as that natural transformation, which to the object $j \in \underline{n}$ $(0 \leq j \leq n-1)$ associates the map in $\mathcal{A}T$ given by the (commutative) triangle



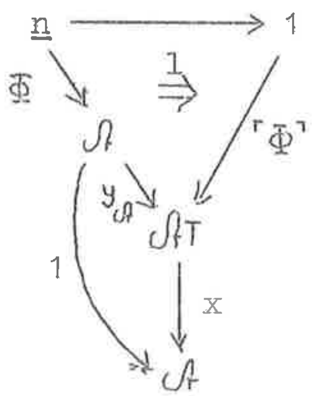
λ is now described as that rule which to an object $\underline{n} \xrightarrow{\Phi} \mathcal{A}$ of $\mathcal{A}T$ associates the morphism 1 in $\mathcal{A}T^2$ (1 as described in (3.2)); so $1 = \lambda_{\Phi}$.

A functor $x : \mathcal{A}T \to \mathcal{A}$, left adjoint to $y_{\mathcal{A}}$ by means of λ (and with $y_{\mathcal{A}} \cdot x = 1$) associates to an object $\Phi : \underline{n} \to \mathcal{A}$ its coproduct

$$\Phi(0) + \Phi(1) + \ldots + \Phi(n-1).$$

For, the statement that $\lambda * xT$ is a front adjunction says that for each $\Phi : \underline{n} \to \mathcal{A}$, the morphism in $\mathcal{A}T$

(3.3)



has the universal mapping property with respect to morphisms from Φ to objects of form $1 \to \mathcal{A}$ in $\mathcal{A}T$, which is precisely to say that (3.3) is a coproduct <u>diagram</u>,

$$\Phi(i) \xrightarrow{incl_i} \coprod_{j \in n} \Phi(j).$$

Conversely, if a functor $x: \mathcal{A}T \to \mathcal{A}$ has the property that it takes a coproduct data $\Phi : \underline{n} \to \mathcal{A}$ into a coproduct in such a way that (3.3) is the coproduct diagram, then $\lambda * xT$ has that universal property required to prove it the reflection morphisms for a (reflection) adjointness

$$x \dashv y_{\mathcal{A}}$$

(again we have assumed $y_{\mathcal{A}} * x = 1$).

We may summarize this discussion in:

A "λ-module" (for the λ-monoid T considered here) of the form $Cat \to 1 \xrightarrow{\ulcorner \mathcal{A} \urcorner} Cat$ is the same as a category $\mathcal{A}$ with finite coproducts; the structure map $x: \mathcal{A}T \to \mathcal{A}$ takes coproduct data into coproducts, and the individual cases 1 of λ give rise to the coproduct diagrams.

Or still shorter: λ is in some sense the generic family of inclusions for coproduct diagrams.

Identifying a coproduct data $\Phi : \underline{n} \to \mathcal{A}$ with a list $[A_0, \ldots, A_{n-1}]$ of objects in $\mathcal{A}$, the m for the λ-monad here can be described (on objects) by a straightforward concatenation; for instance, the list of lists

$$[[A_o, A_1], [B_o, B_1, B_2], [\ ]]$$

goes to the list

$$[A_o, A_1, B_o, B_1, B_2].$$

Actually the λ-monad thus described is strictly associative. In [2], we worked hard to get the more general limit monads strictly associative also, but one cannot avoid the 2-dimensional structure coming in when talking about λ-modules ("weak algebras" in the terminology of [2]). So there is not much point in asking for a strictly associative m either.

REFERENCES

1. B. Day, On adjoint-functor factorization, to appear

2. A. Kock, Limit monads in categories, Aarhus Universitet, Preprint Series 1967/68 No. 6

3. A. Kock, Limit monads in categories (revised version), to appear in Math. Scand.

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January 1973
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