THE STACK QUOTIENT OF A GROUPOID

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The stack quotient of a groupoid

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It is a well known conception, see e.g. [2], that the stack $B(G_\bullet)$ of principal $G_\bullet$ bundles is in some sense a quotient of $G_\bullet$. I intend here to make this into a more precise statement, and to prove it under quite general circumstances - essentially that of a category with pull-backs and a class $\mathcal{D}$ of descent maps (as in [4]).

For an equivalence relation in a category $\mathbf{B}$, it is unambiguous what a quotient of it should be; for a groupoid $G_\bullet$ in $\mathbf{B}$, we argue that and how the stack $B(G_\bullet)$ plays a role of quotient in a certain 2-dimensional sense (expanding the base category $\mathbf{B}$ into the category of stacks over $\mathbf{B}$, which is a 2-category).

The question of quotients of groupoids may be relevant to the formulations of intentional type theory of e.g. [3], [6], who approximate the notion of types-with-an-intentional-equality in terms of groupoids.

This paper is a sequel to [4]. In particular, we assume a base category $\mathbf{B}$ with pull-backs, and ultimately with a good class ($\mathcal{D}$) of "descent epis"; We shall talk about $\mathbf{B}$ as if it were the category of sets. In particular "small groupoid" is synonymous with "groupoid object in $\mathbf{B}$".

1 Basics on fibrations and stacks

This section is mainly to fix notation and terminology. Consider a fibration $P : \mathbf{X} \to \mathbf{B}$ (see e.g. [4]). If $G \in \mathbf{B}$, we take $X \in_G \mathbf{B}$ to be synonymous with $X \in \mathbf{X}_G$. If $G \in \mathbf{B}$, we have the *representable* fibration $y(G)$ over $\mathbf{B}$: it is the slice category $\mathbf{B}/G$, with domain formation ∂_0 as structural map to $\mathbf{B}$. Thus $f \in_H y(G)$ means that $f : H \to G$. Arrows in $y(G)$ are commutative triangles $f = g \circ h$ where f and g have codomain G; we write this morphism in $y(G)$ as $(g; h)$. It is an arrow from f to g, over h.

We consider the 2-category of fibrations over $\mathbf{B}$, denoted $\underline{\underline{Fib}}_{\mathbf{B}}$, or just $\underline{\underline{Fib}}$, since $\mathbf{B}$ will be fixed. Morphisms are functors over $\mathbf{B}$ which preserve the property of being a cartesian arrow. We shall be considering fibrations-in-groupoids, where all arrows are anyway cartesian, so the preservation requirement is void; 2-cells are natural transformations all of whose components are vertical.

We shall be interested in morphisms in $\underline{\underline{Fib}}_{\mathbf{B}}$ whose domain are representable fibrations $y(G)$, $y(H)$, etc. We collect some basic formulas. Note that since no "cleavage" or other arbitrary things are mentioned, the principle "whatever is meaningful, is true" is likely to be applicable. (We refer to these assertions as "Basic Item 1.-4.".)

1. Let $D : y(G) \to \mathbf{X}$, and let $d : H \to G$. The composite $D \circ y(d)$,

$$y(H) \xrightarrow{\;y(d)\;} y(G) \xrightarrow{\;\;D\;\;} \mathbf{X}$$

is given on objects $e \in_K (y(H))$ by

$$(D \circ y(d))(e) = D(d \circ e) \tag{1}$$

and on morphisms $(e; f)$ in $y(H)$ by

$$(D \circ y(d))(e; f) = D(d \circ e; f); \tag{2}$$

it is an arrow in $\mathbf{X}$ over f.

2. Next, we consider a 2-cell

$$y(G) \underset{D'}{\overset{D}{\underset{\downarrow \xi}{\rightrightarrows}}} \mathbf{X}.$$

So for $d \in_H y(G)$, the component $\xi_d : D(d) \to D'(d)$ is an arrow in $\mathbf{X}$, vertical over H. For an arrow $(d; e) : f \to d$ in $y(G)$ (where $f = d \circ e$), the naturality square is

$$\begin{array}{ccc} D(d \circ e) & \xrightarrow{\;D(d;e)\;} & D(d) \\ {\scriptstyle \xi_{d \circ e}} \downarrow & & \downarrow {\scriptstyle \xi_d} \\ D'(d \circ e) & \underset{D'(d;e)}{\longrightarrow} & D'(d) \end{array} \tag{3}$$

3. We next consider the composition ("whiskering") of the form

$$y(H) \xrightarrow{y(d)} y(G) \underset{D'}{\overset{D}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow \xi}}}} \mathbf{X}$$

where $d : H \to G$ in $\mathbf{B}$. For an object $e \in_K y(H)$, the component of the whiskering $\xi \circ y(d)$ at e is given as follows:

$$(\xi \circ y(d))_e = \xi_{d \circ e}; \tag{4}$$

it is a an arrow in $\mathbf{X}$, vertical over K.

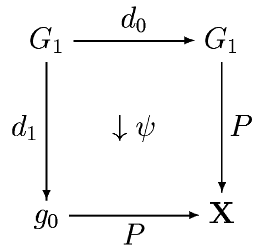
4. Let D and D' be as in item 2. above. From the naturality square exhibited in (3), it is easy to conclude that if the values of D' are cartesian arrows, and if two 2-cells ξ and $\eta : D \to D'$ agree on the object 1_G (identity map of G), then they agree everywhere. For, from the naturality squares (3) for ξ and η with respect to $(1; d) : d \to 1$, it follows that $D'(1; d) \circ \xi_d = D'(1; d) \circ \eta_d$, but two parallel arrows vertical arrows which postcompose with some cartesian arrow to give the same, are equal.

By 2-category, we understand here a 2-category where all 2-cells are invertible; equivalently, a category enriched in the category of groupoids. We have the full embedding y of the category $\mathbf{B}$ into the 2-category $\underline{\underline{Fib}}_{\mathbf{B}}$. It actually factors through the full subcategory of $\mathcal{D}$-stacks $\underline{\underline{S}}(\mathbf{B})$, to be described below. It is full and faithful on as well 1-cells as on 2-cells (viewing $\mathbf{B}$ as a locally discrete 2-category, in the sense that all 2-cells are identities). We sometimes omit the name of the embedding y from the notation. We shall not discuss coequalizers in $\underline{\underline{S}}(\mathbf{B})$ in general, but only coequalizers of groupoids $G_\bullet$ in $\mathbf{B} \subseteq \underline{\underline{S}}(\mathbf{B})$. In the first approximation, this means of course a diagram

$$G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0 \xrightarrow{P} \mathbf{X} \tag{5}$$

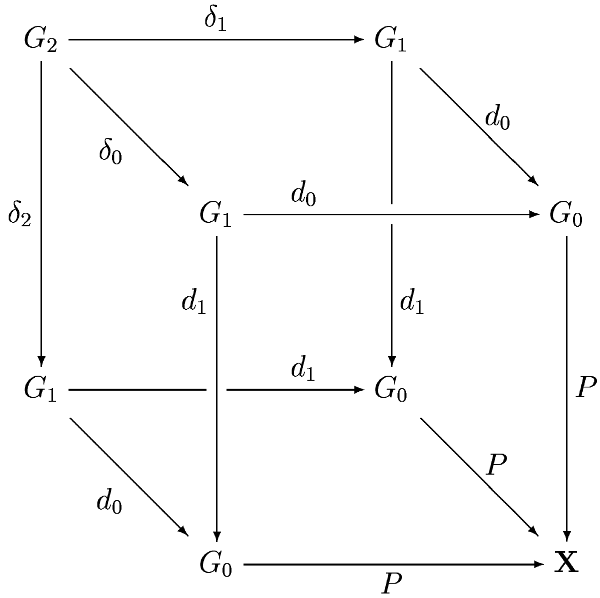
which commutes, and is universal in $\underline{\underline{S}}(\mathbf{B})$ with this property. But since there are 2-cells available between parallel arrows to $\mathbf{X}$, two-dimensional wisdom says that the notion "the two composites are *equal*" should be replaced by "there is a *specified* 2-cell ψ comparing the two composites". But wisdom also says that specifications should come together with equations to be satisfied, and here it is a cocycle condition on ψ, which involves the three maps $G_2 \to$

G_1. To make better room for the pasting geometry involved, we exhibit the fork (5) in terms of a square



(6)

Then the equations to be satisfied are a cocycle condition, and a unit condition. The cocycle condition is expressed in terms of commutativity of the 2-cells in a cube,



(7)

The three faces adjacent to the vertex labelled $\mathbf{X}$ are equal, and are all filled with the (invertible) 2-cell ψ, and the three other faces, adjacent to the vertex labelled G_2, are strictly commutative, and express the three simplicial identities that obtains between the composite face operators $G_2 \to G_0$. As a pasting diagram, it makes sense, (ψ being an oriented 2-cell; there are in fact unique orientations on the three simplicial identities making this cube into a valid pasting scheme, namely $d_0 d_0 \to d_0 d_1$, $d_1 d_0 \to d_0 d_2$, and $d_1 d_1 \to d_1 d_2$). The cocycle condition on ψ says that the pasting diagram commutes.

There is also a unit condition: it says that pasting the 2-cell ψ with $s : G_0 \to G_1$ yields an identity 2-cell,

$$\psi \circ s = 1_P.$$

(If $\mathbf{X}$ is equipped with a cleavage, so that one has functors $d_0^* : \mathbf{X}_{G_0} \to \mathbf{X}_{G_1}$ etc., the cubic cocycle condition can be rendered in the usual form $\delta_2^*(\psi) \circ \delta_0^*(\psi) = \delta_0^*(\psi)$ for descent data.)

The collection of such data form a groupoid $\underline{Coeq}(G_\bullet, \mathbf{X})$, whose arrows are 2-cells $P \to P'$ compatible with the ψ's. (It is, with a choice of cleavage, equivalent to the category of descent data along e, if d_0, d_1 happen to be the kernel pair of some map e.)

We now describe the subcategory $\underline{\underline{S}}(\mathbf{B})$ of $\underline{\underline{Fib}}(\mathbf{B})$. Its objects are fibrations $\pi : \mathbf{X} \to \mathbf{B}$, such that $\mathbf{X}$ is a $\mathcal{D}$-stack. Here $\mathcal{D}$ is a class of "descent epis", as in [4] (and probably the same as in [5]). To say that $\pi : \mathbf{X} \to \mathbf{B}$ is a stack is to say that it has the descent property with respect to all maps p in $\mathbf{B}$; this means in particular that whenever $p : G_0 \to G_{-1}$ is a map in $\mathcal{D}$, and $X_\bullet$ is a simplicial object in $\mathbf{X}$, mapping by π to the simplicial kernel of p, then $X_\bullet$ is the simplicial kernel of a map in $\mathbf{X}$ above $G_0 \to G_{-1}$. Also, we require that the class $\pi^{-1}(\mathcal{D})$ of arrows in $\mathbf{X}$ which map to arrows of $\mathcal{D}$ by π, form a good class of descent epis.

Note that if $\mathbf{Y}$ and $\mathbf{X}$ are stacks over $\mathbf{B}$, then any functor $\mathbf{Y} \to \mathbf{X}$ over $\mathbf{B}$ preserves pull-backs and the given (or constructed) class of $\mathcal{D}$-epis.

2 From coequalizing data to groupoids

We consider a fibration $\pi : \mathbf{X} \to \mathbf{B}$; we assume that all arrows in $\mathbf{X}$ are cartesian, so that the fibres $\mathbf{X}_G$ (for $G \in \mathbf{B}$) are groupoids. We also assume that $\mathbf{B}$ has pull-backs. Then it follows that $\mathbf{X}$ has pull-backs, and that π preserves them. Even more, π *reflects* pull-backs, in the sense that if a commutative square in $\mathbf{X}$ is mapped to a pull-back by π, then it is itself a pull-back.

A groupoid object in $\mathbf{B}$ may be given in terms of its nerve $G_\bullet$; a more economic way of giving the data of a groupoid object $\underline{G}$ is the following standard one: it consists of *truncated simplicial data*,

$$G_2 \;\begin{matrix}\longrightarrow\\\longrightarrow\\\longrightarrow\end{matrix}\; G_1 \;\begin{matrix}\stackrel{s}{\longleftarrow}\\\longrightarrow\\\longrightarrow\end{matrix}\; G_0 \qquad (8)$$

of face maps satisfying the simplicial identities, cf. Appendix, from where the notation is taken, plus a map $s : G_0 \to G_1$, splitting the two face maps $G_1 \to G_0$ (s "picks out identity arrows").

For such truncated data to be a groupoid, the three commutative squares that represent the three simplicial identities among face maps (see Appendix) should be pull-backs; also, with the middle of the three face maps $G_2 \to G_1$ as composition, this composition should be associative and have s as unit. If these conditions are satisfied, its "nerve" $G_\bullet$ may be formed. It is a full-fledeged simplicial object, of which the given data then is a "truncation". The category of small groupoids becomes a full subcategory of the category of simplicial objects. – With the stated assumptions on $\pi : \mathbf{X} \to \mathbf{B}$, we then have

Proposition 1 *Let* $\underline{X}$ *be a truncated simplicial object in* $\mathbf{X}$ *mapping to a groupoid* $\underline{G}$ *in* $\mathbf{B}$*. Then* $\underline{X}$ *is a groupoid.*

Proof. The associativity condition for the composition map $\delta_1 : X_2 \to X_1$ is expressed as an equality between two parallel maps $a_1, a_2 : X_3 \to X_1$ (where $X_3 = X_1 \times_{X_0} X_1 \times_{X_0} X_1$). Now since $X_\bullet$ maps to a groupoid $G_\bullet$, where the associativity condition holds, and since π preserves pull-backs, it follows that $\pi(a_1) = \pi(a_2)$. Since therefore a_1 and a_2 are parallel map over the same map, it suffices to see that they become equal when post-composed with some (cartesian) map. But clearly for instance $d_0 : X_1 \to X_0$ will do this job.

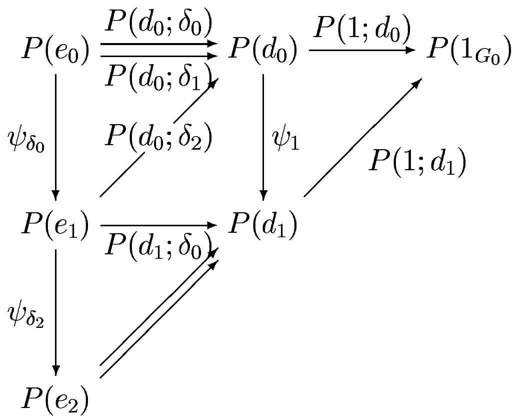
So to construct a groupoid in $\mathbf{X}$ out of coequalizing data $P : G_0 \to \mathbf{X}$, ψ, as in (6), it suffices to construct truncated data X_2, X_1, X_0, with the relevant six maps in between. This is completely explicitly done, and exhibited in the diagram (9) below (as far as the five face maps are concerned). Namely, we take $X_0 := P(1_{G_0})$, $X_1 := P(d_0)$, $X_2 := P(e_0)$, in other words, they are the objects of the upper row in (9). The five face maps are also present in the diagram. Denoting face maps in the $X_\bullet$ under construction by $\tilde{\delta}_i$, $\tilde{d}_j$, etc., we thus put $\tilde{\delta}_0 := P(d_0; \delta_0)$, (note that by $d_0 \circ \delta_0 = e_0$, $(d_0, \delta_0) : e_0 \to d_0$ in $y(G_0)$, and similar for the other "semicolon" expressions. Similarly, we put $\tilde{\delta}_1 := P(d_0; \delta_1)$, but for $\tilde{\delta}_2$, we need to involve ψ:

$$\tilde{\delta}_2 := P(d_0; \delta_2) \circ \psi_{\delta_0};$$

and $\tilde{d}_0 := P(1; d_0)$,

$$\tilde{d}_1 := P(1; d_1) \circ \psi_1.$$

(Here, 1 denotes the identity map of G_0.) Finally, the construction is completed by putting $\tilde{s} : X_0 \to X_1$ equal to $P(d_0; s)$ (note that since $d_0 \circ s = 1$, $(d_0; s)$ is a morphism in $y(G_0)$ from 1 to d_0). The reader will find some of this data exhibited in the diagram



(9)

(the arrow $\psi_{\delta_1} : P(e_0) \to P(e_2)$ on the far left not exhibited; the 1 on ψ refers to 1_{G_1}.

To prove the simplicial identities among the $\tilde{\delta}_i$, $\tilde{d}_j$ and $\tilde{s}$ is easier the fewer ψ's are involved, i.e. the smaller the indices i and j are. The method is in any case the same, so we are only going to present one of them, the "worst" one, – the only one involving the cocycle condition,

$$\tilde{d}_1 \circ \tilde{\delta}_1 = \tilde{d}_1 \circ \tilde{\delta}_2,$$

as well as the identity involving the unit condition,

$$\tilde{d}_1 \circ \tilde{s} = 1.$$

So we calculate

$$\tilde{d}_1 \circ \tilde{\delta}_1 = P(1; d_1) \circ \psi_1 \circ P(d_0; \delta_1) = P(1; d_1) \circ P(d_1; \delta_1) \circ \psi_{\delta_1}$$

(using naturality of ψ with respect to $(\delta_1; d_0) : e_0 \to d_0$)

$$= P(1; e_2) \circ \psi_{\delta_1},$$

using functorality of P on the composite $(1; d_1) \circ (d_1; \delta_1) = (1, e_2)$. On the other hand,

$$\begin{aligned} \tilde{d}_1 \circ \tilde{\delta}_2 &= P(1; d_1) \circ \psi_1 \circ P(d_0; \delta_2) \circ \psi_{d_0} \\ &= P(1; d_1) \circ P(d_1; \delta_2) \circ \psi_{\delta_2} \circ \psi_{\delta_0}, \end{aligned}$$

by naturality of ψ w.r.to $\delta_2 : e_1 \to d_0$. By functorality of P, this is $P(1; e_2) \circ$ $\circ \psi_{\delta_2} \circ \psi_{\delta_0}$, and by the cocycle condition, this equals $P(1; e_2) \circ \psi_{\delta_1}$ as desired.
– To prove the unit condition: $\tilde{d}_0 \circ \tilde{s} = 1$ is trivial by functorality of P; $\tilde{d}_1 \circ \tilde{s}$ uses the unit condition for ψ, namely $\psi_s = 1$.

3 From groupoids to coequalizing data

We consider a groupoid $X_\bullet$ in $\mathbf{X}$, mapping by π to the fixed groupoid $G_\bullet$ in $\mathbf{B}$, and proceed to construct coequalizing data $(P : G_0 \to \mathbf{X}, \psi)$ out of this data. One piece of information is not completely explicit, namely a functor $p : y(G_0) \to \mathbf{X}/X_0$ with $p(1_{G_0}) = 1_{X_0}$; (with $\pi \circ p$ =the identity functor on $\mathbf{B}$. The functor P will just be p followed by the domain formation :$\mathbf{X}/X_0 \to \mathbf{X}$. (One may think of p as a *partial* cleavage of the fibration; but one that is so "minimalistic" that coherence isomorphisms do not come up. If $\mathbf{X} \to \mathbf{B}$ is equipped with a cleavage, one may use it to define $p(d)$ to be the chosen (cartesian) lift of d with codomain X_0, where d is an arrow with codomain G_0.) The domain of $p(d)$, for $d : H \to G_0$ in $y(G_0) = \mathbf{B}/G_0$ will be denoted $d^*(X_0)$ (or $P(d)$). (Note that we do not have $e^*(X)$ for objects X in $\mathbf{X}$ in general, which we would have, if we had had a clevage instead of just a partial cleavage.)

Since thus we have already p, we just have to provide the natural transformation $\psi : p \circ y(d_0) \to p \circ y(d_1)$ between the indicated functors $y(G_1) \to \mathbf{X}$. (The simplicial operators on $G_\bullet$ consist of maps d_i, δ_j, and e_k, as before; the simplicial operators on $X_\bullet$ are denoted similarly, but with a tilde: $\tilde{d}_i$, etc.)

So consider an object $\delta : H \to G_1$ in $y(G_1)$, then ψ_δ should be a vertical arrow in $\mathbf{X}$ over H,

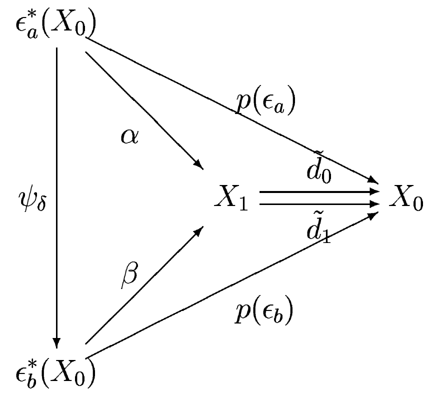
$$\psi_\delta : P(d_0 \circ \delta) \to P(d_1 \circ \delta);$$

denoting $d_0 \circ \delta$ by ϵ_a and $d_1 \circ \delta$ by ϵ_b, we then construct ψ_δ by the following recipe: Consider $p(\epsilon_a) : \epsilon_a^*(X_0) \to X_0$; then use that $\tilde{d}_0$ is cartesian, so that we may consider the comparison arrow $\alpha : \epsilon_a^*(X_0) \to X_1$ over δ, arising form the factorization $\epsilon_a = d_0 \circ \delta$; similarly, consider $p(\epsilon_b) : \epsilon_b^*(X_0) \to X_0$: then use that $\tilde{d}_1$ is cartesian, so that we may consider the comparison arrow

$\beta : \epsilon_b^* \to X_1$ over δ arising from the factorization $\epsilon_b = d_1 \circ \delta$. Since both α and β live over δ, and have common codomain X_1, we may use that β is cartesian, to get a unique vertical comparison from α to β, and this is to be our ψ_δ, so

$$\beta \circ \psi_\delta = \alpha. \tag{10}$$

For the convenience of the reader, we record the recipe in a diagram:



(11)

The unit condition $\psi \circ y(s) = 1$ follows by contemplating this diagram, with $\delta = s$, then the long sloping arrows will be 1_{X_0}; so $\alpha = \tilde{s}$ and $\beta = \tilde{s}$ by uniqueness of cartesian factorization, and so $\psi \circ y(s)$ is the identity 2-cell of 1_{X_0}.

To prove the cocycle condition (in the "cube" form, (7)), we need to calculate the whiskerings $\psi \circ y(\delta_i)$ for $i = 0, 1, 2$.

We claim that, for their components at the object id_{G_2} (for brief denoted I), we have, for certain canonical vertical arrows $c_0, c_1 and c_2$ to be given below,

$$(\psi \circ y(\delta_0))_I = c_1 \circ c_0^{-1} \tag{12}$$

$$(\psi \circ y(\delta_1))_I = c_2 \circ c_0^{-1} \tag{13}$$

$$(\psi \circ y(\delta_2))_I = c_2 \circ c_1^{-1} \tag{14}$$

Since natural transformations in this case are determined by their component at the identity of the domain, these equations will establish the cocycle condition for ψ,

$$(\psi * y(\delta_2)) \circ (\psi * y(\delta_0)) = \psi * y(\delta_1),$$

(where we used $*$ rather than $\circ$ to denote horizontal composition (whiskering). The three calculations proceed in the same way, so we shall give only the one for (13). We use the cartesian property of $\tilde{d}_0$ to lift the factorization $d_0 \circ \delta_1 = e_0$ to a factorization of $p(e_0)$ over $\tilde{d}_0$, say

$$p(e_0) = \tilde{d}_0 \circ \alpha, \tag{15}$$

with $\pi(\alpha) = \delta_1$, and similarly, the factorization $d_1 \circ \delta_1 = e_2$ lifts to a factorization of $p(e_2)$ over $\tilde{d}_1$

$$p(e_2) = \tilde{d}_1 \circ \beta. \tag{16}$$

with $\pi(\beta) = \delta_1$. Also, by the definition of ψ_{δ_1},

$$\beta \circ \psi_{\delta_1} = \alpha \tag{17}$$

with ψ_{δ_1} vertical, for the α and β of (15) and (16). Let c_i denote the unique vertical comparison $X_2 \to e_i^*(X_0)$ with

$$p(e_i) \circ c_i = \tilde{e}_i \tag{18}$$

Then we have

$$\alpha \circ c_0 = \tilde{\delta}_1. \tag{19}$$

These are parallel arrows over the same arrow δ_1 in $\mathbf{B}$, so it suffices to prove that they become equal by post-composition with some (cartesian) arro; here, $\tilde{d}_0$ will do the job, since, by (15) $\tilde{d}_0 \circ \alpha \circ c_0 = p(e_0) \circ c_0 = \tilde{e}_0 = \tilde{d}_0 \circ \tilde{\delta}_1$. We can now prove

$$\psi_{\delta_1} \circ c_0 = c_2.$$

Since both sides of this equation are vertical, it suffices to prove that post-composing them with some (cartesian) arrow give same result; we shall utilize $p(e_2)$, so we intend to prove

$$p(e_2) \circ \psi_{\delta_1} \circ c_0 = p(e_2) \circ c_2.$$

We calculate

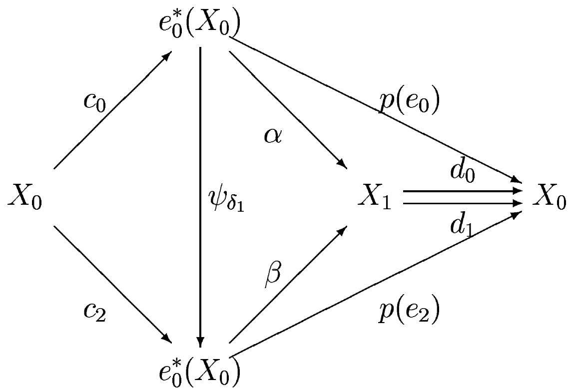
$$p(e_2) \circ \psi_{\delta_1} \circ c_0 = \tilde{d}_1 \circ \beta \circ \psi_{\delta_1} \circ c_0 \text{ by (16)}$$

$$= \tilde{d}_1 \circ \alpha \circ c_0 \text{ by (17)}$$

$$= \tilde{d}_1 \circ \tilde{\delta}_1 \text{ by (19)}$$

$$= \tilde{e}_2 = p(e_2) \circ c_2.$$

This finishes the proof of (13). For the convenience of the reader, we compile the data of the proof of (13) into a diagram. Note that the similar diagrams for (12) and 14) will look similar, but that the α and β will denote something different (whereas the c_i's remain the same).



We now prove that the two processes (of Section 2 and the current part of the present Section) are inverse of each other, up to canonical isomorphism. If we start with coequalizing data (P, ψ), $P : y(G_0) \to \mathbf{X}$ in particular provides a partial cleavage p of $\mathbf{X}$ with codomain X_0 (defined as $p(1_{G_0})$); so the groupoid constructed gives rise to, possibly new, coequalizing data (P', ψ'), whose construction starts out with choosing a partial cleavage $\overline{p}$ with $\overline{p}(1_{G_0}) = X_0 = p(1_{G_0})$. Hence there is a unique isomorphism between them, and the compatibility with ψ means an assertion of equality of two natural transformations with domain $y(G_1)$. From Basic Item 4, it suffices to see agreement on 1_{G_1}, which is easy.

Conversely, if we have a groupoid $X_\bullet$ in $\mathbf{X}$ over $G_\bullet$ in $\mathbf{B}$, and produce coequalizing data, by some partial cleavage p, (with $p(1) = X_0$) then we have the vertical comparisons $c_0 : X_1 \to d_0^*(X_0)1$ and $c_0 : X_2 \to e_0^*(X_0)$; and by the construction, these comparisons are immediately compatible with the face maps, except possibly with the last one $\tilde{d}_1$ and $\tilde{\delta}_2$, whose definition involved ψ, cf. the display in (9). But contemplate the construction of ψ_1 in terms of the groupoid, cf. (11): in that diagram, the comparison α is just the inverse of the comparison c_0, and β similarly for c_1, the unique comparison for $\tilde{d}_1$, so $\psi_1 \circ c_0 = c_1$, and then the desired compatibility is clear. For the compatibility of the δ's, one can utilize that we are dealing with groupoids over the same groupoid $G_\bullet$, and prove the desired equality by post-composition with some suitable (cartesian) arrow $X_1 \to X_0$ (take $\tilde{d}_1$).

4 Dec

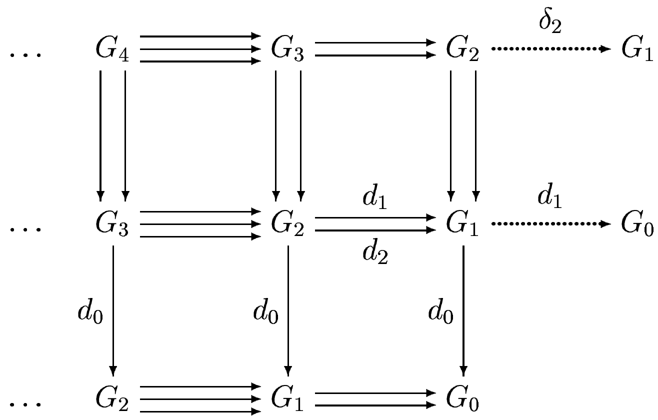
If $G_\bullet$ is a small groupoid (identified with its nerve, which is a simplicial set), a principal $G_\bullet$ bundle is a simplicial set over $p : G_\bullet$, $E_\bullet \to G_\bullet$ such that

1) all the squares, expressing that p commutes with the face- operators, are pull-backs, and

2) $E_\bullet$ is the (nerve of) an equivalence relation, with coequalizer $E_0 \to E_{-1}$, say, called the augmentation.

We say that $E_\bullet$ is a principal $G_\bullet$-bundle *on* E_{-1}. The category of principal $G_\bullet$-bundles, with augmentation, $E_\bullet \to E_{-1}$ as part of the data, form a fibered category over $\mathbf{B}$, $\pi : B(G_\bullet) \to \mathbf{B}$, where $\pi(E_\bullet \to E_{-1}) = E_{-1}$. All arrows in $B(G_\bullet)$ are cartesian (equivalently, the fibres are (large) groupoids.) It is actually even a *stack*, provide the structural maps (face operators) of $G_\bullet$ are $\mathcal{D}$ -epis.

A particular object in $B(G_\bullet)$ is Illusie's $Dec^1(G_\bullet)$, or Dec^1, for short, since $G_\bullet$ will be fixed; it is a principal bundle over G_0, and is given by $Dec^1_n = G_{n+1}$. It is depicted in row number two from below in the diagram



The row about that is called Dec^2, and above that (not depicted) Dec^3, etc. Although there are three maps from Dec^3 to Dec^2, and two maps from Dec^2 to Dec^1, they all compose to give, for each n, exactly one map from

Dec^n to $G_\bullet$. In fact this map makes Dec^n into a principal $G_\bullet$-bundle over G_{n-1} for $n \geq 1$. Altogether, the various Dec^n's fit together into a simplicial object of principal bundles, augmented over the simplicial object $G_\bullet$ in the right hand column. Since all squares in sight are pull-backs, this means that the Dec^n's form a groupoid in $B(G_\bullet)$, over the groupoid $G_\bullet$ in $\mathbf{B}$.

5 The coequalizer

We are now going to make precise in which sense and why $B(G_\bullet)$ is a coequalizer of the groupoid $G_\bullet$. This first of all means that one should specify the 2-category in which things take place; this is the 2-category $\underline{\underline{S}}(\mathbf{B})$, the subcategory of stacks inside the 2-category $\underline{\underline{Fib}}(\mathbf{B})$. Secondly, on should specify the map $q : G_0 \to B(G_\bullet)$, which is to be the "coequalizing map", together with a 2-cell ϕ between $q \circ d_0$ and $q \circ d_1$. The map q is going to be Dec, more precisely, some partial cleavage of $B(G_\bullet)$ with codomain $Dec^1(G_\bullet)$. And q, ϕ is going to be the object in $\underline{Coeq}(G_\bullet, B(G_\bullet))$ which corresponds to the groupoid $Dec^\bullet(G_\bullet)$ over $G_\bullet$ in $B(G_\bullet)$ under the correspondence of Sections 2 and 3.

So consider, a fixed fibration-in-groupoids $\pi : \mathbf{X} \to \mathbf{B}$, and also a fixed ("small") groupoid $G_\bullet$ in $\mathbf{B}$. We have the following categories and functors

$$\underline{Coeq}(G_\bullet, \mathbf{X}) \rightleftarrows \underline{Grpd}(\mathbf{X})/G_\bullet \leftrightarrows \underline{hom}_S(B(G_\bullet), \mathbf{X}) \tag{20}$$

The categories are, respectively, the category of coequalizing data $(p : G_0 \to \mathbf{X}, \psi)$, as explained in Section 2, the category of groupoid objects $X_\bullet$ in $\mathbf{X}$, over $G_\bullet$, and $\underline{hom}_{Fib}(B(G_\bullet), \mathbf{X})$ is the category of (cartesian) functors between fibrations-in-groupoids, over $\mathbf{B}$. All three categories are in fact (large) groupoids.

The functors displayed are all equivalences; the full arrows are explicit, the dotted ones are quasi-inverses, and depend on choice (say, of a partial cleavage); the functor

$$\underline{Grpd}(\mathbf{X})/G_\bullet \dashrightarrow \underline{hom}_{stack}(B(G_\bullet), \mathbf{X}) \tag{21}$$

requires for its construction that $\mathbf{X}$ is a stack. The two functors on the left in (20) are those that have been expounded in the previous sections. The functorality is that pasting with $F : \mathbf{X} \to \mathbf{Y}$ on the left corresponds to applying F on groupoid objects in $\mathbf{X}$. The explicit functor on the right is just "evaluate at $Dec^\bullet$"; for, a functor over $\mathbf{B}$, say $\mathbf{Y} \to \mathbf{X}$, clearly takes groupoid objects over $G_\bullet$ in $\mathbf{Y}$ to groupoid objects over $G_\bullet$ in $\mathbf{X}$. This in particular applies to the groupoid object $Dec^\bullet(G_\bullet)$ in $B(G_\bullet)$.

So the remaining task is to provide the functor (21), provided that $\mathbf{X}$ is a stack, and prove it to be quasi inverse to the evaluation at $Dec^\bullet$.

When this has been carried out, we have the right to assert

Theorem 1 *Let $q : G_0 \to B(G_\bullet), \phi$ be the coequalizing data, corresponding under the left side equivalence of (20) to the groupoid object $Dec^\bullet(G_\bullet)$ in $B(G_\bullet)$. Then for any stack $\mathbf{X}$ over $\mathbf{B}$, pasting with ϕ provides an equivalence*

$$\underline{hom}_S(B(G_\bullet), \mathbf{X}) \longrightarrow \underline{Coeq}(G_\bullet, \mathbf{X}).$$

This is exactly to say that such q, ϕ is a coequalizer in the 2-dimensional sense, of $G_\bullet$, recalling that universal properties 2-dimensionally should be expected to classify "up to equivalence", not "up to isomorphism".

So let us construct a functor (21). Let $X_\bullet$ be a groupoid over $G_\bullet$, in $\mathbf{X}$, assumed to be a stack. To construct its image under the functor (21) means to construct a functor over $\mathbf{B}$,

$$B(G_\bullet) \to \mathbf{X}. \qquad (22)$$

The conctruction is going to involve some choosing of (cartesian) lifts; a partial cleavage $\mathbf{B}/G_0 \to \mathbf{X}/X_0$ will suffice. So we assume given an object $E_\bullet \to E_{-1}$ on the left hand side, i.e., a principal $G_\bullet$-bundle with quotient E_{-1}; so there is in particular a simplicial map $a_\bullet : E_\bullet \to G_\bullet$. For each n, we take a cartesian lift over a_n with codomain X_n, say $\tilde{a}_n : X'_n \to X_n$. (Such lifts can be obtained canonically by comparison with the chosen lift of $d \circ a_n : E_n \to G_n \to G_0$, where $d : G_n \to G_0$ is the composite of a string of d_0's say.) Now by using the cartesian property of the $\tilde{a}_n$'s, and comparing with the simplicial map $E_\bullet \to G_\bullet$, one obtains a series of face operators between the X'_n's, making $X'_\bullet$ into a simplicial object in $\mathbf{X}$ above the groupoid $E_\bullet$. But such data is now precisely descent data (in the sense explit in [4]) for descent along the augmentation $E_0 \to E'_{-1}$, so since $\mathbf{X}$ is a stack, $X'_\bullet$ descends to an object X'_{-1} in $\mathbf{X}_{E_{-1}}$. The process $E_\bullet \mapsto X'_{-1}$ thus described is the requisite functor $B(G_\bullet) \to \mathbf{X}$.

We now prove that the two processes are inverse to each other, up to isomorphism. If we start with a groupoid $X_\bullet$ over $G_\bullet$ in the stack $\mathbf{X}$, and want to evaluate the resulting functor $B(G_\bullet) \to \mathbf{X}$ on $Dec^\bullet(G_\bullet)$. It is easy to see that the value, say in dimension 0, comes about (up to isomorphism) as coequalizer of a particular equivalence relation in $\mathbf{X}$, namely $Dec^1(X_\bullet)$, which is X_0 Similarly, in dimension 1, we end up with the coequalizer of $Dec^2(X_\bullet)$, which is X_1, etc., so we end up with a groupoid isomorphic to the $X_\bullet$ with which we started.

Conversely, let us start with a functor $P : B(G_\bullet) \to \mathbf{X}$, and evaluate it at $Dec^\bullet(G_\bullet)$, so as to get a groupoid $P(Dec^\bullet(G_\bullet))$; by the recipe provided, this groupoid gives rise to a functor $\overline{P} : B(G_\bullet) \to \mathbf{X}$, whose value at a principal $G_\bullet$-bundle $E_\bullet \to E_1$ may be desribed as follows: it amounts to use the stack property of $\mathbf{X}$ to descend a certain equivalence relation in $\mathbf{X}$ along $E_0 \to E_{-1}$ in $\mathbf{B}$, and this equivalence relation is described in terms of its nerve, which is simply

$$\equiv\!\!\!\rightarrow P(Dec^2(E_\bullet)) \rightrightarrows P(Dec^1(E_\bullet)),$$

but since P preserves pull-backs and coequalizers of equivalence relations, this coequalizer is (isomorphic to) $P(E_\bullet)$. (Note that a principal $G_\bullet$-bundle $E_\bullet$ always sits as coequalizer in $B(G_\bullet)$ of $Dec^2(E_\bullet) \rightrightarrows Dec^1(E_\bullet)$.)

Appendix. The faces of a triangle

For a simplicial object $X_\bullet$ in any category, we shall be interested in its lowest dimensional parts,

$$X_2 \equiv\!\!\!\rightarrow X_1 \rightrightarrows X_0. \tag{23}$$

The three face operators $X_2 \to X_1$ we denote δ_0, δ_1 and δ_2, and the two face operators $X_1 \to X_0$, we denote d_0 and d_1. For the calculations, it is also convenient to have names for the three composites $X_2 \to X_0$, we call them e_0, e_1 and e_2, they are defined by the following basic equations

$$\begin{aligned} e_0 &= d_0 \circ \delta_0 = d_0 \circ \delta_1 \\ e_1 &= d_0 \circ \delta_2 = d_1 \circ \delta_0 \\ e_2 &= d_1 \circ \delta_1 = d_1 \circ \delta_2 \end{aligned}$$

For the case where $X_\bullet$ is the nerve of a small category and we consider a 2-simplex x, i.e., a composable pair

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

$\delta_0(x) = f$, $\delta_1(x) = g \circ f$, $\delta_2(x) = g$, and for instance the middle equation can be rendered verbally: *the domain of the second arrow g is the codomain of the first arrow f* – and $e_0(x) = A$, $e_1(x) = B$, $e_2(x) = C$. The commutative square expressed by the middle equation is a pull-back, by definition of "composable pair"; the commutative squares expressed by the two other equations are pull-backs precisely when $X_\bullet$ is a groupoid.

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