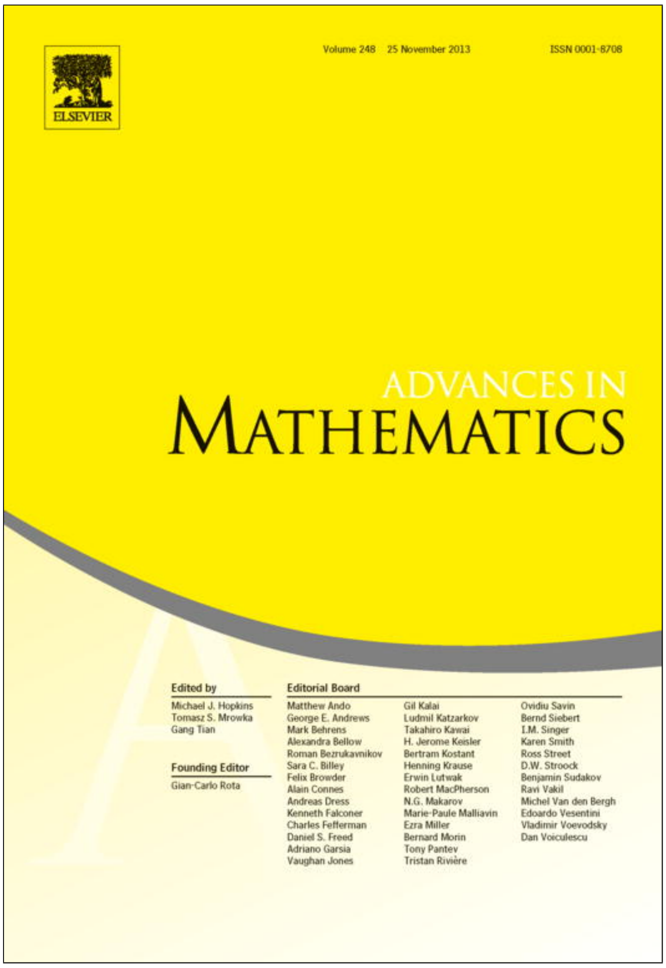
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Absence of embedded eigenvalues for Riemannian Laplacians ☆

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Abstract

In this paper we study absence of embedded eigenvalues for Schrödinger operators on non-compact connected Riemannian manifolds. A principal example is given by a manifold with an end (possibly more than one) in which geodesic coordinates are naturally defined. In this case one of our geometric conditions is a positive lower bound of the second fundamental form of angular submanifolds at infinity inside the end. Another condition is an upper bound of the trace of this quantity, while a third one is a bound of the derivatives of part of the trace (some oscillatory behaviour of the trace is allowed). In addition to geometric bounds we need conditions on the potential, a regularity property of the domain of the Schrödinger operator and the unique continuation property. Examples include ends endowed with asymptotic Euclidean or hyperbolic metrics.

Keywords: Spectral theory; Schrödinger operator; Riemannian geometry

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Contents

1. Introduction and results

Let (M, g) be a non-compact connected Riemannian manifold of dimension $d \geqslant 1$ (possibly incomplete), and H the Schrödinger operator on the Hilbert space $\mathcal{H} = L^2(M)$:

$$H = H_0 + V; \quad H_0 = -\frac{1}{2}\Delta = \frac{1}{2}p_i^* g^{ij} p_j, \;\; p_i = -\mathrm{i}\partial_i.$$

We introduce four conditions under which we prove that a self-adjoint realization of H does not have eigenvalues greater than some computable constant. Our conditions appear rather weak and allow for application to manifolds with boundary (possibly caused by metric or potential singularities). In particular, to our knowledge, they are weaker than conditions used so far in the literature on the subject, cf. e.g. [12,13,3,10,11]. The present work is applied in a companion paper [7] in which scattering theory is studied for a general class of metrics. Our conditions are also weaker than the conditions of [7].

For the Euclidean case (and a particular subclass of potentials) the theory amounts to absence of positive eigenvalues which is a very well studied subject, see e.g. [16,6,8,9]. More precisely we recover then [16, Theorem XIII.58]. On the other hand, it does not cover absence of positive eigenvalues for N-body Schrödinger operators, cf. [5]. The aim of this paper is rather to study absence of embedded (possibly only high energy) eigenvalues of Schrödinger operators in a general geometric framework.

The first condition we impose guarantees intuitively that (M, g) has at least one "expanding end". Recall the definition of the geometric Hessian ∇^2, i.e. for $f \in C^2(M)$ in local coordinates

$$\big(\nabla^2 f\big)_{ij} = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f; \quad \Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}).$$

We denote the gradient vector field for $r \in C^1(M)$ by ∂^r, i.e. $\partial^r f = (\partial_i r)g^{ij}(\partial_j f)$ for $f \in C^1(M)$. For functions $f, r : M \to \mathbb{R}$ we introduce the limits:

$$\liminf_{r\to\infty} f = \lim_{\nu\to\infty}\big(\inf\{f(x) \,\big|\, r(x) \geqslant \nu\}\big),$$
$$\limsup_{r\to\infty} f = \lim_{\nu\to\infty}\big(\sup\{f(x) \,\big|\, r(x) \geqslant \nu\}\big).$$

Condition 1.1. There exist an unbounded real-valued function $r \in C^4(M)$, $r(x) \geqslant 1$, constants $c_1 > c_2 > 0$, and a decomposition as a sum of C^2-functions, $\Delta r^2 = \rho_1 + \rho_2 + \rho_3$, such that:

(1) There exists a constant $r_0 \geqslant 1$ such that, as quadratic forms on TM,

$$\nabla^2 r^2 \geqslant \left(c_1 + \frac{1}{2}\rho_1\right) g \quad \text{and} \quad \rho_1 \geqslant 0 \quad \text{for all } x \in M \quad \text{with } r(x) \geqslant r_0. \tag{1.1a}$$

Moreover

$$\liminf_{r\to\infty}\left(r\partial^r|\mathrm{d}r|^2 + \left(c_2 + \frac{1}{2}\rho_1\right)|\mathrm{d}r|^2\right) > 0, \qquad \limsup_{r\to\infty}|\mathrm{d}r| < \infty. \tag{1.1b}$$

(2) The following bounds hold

$$\limsup_{r\to\infty}\left|r^{-1}\Delta r^2\right| < \infty, \tag{1.2a}$$

$$\limsup_{r\to\infty}\rho_1 < \infty, \qquad \limsup_{r\to\infty}|\mathrm{d}\rho_2| < \infty, \qquad \limsup_{r\to\infty}\Delta\rho_3 < \infty. \tag{1.2b}$$

Most of the above quantities along with the potential in Condition 1.2 below will be used quantitatively, in particular to define a certain energy E_0 (see (1.4a)) above which we will show absence of eigenvalues. The regularity assumption on the metric is implicitly included in Condition 1.1, and we require, at most, C^3. The (highest) third derivatives of g could show up in $\Delta\rho_3$ of (1.2b), because $\Delta = \operatorname{tr}\nabla^2$ and the Christoffel symbols Γ_{ij}^k contain the first derivatives of g. If we choose $\rho_3 \equiv 0$, then the metric may be C^2, but this might give a worse critical energy in (1.4a). Note that the subsets $\{x \in M |\ r(x) \leqslant \tilde{r}\}$, $\tilde{r} \geqslant 1$, may not be compact (this is similar to [10,11], see Section 2.2). The function r could model a distance function within a fixed single *end* of M extended to be bounded outside, in particular bounded in other ends of M. Note that for an exact distance function (1.1b) is trivially fulfilled (for any $c_2 > 0$), and in that case the above operator ∂^r is identified as the geodesic radial derivative ∂_r, see Section 2.2. Also note that (1.1a) implies the convexity $\nabla^2 r^2 \geqslant c_1 g > 0$ for $r \geqslant r_0$, and that (1.1b) imposes further lower boundedness for the $\mathrm{d}r \otimes \mathrm{d}r$ component since

$$\left(\nabla^2 r^2\right)^{ij}(\partial_i r)(\partial_j r) = 2|\mathrm{d}r|^4 + 2r\left(\nabla^2 r\right)^{ij}(\partial_i r)(\partial_j r) = 2|\mathrm{d}r|^4 + r\partial^r|\mathrm{d}r|^2.$$

In particular the geodesics in this region are non-trapped, more precisely $r^2 \geqslant ct^2$ for $t \to \infty$. Another immediate consequence is the lower bound $\Delta r^2 \geqslant c_1 d$ for $r \geqslant r_0$. Finally it is worth noting that one could think about ρ_1 as a small oscillatory function. This is motivated by examples, see the discussion before Corollary 2.4.

Condition 1.2. There exists a decomposition $V = V_1 + V_2$, $V_1 \in L^2_{\mathrm{loc}}(M)$, $V_2 \in C^1(M)$ and V_1, V_2 real-valued, such that uniformly in $x \in M$:

$$\limsup_{r\to\infty}|V| < \infty, \qquad \limsup_{r\to\infty} r|V_1| < \infty, \qquad \limsup_{r\to\infty} r\partial^r V_2 < \infty. \tag{1.3}$$

The decomposition of Δr^2 in Condition 1.1 as well as that of V in Condition 1.2 represent a trade-off relation between regularity and decaying properties for perturbations. Note that under Condition 1.2 the subspace $C_c^\infty(M) \subseteq \mathcal{D}(V)$ and hence H is defined at least on $C_c^\infty(M)$. However under Conditions 1.1 and 1.2 this operator is not necessarily essentially self-adjoint. Note that (M, g) is allowed to be incomplete and that V is allowed to be unbounded. For instance (M, g) could be the interior of a Riemannian manifold with boundary and for essential self-adjointness we would then need a symmetric boundary condition. Lack of essential self-adjointness could also originate from unboundedness of V in some end. To fix a self-adjoint extension we first choose a non-negative $\chi \in C^\infty(\mathbb{R})$ with

$$\chi(r) = \begin{cases} 0 & \text{for } r \leqslant 1, \\ 1 & \text{for } r \geqslant 2, \end{cases}$$

and then set

$$\chi_\nu(r) = \chi(r/\nu), \quad \nu \geqslant 1.$$

We shall henceforth consider the function χ_ν as being composed with the function r from Condition 1.1. In this sense particularly $\chi_\nu \in C^4(M)$.

Condition 1.3. The operator H defined on $C_c^\infty(M)$ (by Condition 1.2) has a self-adjoint extension, denoted by H again, such that for any $\psi \in \mathcal{D}(H)$ there exists a sequence $\psi_n \in C_c^\infty(M)$ such that for all large $\nu \geqslant 1$

$$\|\chi_\nu(\psi - \psi_n)\| + \|\chi_\nu(H\psi - H\psi_n)\| \to 0 \quad \text{as } n \to \infty.$$

Note that Condition 1.3 is fulfilled if (M, g) is complete and V is bounded. In that case indeed H is essentially self-adjoint on $C_c^\infty(M)$, see Proposition 2.1 for a more general result.

As a global condition we impose for this self-adjoint extension *the unique continuation property*.

Condition 1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E \in \mathbb{R}$, and $\phi(x) = 0$ in some open subset, then $\phi(x) = 0$ in M.

In Section 2 we shall discuss various models satisfying Conditions 1.1–1.4.

We define a "critical" energy,

$$E_0 = \inf_{c \in (0, c_1 - c_2]} \limsup_{r \to \infty} \left(V + \frac{|\beta|^2 - c\gamma}{2c\alpha_c} \right); \tag{1.4a}$$

$$\alpha_c = c_1 - c + \frac{1}{2}\rho_1, \tag{1.4b}$$

$$\beta = \frac{1}{4}\,\mathrm{d}\rho_2 + V_1\,\mathrm{d}r^2, \tag{1.4c}$$

$$\gamma = -\frac{1}{4}\Delta\rho_3 + (\Delta r^2)V_1 - 2r\partial^r V_2. \tag{1.4d}$$

For some examples in Section 2.2 (where for simplicity $V = 0$) we compute that the essential spectrum $\sigma_{\mathrm{ess}}(H_0) = [E_0, \infty)$, see Examples 2.2 and Remark 2.3(1). Whence for these examples indeed E_0 is critical regarding absence of eigenvalues as stated more generally in the following theorem.

Theorem 1.5. *Suppose Conditions 1.1–1.4. Then the eigenvalues of H are absent above E_0, i.e. $\sigma_{\mathrm{pp}}(H) \cap (E_0, \infty) = \emptyset$.*

Under the above conditions embedded eigenvalues can occur. It is well known in Schrödinger operator theory that the von Neumann–Wigner potential, see for example [5] or [16, Section XIII.3], provides an example of a positive eigenvalue for a decaying potential $O(r^{-1})$, $r = |x|$. Whence the conclusion of Theorem 1.5 is in general false above the bottom of the essential spectrum. An example of a Laplace–Beltrami operator having an embedded eigenvalue is constructed in [10]. This is for a hyperbolic metric, and the example shows similarly that the conclusion of Theorem 1.5 in general is false above the bottom of the essential spectrum, see also Remark 2.5(1). (Actually Kumura uses the von Neumann–Wigner potential in his construction.)

The proof of Theorem 1.5 follows the scheme of [6,5,2,14] employing in particular a Mourre-type commutator estimate and super-exponential decay estimates of a priori eigenstates. In our geometric setting the "Mourre commutator" can be very singular (in particular not bounded relatively to H in any usual sense). Consequently we only have a weak (however sufficient) version of the commutator estimate, see Lemmas 3.3, 4.1, 5.4 and 5.5.

Aside from Theorem 1.5 itself we also generalize [17] showing absence of super-exponentially decaying eigenstates under somewhat weaker conditions than those of Theorem 1.5 as well as those of [17], see Section 5. Our proof is in the spirit of the well known Carleman estimate, cf. [8,9,16,17].

We use throughout the paper the standard notation $\langle\sigma\rangle = (1 + |\sigma|^2)^{1/2}$ and (as above) d for exterior differentiation (acting on functions on M). Note that in local coordinates $p := -\mathrm{id}$ takes the form $p = (p_1, \ldots, p_d)$. We shall slightly abuse notation writing for example $p\psi \in \mathcal{H} = L^2(M)$ for $\psi \in C_c^\infty(M)$ even though the correct meaning here is a section of the (complexified) cotangent bundle, i.e. $p\psi \in \Gamma(T^*M)$. Note at this point that $\|p\psi\| := \|p\psi\|_{\Gamma(T^*M)} = \||p\psi|\|_{\mathcal{H}}$. If A is an operator on $\mathcal{H}$ and $\psi \in \mathcal{D}(A)$ we denote the expectation $\langle\psi, A\psi\rangle$ by $\langle A\rangle_\psi$. Unimportant positive constants are denoted by C, in particular C may vary from occurrence to occurrence. The dependence on other variables is sometimes indicated by subscripts such as C_ν.

2. Discussion and examples

In this section we investigate how general our conditions are by looking at several examples.

2.1. Global conditions

We recall some general criteria for self-adjointness and the unique continuation property.

Proposition 2.1. *Let (M, g) be a complete Riemannian manifold of dimension $d \geqslant 1$. Then the free Schrödinger operator H_0 is essentially self-adjoint on $C_c^\infty(M)$. Suppose V is real-valued, measurable, bounded outside a compact set and in addition: $V \in L^2_{\mathrm{loc}}(M)$ for $d = 1, 2, 3$, $V \in L^p_{\mathrm{loc}}(M)$ for some $p > 2$ if $d = 4$ while $V \in L^{d/2}_{\mathrm{loc}}(M)$ for $d \geqslant 5$. Then V is infinitesimally relatively small. In particular H is essentially self-adjoint on $C_c^\infty(M)$.*

We refer to [1] and [15, Theorems X.20 and X.21]. We can generalize the class of potentials to the Stummel class, see e.g. [4].

As for the unique continuation property, Condition 1.4, there is an extensive literature although mostly for Schrödinger operator theory, see e.g. [8]. For general connected manifolds we refer to [18] and references therein, quoting here the following sufficient conditions supplementing connectivity and the conditions in Proposition 2.1: 1) $d = 2, 3, 4$ and V is globally bounded, or 2) $d \geqslant 5$. One could (of course) add 3) $d = 1$.

2.2. Conditions inside an end

In the sequel we consider a connected and complete (M, g) of dimension $d \geqslant 2$ and take (for simplicity) $V = 0$. We shall examine the meaning of Condition 1.1 in the case where, in addition, (M, g) has the following explicit *end* structure: There exists an open subset $E \subset M$ such that isometrically the closure $\bar{E} \cong [0, \infty) \times S$ for some $(d-1)$-dimensional manifold S where the metric on $[0, \infty) \times S$ has the form

$$g = g_{rr}\,\mathrm{d}r \otimes \mathrm{d}r + g_{\alpha\beta}\,\mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta; \quad g_{rr} = 1,\ g_{r\alpha} = g_{\alpha r} = 0,\ g_{\alpha\beta} = g_{\alpha\beta}(r, \sigma). \tag{2.1}$$

Here $(r, \sigma) \in [0, \infty) \times S$ denotes local coordinates and the Greek indices run over $2, \dots, d$. Whence actually r is globally defined in E and it is a smooth distance function (here given as the distance to $\{0\} \times S$). In particular we have $|\mathrm{d}r| = 1$ which obviously implies (1.1b) for any $c_2 > 0$. Notice here that Condition 1.1 involves only the part of the function r at large values, so in agreement with Condition 1.1 we can cut and extend it to a smooth function on M obeying $r \geqslant 1$. This is tacitly understood below. To examine the remaining statements (1.1a), (1.2a) and (1.2b) of Condition 1.1 we compute

$$\nabla^2 r^2 = 2\,\mathrm{d}r \otimes \mathrm{d}r + r(\partial_r g_{\alpha\beta})\,\mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta, \tag{2.2a}$$

$$\Delta r^2 = g^{ij}\left(\nabla^2 r^2\right)_{ij} = 2 + r g^{\alpha\beta}(\partial_r g_{\alpha\beta}). \tag{2.2b}$$

2.2.1. End of warped product type

If we consider the *warped product* case where $g_{\alpha\beta}(r, \sigma) = f(r) h_{\alpha\beta}(\sigma)$ we obtain, using (2.2a) and (2.2b), the following examples fulfilling also (1.1a), (1.2a) and (1.2b) of Condition 1.1.

Example 2.2.

(1) Let $f = r^{2a}$ with $a > 0$. Then (1.1a), (1.2a) and (1.2b) hold with $c_1 = \min\{2, 2a\}$ and $\rho_1 = 0$, $\rho_2 = 2 + 2a(d-1)$, $\rho_3 = 0$, and the critical energy $E_0 = 0$.
(2) Let $f = \exp(2\kappa r^q)$ with $\kappa > 0$ and $q \in (0, 1)$. Then (1.1a), (1.2a) and (1.2b) hold with $c_1 = 2$ and $\rho_1 = 0$, $\rho_2 = 2 + 2\kappa q(d-1) r^q$, $\rho_3 = 0$, and $E_0 = 0$.
(3) Let $f = \exp(2\kappa r)$ with $\kappa > 0$. Then (1.1a), (1.2a) and (1.2b) hold with $c_1 = 2$ and $\rho_1 = 0$, $\rho_2 = 2 + 2\kappa(d-1)r$, $\rho_3 = 0$, and $E_0 = \kappa^2(d-1)^2/8$.

Remark 2.3.

(1) For all of these examples it is easy to compute that the essential spectrum $\sigma_{\mathrm{ess}}(H_0) \supseteq [E_0, \infty)$. If in addition $M \setminus E$ and S are compact then we have $\sigma_{\mathrm{ess}}(H_0) = [E_0, \infty)$. Whence indeed the absence of eigenvalues in (E_0, ∞) as stated in Theorem 1.5 is optimal under these additional conditions for the above examples (except possibly that the threshold energy $E = E_0$ in a concrete situation might not be an eigenvalue either).
(2) It is not required in Condition 1.1 that r is an exact distance function so we may still have this condition fulfilled in perturbed situations (letting r be the unperturbed distance function). This is also the spirit of [3,12,13] where (roughly) perturbations of the Euclidean metric (corresponding to $a = 1$ in (1)) are studied. The authors show absence of positive eigenvalues for these models. More generally, but roughly still in the framework of perturbations of (1), absence of embedded eigenvalues was obtained in [11], and for hyperbolic models (roughly for perturbations of (3)) it was done in [10]. However Kumura's results are stated in terms of

an exact distance function and all results involve conditions on a radial curvature (possibly including here the radial Ricci curvature). Whence his framework is seemingly somewhat different. It turns out, however, that some of Kumura's conditions appear too strong, in particular curvature conditions are not needed. In Corollary 2.4 below we state a simplified and extended result, see Remark 2.5(2) for further discussion and Corollary 2.6 for an application recovering a main result of [10].

(3) Under the condition of warped product metrics growth rates between $f = r^{2a}$ with $a > 1/2$ and $f = \exp(2\kappa r^q)$ with $\kappa > 0$ and $q \in (0, 1/2)$ define a class of metrics for which the scattering theory [7] applies. More generally Conditions 1.1–1.4 are weaker than the conditions used in [7].

2.2.2. *Volume growth and curvature*

Here we describe the meaning of Condition 1.1 in terms of geometric quantities, and then relate the critical energy E_0 to them. We continue to assume (2.1) in the end E although without warped product structure.

Suppose Condition 1.1 (note that $c_1 \leqslant 2$ is necessary). Then, by (1.1a)

$$\nabla^2 r^2 \geqslant c_1 g \quad \text{for } r \geqslant r_0. \tag{2.3}$$

In the coordinates $(r, \sigma) \in [0, \infty) \times S$ used in (2.1) we have (2.2a), so that the inequality (2.3) is equivalent to

$$(r\partial_r g_{\alpha\beta} - c_1 g_{\alpha\beta})_{\alpha,\beta} \geqslant 0 \quad \text{for } r \geqslant r_0. \tag{2.4}$$

Hence, (1.1a) implies that the induced metric on the angular manifold $S_{\tilde r} = \{x \in \bar E \mid r = \tilde r\}$ grows as a function of $\tilde r$, and $c_1 \in (0, 2]$ gives a lower growth rate of the metric depending on directions. On the other hand, since we have

$$\Delta r^2 = 2 + 2r\Delta r, \qquad \Delta r = \partial_r \ln \sqrt{\det g},$$

and we can measure the volume growth in the radial direction in terms of $\partial_r \ln \sqrt{\det g}$, the bounds (1.2a) and (1.2b) yield an upper bound for the volume growth. We note that, by taking the trace of (2.4),

$$2r\Delta r \geqslant c_1(d-1) \quad \text{for } r \geqslant r_0,$$

and this implies that the volume has to grow, at least.

Next we assume the "lower metric growth rate" (2.3) with c_1 there replaced by some $\tilde c_1 \in (0, 2]$. Assume also the existence of "asymptotic volume growth rate": There exist constants ρ_0, $\omega_\pm$ such that for large $r \geqslant 1$

$$\Delta r = \rho_0 + \frac{\omega(r)}{r}; \qquad \rho_0 \geqslant 0,\ \omega_- \leqslant \omega \leqslant \omega_+,\ \omega_+ - \omega_- < \tilde c_1.$$

Then we can verify Condition 1.1: By setting

$$\rho_1 = 2\omega - 2\omega_-, \qquad \rho_2 = 0, \qquad \rho_3 = 2 + 2\omega_- + 2r\rho_0,$$

and choosing $c_1 = \tilde c_1 - (\omega_+ - \omega_-)$ and sufficiently small $c_2 > 0$ indeed Condition 1.1 is fulfilled. Hence we can estimate $\inf\{E_0 \mid \Delta r^2 = \rho_1 + \rho_2 + \rho_3\}$ in terms of the lower metric growth rate and the volume growth rate:

$$\inf\{E_0 \bigm| \Delta r^2 = \rho_1 + \rho_2 + \rho_3\} \leqslant \rho_0^2/(4c_1) = \rho_0^2/\big(4(\tilde c_1 - \omega_+ + \omega_-)\big). \tag{2.5}$$

Note that above $\tilde{c}_1$ is taken as a bound of the amplitude of oscillation allowed in Δr^2 (i.e. a bound of the term $\rho_1 = 2\omega - 2\omega_-$ with allowed "bad" derivatives). However, also note that in general $\tilde{c}_1$ is just a rough bound because there can be some cancellation in $\nabla^2 r^2 - \frac{1}{2}\rho_1 g$ (an example of this occurs in Corollary 2.4 below).

Now we recover and extend various results of [10,11].

Corollary 2.4. *Suppose (M, g) is connected and complete having an end E with metric of the form (2.1). Suppose there exist $\kappa \geqslant 0$ and real numbers $a \leqslant b$, $a > 0$ if $\kappa = 0$, such that for large $r \geqslant 1$*

$$\left(\kappa + \frac{a}{r}\right)(g - \mathrm{d}r \otimes \mathrm{d}r) \leqslant \nabla^2 r_{|S_r} \leqslant \left(\kappa + \frac{b}{r}\right)(g - \mathrm{d}r \otimes \mathrm{d}r), \tag{2.6}$$

and that

$$A := (d-1)(b-a) < B, \quad B := \begin{cases} \min\{2, a+b\} & \text{if } \kappa = 0, \\ 2 & \text{if } \kappa > 0. \end{cases}$$

Then

$$\sigma_{\mathrm{pp}}(H_0) \cap \left(\frac{\kappa^2}{4}(d-1)^2/(2-A), \infty\right) = \emptyset. \tag{2.7}$$

Proof. Taking the trace of (2.6), we define ω by

$$\Delta r = \kappa(d-1) + \frac{\omega(r)}{r},$$

and set

$$\rho_1 = 2\omega - 2a(d-1), \qquad \rho_2 = 0, \qquad \rho_3 = 2 + 2a(d-1) + 2r\kappa(d-1).$$

Then, noting for $\kappa = 0$ that there is cancellation of the smallest eigenvalue of $\nabla^2 r|_{S_r}$, we obtain with $c_1 := B - (d-1)(b-a)$

$$\nabla^2 r^2 - \frac{1}{2}\rho_1 g \geqslant c_1 g, \quad \rho_1 \geqslant 0.$$

Thus the result follows by applying Theorem 1.5. □

Remark 2.5.

(1) In [10] Kumura constructed an example, fulfilling the conditions of Corollary 2.4 with $\kappa > 0$, for which $\sigma_{\mathrm{ess}}(H_0) = [\frac{\kappa^2}{8}(d-1)^2, \infty)$ and $\sigma_{\mathrm{pp}}(H_0) \cap (\frac{\kappa^2}{8}(d-1)^2, \infty) \neq \emptyset$. Whence in such case (2.7) is an upper bound of the set of embedded eigenvalues. Clearly, as a general feature, the bound is better the smaller $A \geqslant 0$ can be chosen. In the extreme case, imagining here the quantities a and b being depending on r, where $\liminf a = \limsup b \in \mathbb{R}$ we get an even better bound. We give below an application of Corollary 2.4 to this situation stated in terms of the radial curvature, cf. [10, Theorems 1.4 and 1.7]. Note that the radial curvature can control the second fundamental form $\nabla^2 r_{|S_r}$ by a standard comparison argument, see e.g. [7, Remark 1.13] for a reference.

(2) The bound (1.1a) and parts of the bounds (1.2a) and (1.2b) may be viewed as bounds on the minimal and the mean curvatures of S_r, respectively, whereas (2.6) certainly is a uniform asymptotic result for all the principal curvatures.

Corollary 2.6. *Suppose (M, g) is connected and complete having an end E with metric of the form (2.1). Suppose there exists $\kappa > 0$ such that the radial curvature R_{rad}, defined in local coordinates by $(R_{\mathrm{rad}})_{ij} = (\partial_k r)(\partial_l r)R^k{}_i{}^l{}_j$, satisfies*

$$R_{\mathrm{rad}} = -\left(\kappa^2 + o\left(\frac{1}{r}\right)\right)g \quad \text{on } S_r \text{ (uniformly in } x \in E),$$

and there exists $r_1 \geqslant 0$ such that

$$R_{\mathrm{rad}} \leqslant 0 \quad \text{on } S_{\tilde r} \text{ for all } \tilde r \geqslant r_1 \quad \text{and} \quad \nabla^2 r \geqslant 0 \quad \text{on } S_{r_1}.$$

Then

$$\sigma_{\mathrm{pp}}(H_0) \cap \left(\kappa^2(d-1)^2/8, \infty\right) = \emptyset.$$

We note that although the radial curvatures R_{rad} and K_{rad} of [7] and [10,11], respectively, are different objects they contain equivalent information. Whence in fact the results Corollary 2.6 and [10, Theorem 1.4] (almost) coincide.

3. Preliminaries

The proof of Theorem 1.5 begins from this section. Obviously, Theorem 1.5 is a consequence of the following two propositions, a priori super-exponential decay estimates for eigenfunctions and the absence of super-exponentially decaying eigenfunctions.

Proposition 3.1. *Suppose Conditions 1.1–1.3. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E > E_0$, then $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$ for any $\sigma \geqslant 0$.*

Proposition 3.2. *Suppose Conditions 1.1–1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E \in \mathbb{R}$ and $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$ for any $\sigma \geqslant 0$, then $\phi(x) = 0$ in M.*

We shall prove Propositions 3.1 and 3.2 in Sections 4 and 5, respectively. In fact we prove a little generalized version of Proposition 3.2. This generalized version recovers the result of [17], see a short discussion in Section 5. The rest of the present section is devoted to preliminary steps for the proofs: We derive a commutator formula which will be our substitute for the so-called Mourre commutator, and we rewrite Condition 1.3 in a more practical form.

3.1. Mourre-type commutator

We shall use the Mourre-type commutator with respect to the "conjugate operator"

$$A = \mathrm{i}\left[H_0, r^2\right] = \frac{1}{2}\left\{\left(\partial_i r^2\right)g^{ij}p_j + p_i^* g^{ij}\left(\partial_j r^2\right)\right\} = rp^r + \left(p^r\right)^* r; \quad p^r = -\mathrm{i}\partial^r.$$

While not necessarily being self-adjoint this operator is certainly symmetric as defined on $C_c^\infty(M)$, and that suffices for our applications.

Lemma 3.3. *As a quadratic form on $C_c^\infty(M)$,*

$$\mathrm{i}[H, A] = p_i^*\left(\nabla^2 r^2 - \alpha_c g\right)^{ij} p_j + 2\,\mathrm{Re}(\alpha_c H_0) - 2\,\mathrm{Im}\left(\beta^i p_i\right) + \gamma,$$

where α_c, β, γ are defined by (1.4b), (1.4c), (1.4d), respectively.

Proof. We note the commutator formulas, valid for any $\phi \in C^\infty(M)$,

$$-\big[H_0, [H_0, \phi]\big] = p_i^*\big(\nabla^2\phi\big)^{ij} p_j - \frac{1}{4}\big(\Delta^2\phi\big), \tag{3.1a}$$

$$p_i^*\phi g^{ij} p_j = \phi H_0 + H_0\phi + \frac{1}{2}(\Delta\phi), \tag{3.1b}$$

$$0 = \mathrm{i}\big\{(\partial_i\phi)g^{ij} p_j - p_i^* g^{ij}(\partial_j\phi)\big\} + (\Delta\phi). \tag{3.1c}$$

As for (3.1a) we refer to [3, Lemma 2.5] or [7, Corollary 4.2]. The lemma follows by first using (3.1a) with $\phi = r^2$ and then using (3.1b) and (3.1c) with $\phi = \alpha_c$ and $\phi = \frac{1}{4}\rho_2$, respectively. □

3.2. Approximate sequences

To implement Condition 1.3 efficiently we need to strengthen the stated approximation property in $\mathcal{H} = L^2(M)$ under some additional conditions.

Lemma 3.4. *Suppose the second bound of (1.1b), (1.2a), the first bound of (1.3) and Condition 1.3. Let $\psi \in \mathcal{D}(H)$. There exists $\nu_0 \geqslant 1$ such that for $\nu \geqslant \nu_0$ and for any $\sigma \geqslant 0$ such that $\mathrm{e}^{\sigma r}\psi, \mathrm{e}^{\sigma r}H\psi \in \mathcal{H}$ the following properties hold: The states $\chi_\nu \mathrm{e}^{\sigma r}p\psi, \mathrm{e}^{\sigma r}p\chi_\nu\psi \in \mathcal{H}$ and there exists a sequence $\psi_n \in C_{\mathrm{c}}^\infty(M)$ (possibly depending on σ) such that as $n \to \infty$*

$$\big\|\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_n)\big\| + \big\|\chi_\nu \mathrm{e}^{\sigma r}(p\psi - p\psi_n)\big\| + \big\|\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_n)\big\| \to 0. \tag{3.2}$$

Proof. *Step I.* Note the distributional identity

$$\chi_\nu \mathrm{e}^{\sigma r} p\psi = \mathrm{e}^{\sigma r} p\chi_\nu\psi + \mathrm{i}\mathrm{e}^{\sigma r}\psi\chi_\nu' \,\mathrm{d}r.$$

Applied to the given ψ we see that $\chi_\nu \mathrm{e}^{\sigma r}p\psi \in \mathcal{H}$ if and only if $\mathrm{e}^{\sigma r}p\chi_\nu\psi \in \mathcal{H}$.

Step II. We claim that there exists $C > 0$ such that, if $\nu \geqslant 1$ is large, then for any $\psi \in C_{\mathrm{c}}^\infty(M)$ and $\sigma \geqslant 0$

$$\big\|\chi_\nu \mathrm{e}^{\sigma r}|p\psi|\big\|^2 \leqslant \big\|\chi_\nu \mathrm{e}^{\sigma r}H\psi\big\|^2 + C\langle\sigma\rangle^2\big\|\chi_{\nu/2}\mathrm{e}^{\sigma r}\psi\big\|^2. \tag{3.3}$$

In fact by (3.1b)

$$\begin{aligned}\big\|\chi_\nu \mathrm{e}^{\sigma r}|p\psi|\big\|^2 &= 2\,\mathrm{Re}\big\langle\chi_\nu \mathrm{e}^{\sigma r}\psi, \chi_\nu \mathrm{e}^{\sigma r}H\psi\big\rangle + \frac{1}{2}\big\langle\psi, \big(\Delta\chi_\nu^2 \mathrm{e}^{2\sigma r}\big)\psi\big\rangle - 2\big\langle\chi_\nu \mathrm{e}^{\sigma r}\psi, V\chi_\nu \mathrm{e}^{\sigma r}\psi\big\rangle \\ &\leqslant \big\|\chi_\nu \mathrm{e}^{\sigma r}H\psi\big\|^2 + C\langle\sigma\rangle^2\big\|\chi_{\nu/2}\mathrm{e}^{\sigma r}\psi\big\|^2.\end{aligned}$$

Here we used the second bound of (1.1b), (1.2a) and the following consequence

$$|\Delta r| = \frac{1}{2r}\big|\Delta r^2 - 2|\mathrm{d}r|^2\big| \leqslant C \quad \text{for } r = r(x) \text{ large.} \tag{3.4}$$

Step III. We consider the case $\sigma = 0$, and hence suppose only $\psi \in \mathcal{D}(H)$. Let $\psi_n \in C_{\mathrm{c}}^\infty(M)$ and large $\nu \geqslant 1$ be as in Condition 1.3. Then, regarding (3.2), it suffices to consider the middle term. By (3.3) we have

$$\big\|\chi_\nu(p\psi_n - p\psi_{n'})\big\|^2 \leqslant C\big(\big\|\chi_\nu(H\psi_n - H\psi_{n'})\big\|^2 + \big\|\chi_{\nu/2}(\psi_n - \psi_{n'})\big\|^2\big).$$

This implies $\chi_\nu p\psi_n$ converges strongly. Since also $\chi_\nu p\psi_n$ converges in distributional sense to $\chi_\nu p\psi$, we obtain that the limit $\chi_\nu p\psi \in \mathcal{H}$ and then in turn, by letting $n' \to \infty$ above, (3.2) for $\sigma = 0$.

Step IV. We let $\sigma > 0$ and suppose $\mathrm{e}^{\sigma r}\psi, \mathrm{e}^{\sigma r}H\psi \in \mathcal{H}$. Choose $\psi_n \in C_{\mathrm{c}}^\infty(M)$ and large $\nu \geqslant 1$ as in Condition 1.3, again. As for the first and the third terms of (3.2), we compute as follows: Put $\psi_{n,\nu'} = \bar{\chi}_{\nu'}\psi_n$ for $\nu' \geqslant 2\nu$ with $\bar{\chi}_{\nu'} := 1 - \chi_{\nu'}$. Then we decompose

$$\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_{n,\nu'}) = \bar{\chi}_{\nu'}\mathrm{e}^{\sigma r}\chi_\nu(\psi - \psi_n) + \chi_{\nu'}\mathrm{e}^{\sigma r}\psi. \tag{3.5}$$

We put

$$R_{\nu'} = \mathrm{i}[H, \chi_{\nu'}] = \frac{1}{2}\big(\chi'_{\nu'}p^r + \big(p^r\big)^*\chi'_{\nu'}\big) = \chi'_{\nu'}p^r - \frac{\mathrm{i}}{2}\big(\chi''_{\nu'}|\mathrm{d}r|^2 + \chi'_{\nu'}\Delta r\big), \tag{3.6}$$

and decompose similarly

$$\begin{aligned}&\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_{n,\nu'})\\ &\quad = \bar{\chi}_{\nu'}\mathrm{e}^{\sigma r}\chi_\nu(H\psi - H\psi_n) + \chi_{\nu'}\mathrm{e}^{\sigma r}H\psi + \mathrm{i}\mathrm{e}^{\sigma r}R_{\nu'}(\psi - \psi_n) - \mathrm{i}\mathrm{e}^{\sigma r}R_{\nu'}\psi.\end{aligned} \tag{3.7}$$

The norm of the right-hand side of (3.5) can be arbitrarily small by first letting ν' be large and then n large accordingly (using that $\bar{\chi}_{\nu'}\mathrm{e}^{\sigma r}$ is bounded). Similarly the norm of first three terms on the right-hand side of (3.7) can be arbitrarily small by first letting ν' be large and then n large accordingly (for the third term we use Step III, i.e. (3.2) with $\sigma = 0$). It remains to consider the last term on the right-hand side of (3.7). We claim that

$$\big\|\mathrm{e}^{\sigma r}R_{\nu'}\psi\big\| \leqslant C/\nu'. \tag{3.8}$$

To show this we use again Step III to write

$$\big\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi\big\|^2 = \lim_{m\to\infty}\big\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi_m\big\|^2.$$

On the other hand, by the derivation of (3.3)

$$\big\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi_m\big\|^2 \leqslant C\bigg(\big\|\chi'_{\nu'}\mathrm{e}^{\sigma r}H\psi_m\big\|^2 + \bigg(\frac{\langle\sigma\rangle}{\nu'}\bigg)^2\big\|\chi_{\nu/2}\bar{\chi}_{2\nu'}\mathrm{e}^{\sigma r}\psi_m\big\|^2\bigg),$$

and hence we conclude by taking the limit that

$$\begin{aligned}\big\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi\big\|^2 &\leqslant \bigg(\frac{C_\sigma}{\nu'}\bigg)^2\big(\big\|\chi_\nu\bar{\chi}_{2\nu'}\mathrm{e}^{\sigma r}H\psi\big\|^2 + \big\|\chi_{\nu/2}\bar{\chi}_{2\nu'}\mathrm{e}^{\sigma r}\psi\big\|^2\big)\\ &\leqslant \bigg(\frac{C_\sigma}{\nu'}\bigg)^2\big(\big\|\mathrm{e}^{\sigma r}H\psi\big\|^2 + \big\|\mathrm{e}^{\sigma r}\psi\big\|^2\big).\end{aligned} \tag{3.9}$$

A consequence of (3.9) is indeed (3.8), and whence in turn also the last term on the right-hand side of (3.7) is small for ν' sufficiently large.

We conclude that there exists a sequence of indices $(\nu'(m), n(m))$ so that with $\psi_m := \psi_{n(m),\nu'(m)}$ (here and henceforth slightly abusing notation)

$$\big\|\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_m)\big\| + \big\|\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_m)\big\| \to 0.$$

In particular, using here (3.3), the right-hand side of

$$\big\|\chi_{2\nu}\mathrm{e}^{\sigma r}p(\psi_n - \psi_{n'})\big\|^2 \leqslant C\big(\big\|\chi_{2\nu}\mathrm{e}^{\sigma r}H(\psi_n - \psi_{n'})\big\|^2 + \big\|\chi_\nu \mathrm{e}^{\sigma r}(\psi_n - \psi_{n'})\big\|^2\big)$$

is small for $n, n' \to \infty$. We can from this point mimic the last part of Step III. □

4. A priori super-exponential decay estimates of eigenstates

Now we prove Proposition 3.1. Suppose Conditions 1.1–1.3 from this point. We introduce the regularized weights

$$\Theta(r) = \Theta_m^{\sigma,\delta}(r) = \sigma r + \delta r\left(1 + \frac{r}{m}\right)^{-1}$$

for $\sigma, \delta \geqslant 0$ and $m \geqslant 1$, and denote the first and the second derivatives in r by

$$\Theta' = \sigma + \delta\left(1 + \frac{r}{m}\right)^{-2}, \qquad \Theta'' = -\frac{2\delta}{m}\left(1 + \frac{r}{m}\right)^{-3}. \tag{4.1}$$

Set

$$H_\Theta = \mathrm{e}^{\Theta} H \mathrm{e}^{-\Theta} = H - \frac{1}{2}|\mathrm{d}\Theta|^2 + \mathrm{i}\,\mathrm{Re}\, p^{\Theta}; \quad p^{\Theta} = (\partial_i \Theta) g^{ij} p_j = \Theta' p^r. \tag{4.2}$$

We shall consider H_Θ as an operator defined on $C_c^\infty(M)$ only.

Lemma 4.1. *Let $E > E_0$ and $\sigma_0 \geqslant 0$. Then there exist $c \in (0, c_1 - c_2]$, $\epsilon > 0$ and $\delta_0 > 0$ such that for large $\nu \geqslant 1$ and for all $\sigma \in [0, \sigma_0]$, $\delta \in (0, \delta_0]$ and $m \geqslant 1$, as quadratic forms on $C_c^\infty(M)$,*

$$\chi_\nu\{2\,\mathrm{Im}(A(H_\Theta - E))\}\chi_\nu \geqslant \chi_\nu\{\epsilon + 2\,\mathrm{Re}(\alpha_c(H_\Theta - E))\}\chi_\nu. \tag{4.3}$$

Proof. By (4.2) and Lemma 3.3 we obtain

$$\begin{aligned} &2\,\mathrm{Im}(A(H_\Theta - E)) \\ &\quad = 2\,\mathrm{Re}(\alpha_c(H_\Theta - E)) + 2\alpha_c(E - V) + p_i^*(\nabla^2 r^2 - \alpha_c g)^{ij} p_j - 2\,\mathrm{Im}(\beta^i p_i) + \gamma \\ &\qquad + \alpha_c|\mathrm{d}\Theta|^2 + 2\,\mathrm{Im}(\alpha_c\,\mathrm{Re}\, p^{\Theta}) + r(\partial^r|\mathrm{d}\Theta|^2) + 2\,\mathrm{Re}(A\,\mathrm{Re}\, p^{\Theta}). \end{aligned} \tag{4.4}$$

By (1.4a) we can find $c \in (0, c_1 - c_2]$, $\epsilon > 0$ and $\nu \geqslant 1$ such that for $r \geqslant \nu$

$$2\alpha_c\left\{E - \left(V + \frac{|\beta|^2 - c\gamma}{2c\alpha_c}\right)\right\} \geqslant 4\epsilon.$$

Hence, if we consider large $r \geqslant \nu$ only, omitting the cutoff χ_ν for the moment, this implies by (1.1a) and the Cauchy–Schwarz inequality

$$\begin{aligned} &2\alpha_c(E - V) + p_i^*(\nabla^2 r^2 - \alpha_c g)^{ij} p_j - 2\,\mathrm{Im}(\beta^i p_i) + \gamma \\ &\quad \geqslant 2\alpha_c(E - V) - \frac{|\beta|^2}{c} + \gamma \geqslant 4\epsilon. \end{aligned} \tag{4.5}$$

To complete the proof it suffices to demonstrate the lower bound -3ϵ for sum of the last four terms on the right-hand side of (4.4). By (1.1b) and (4.1)

$$\begin{aligned} \alpha_c|\mathrm{d}\Theta|^2 + r(\partial^r|\mathrm{d}\Theta|^2) &= (\Theta')^2(r\partial^r|\mathrm{d}r|^2 + \alpha_c|\mathrm{d}r|^2) + 2r\Theta''\Theta'|\mathrm{d}r|^4 \\ &\geqslant -4\delta_0(\sigma_0 + \delta_0)|\mathrm{d}r|^4. \end{aligned}$$

This implies for sufficiently small $\delta_0 > 0$

$$\alpha_c|\mathrm{d}\Theta|^2 + r(\partial^r|\mathrm{d}\Theta|^2) \geqslant -\epsilon. \tag{4.6}$$

Next, noting the expressions

$$A = rp^r + (p^r)^* r = 2(\operatorname{Re} p^r)r - \frac{1}{\mathrm{i}}|\mathrm{d}r|^2,$$
$$\operatorname{Re} p^\Theta = \frac{1}{2}\big(p^\Theta + (p^\Theta)^*\big) = \Theta' \operatorname{Re} p^r + \frac{1}{2\mathrm{i}}|\mathrm{d}r|^2\Theta'', \tag{4.7}$$

we compute with $\eta := |\mathrm{d}r|^2\Theta' + r|\mathrm{d}r|^2\Theta'' - \alpha_c\Theta'$

$$\begin{aligned}
&2\operatorname{Re}\big(A \operatorname{Re} p^\Theta\big) + 2\operatorname{Im}\big(\alpha_c \operatorname{Re} p^\Theta\big)\\
&\quad = 4\big(\operatorname{Re} p^r\big)r\Theta' \operatorname{Re} p^r - 2\operatorname{Im}\big(\eta \operatorname{Re} p^r\big) + \big(|\mathrm{d}r|^4 - \alpha_c|\mathrm{d}r|^2\big)\Theta''\\
&\quad \geqslant 4\big(\operatorname{Re} p^r\big)\left\{r\Theta' - \frac{1}{4\epsilon}\eta^2\right\}\operatorname{Re} p^r - \epsilon + \big(|\mathrm{d}r|^4 - \alpha_c|\mathrm{d}r|^2\big)\Theta''.
\end{aligned}$$

By using (4.1) we see that in the regime $r \to \infty$ the first term on the right-hand side is non-negative and the third term is arbitrarily small. Hence,

$$2\operatorname{Re}\big(A \operatorname{Re} p^\Theta\big) + 2\operatorname{Im}\big(\alpha_c \operatorname{Re} p^\Theta\big) \geqslant -2\epsilon. \tag{4.8}$$

Thus by (4.4)–(4.8) the asserted inequality (4.3) follows. □

Proof of Proposition 3.1. We let E and ϕ be as in the proposition. Set

$$\sigma_0 = \sup\big\{\sigma \geqslant 0 \,\big|\, \mathrm{e}^{\sigma r}\phi \in \mathcal{H}\big\},$$

and assume $\sigma_0 < \infty$. We fix $c \in (0, c_1 - c_2]$, $\epsilon > 0$, $\delta_0 > 0$ and $\nu \geqslant 1$ in agreement with Lemma 4.1. If $\sigma_0 > 0$, we choose $\sigma \in [0, \sigma_0)$ and $\delta \in (0, \delta_0]$ such that $\sigma + \delta > \sigma_0$. If $\sigma_0 = 0$, we set $\sigma = 0$ and choose any $\delta \in (0, \delta_0]$. In any case we have $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$.

With these values of σ and δ we set for any $\psi \in C_c^\infty(M)$, $m \geqslant 1$, $\nu' \geqslant 2\nu$

$$\psi_\Theta = \chi_{\nu,\nu'}\mathrm{e}^{\Theta}\psi = \chi_{\nu,\nu'}\mathrm{e}^{\Theta_m^{\sigma,\delta}}\psi,$$

where $\chi_{\nu,\nu'} = \chi_\nu\bar{\chi}_{\nu'}$, $\bar{\chi}_{\nu'} = 1 - \chi_{\nu'}$. We note, putting $R_\nu = \mathrm{i}[H, \chi_\nu]$ as in (3.6),

$$\mathrm{i}(H_\Theta - E)\psi_\Theta = \mathrm{i}\chi_{\nu,\nu'}\mathrm{e}^{\Theta}(H - E)\psi + \mathrm{e}^{\Theta}(R_\nu - R_{\nu'})\psi. \tag{4.9}$$

Due to Lemma 4.1,

$$\epsilon\|\psi_\Theta\|^2 \leqslant 2\operatorname{Im}\big\langle A(H_\Theta - E)\big\rangle_{\psi_\Theta} - 2\operatorname{Re}\big\langle \alpha_c(H_\Theta - E)\big\rangle_{\psi_\Theta}, \tag{4.10}$$

where, recalling from Section 1, in general $\langle A\rangle_\psi = \langle\psi, A\psi\rangle$ denotes expectation. Let us compute the right-hand side. We indicate below the dependence of constants using subscripts. For the first term of (4.10) we use (4.9) and

$$\begin{aligned}
2\operatorname{Im}\big\langle A(H_\Theta - E)\big\rangle_{\psi_\Theta} &= -\big\langle A\psi_\Theta, \mathrm{i}\chi_{\nu,\nu'}\mathrm{e}^{\Theta}(H - E)\psi\big\rangle - \big\langle A\psi_\Theta, \mathrm{e}^{\Theta}(R_\nu - R_{\nu'})\psi\big\rangle + \text{h.c.}\\
&\leqslant 2\|A\psi_\Theta\|\big\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta}(H - E)\psi\big\| + C_\nu\big(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2\big)\\
&\quad + C_m\Big(\big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\big\|^2 + \big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\big\|^2\Big)\\
&\leqslant (\nu')^2\big\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta}(H - E)\psi\big\|^2 + C_\nu\big(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2\big)\\
&\quad + C_m\Big(\big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\big\|^2 + \big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\big\|^2\Big),
\end{aligned}$$

where we used that $r/\nu' \leqslant 2\sqrt{r/\nu'}$ on supp $\chi_{\nu,2\nu'}$ to estimate $(\nu')^{-2}\|A\psi_\Theta\|^2$. Similarly we can estimate the second term of (4.10), and

$$-2\,\mathrm{Re}\langle\alpha_c(H_\Theta - E)\rangle_{\psi_\Theta} \leqslant (\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta}(H-E)\psi\|^2 + C_\nu\big(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2\big) + C_m\big(\big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\big\|^2 + \big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\big\|^2\big).$$

Hence

$$\frac{\epsilon}{2}\|\psi_\Theta\|^2 \leqslant (\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + C_\nu\big(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2\big) + C_m\big(\big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\big\|^2 + \big\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\big\|^2\big). \tag{4.11}$$

By Lemma 3.4 we can replace ψ of (4.11) by ϕ. This makes the first term on the right-hand side disappear. Next let $\nu' \to \infty$ invoking Lebesgue's dominated convergence theorem. This makes the third term disappear, and consequently we are left with the bound

$$\big\|\chi_\nu \mathrm{e}^{\Theta_m^{\sigma,\delta}}\phi\big\|^2 \leqslant \frac{2C_\nu}{\epsilon}\big(\|\chi_{\nu/2}\phi\|^2 + \|\chi_{\nu/2}p\phi\|^2\big). \tag{4.12}$$

By letting $m \to \infty$ in (4.12) invoking Lebesgue's monotone convergence theorem we conclude that $\chi_\nu \mathrm{e}^{(\sigma+\delta)r}\phi \in \mathcal{H}$. This is a contradiction since $\sigma + \delta > \sigma_0$. □

5. Absence of super-exponentially decaying eigenstates

We complete the proof of Theorem 1.5 by proving Proposition 3.2 in this section. The proof relies on similar techniques as the one of Proposition 3.1. We will consider a little generalized setting replacing Conditions 1.1 and 1.2 by the following ones stated in terms of a parameter $\tau \leqslant 1$:

Condition 5.1. There exist an unbounded real-valued function $r \in C^4(M)$, $r(x) \geqslant 1$, constants $c_1 > c_2 > 0$ and a decomposition $\Delta r^2 = \rho_1 + \rho_2 + \rho_3$ such that uniformly in $x \in M$:

(1) There exist constants $r_0 \geqslant 1$ and $C > 0$ such that

$$\nabla^2 r^2 \geqslant \Big(c_1 r^\tau + \frac{1}{2}\rho_1\Big)g - Cr^\tau\,\mathrm{d}r\otimes\mathrm{d}r \quad \text{and} \quad \rho_1 \geqslant 0 \quad \text{for } r \geqslant r_0. \tag{5.1a}$$

Moreover

$$\liminf_{r\to\infty} r^{-\tau}\Big(r\partial^r|\mathrm{d}r|^2 + \Big(c_2 r^\tau + \frac{1}{2}\rho_1\Big)|\mathrm{d}r|^2\Big) > 0, \qquad \limsup_{r\to\infty}|\mathrm{d}r| < \infty. \tag{5.1b}$$

(2) The following bounds hold

$$\limsup_{r\to\infty}\big|r^{-1}\Delta r^2\big| < \infty, \tag{5.2a}$$

$$\limsup_{r\to\infty} r^{-\tau}\rho_1 < \infty, \qquad \limsup_{r\to\infty} r^{-\tau}|\mathrm{d}\rho_2| < \infty, \qquad \limsup_{r\to\infty} r^{-\tau}\Delta\rho_3 < \infty. \tag{5.2b}$$

Condition 5.2. There exists a decomposition $V = V_1 + V_2$, $V_1 \in L^2_{\mathrm{loc}}(M)$, $V_2 \in C^1(M)$ and V_1, V_2 real-valued, such that uniformly in $x \in M$:

$$\limsup_{r\to\infty}|V| < \infty, \qquad \limsup_{r\to\infty} r^{1-\tau}|V_1| < \infty, \qquad \limsup_{r\to\infty} r^{1-\tau}\partial^r V_2 < \infty. \tag{5.3}$$

The case $\tau = 0$ corresponds to Conditions 1.1 and 1.2 although even in this case (5.1a) is somewhat weaker than (1.1a) since now possibly some negativity of $\nabla^2 r^2$ along the $\mathrm{d}r \otimes \mathrm{d}r$ component occurs. The weakening of these conditions will be compensated by the assumption of super-exponential decay for the considered eigenfunction. Another remark here is that the negative case, $\tau < 0$, is also allowed. With Examples 2.2 in mind, this means that an end of very slow expansion, which is so slow that the end might be asymptotic to a straight cylinder, could be treated. In fact the stronger decay properties (5.2b) and (5.3) appear somewhat harmless in this case. In the other extreme case $\tau = 1$ the bounds (5.2b) and (5.3) are relaxing (1.2b) and (1.3), respectively.

Under these conditions we prove

Proposition 5.3. *Suppose Conditions 5.1 and 5.2 for some $\tau \leqslant 1$. Suppose Conditions 1.3 and 1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E \in \mathbb{R}$ and $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$ for any $\sigma \geqslant 0$, then $\phi(x) = 0$ in M.*

Proposition 5.3 obviously generalizes Proposition 3.2. We note that Proposition 5.3 generalizes [17] when $\tau = 1$ while Proposition 3.2 does not. For a manifold of bounded geometry and pinched negative curvature is always endowed with a metric of the form (2.1) with uniformly and strictly positive $\nabla^2 r_{|S_r}$ and with bounded derivatives of Δr (for r large). Then the verification of Condition 5.1 is straightforward. For these geometric terminologies we refer to [17] and references therein.

For the proof of Proposition 5.3 we first rewrite Lemma 3.3 as follows.

Lemma 5.4. *As a quadratic form on $C_c^\infty(M)$,*

$$\begin{aligned}
&\mathrm{i}[H, A] = p_i^*\big(\nabla^2 r^2 - \alpha g\big)^{ij} p_j + 2\,\mathrm{Re}(\alpha H_0) - 2\,\mathrm{Im}\big(\beta^i p_i\big) + \gamma;\\
&\alpha = c_2 r^\tau + \frac{1}{2}\rho_1,\\
&\beta = -\frac{1}{2}c_2\,\mathrm{d}r^\tau + \frac{1}{4}\,\mathrm{d}\rho_2 + V_1\,\mathrm{d}r^2,\\
&\gamma = -\frac{1}{4}\Delta\rho_3 + \big(\Delta r^2\big)V_1 - 2r\partial^r V_2.
\end{aligned}$$

We omit the proof which goes along the same pattern as the one of Lemma 3.3. Note that by Conditions 5.1 and 5.2 for large $r \geqslant 1$ the coefficients α, β and γ satisfy

$$c_2 r^\tau \leqslant \alpha < Cr^\tau, \qquad |\beta| < Cr^\tau \quad \text{and} \quad \gamma > -Cr^\tau. \tag{5.4}$$

As in (4.2) we introduce for $\sigma \geqslant 0$ the conjugated operator

$$H_\sigma = H_{\Theta_m^{\sigma,0}} = \mathrm{e}^{\sigma r} H \mathrm{e}^{-\sigma r} = H - \frac{\sigma^2}{2}|\mathrm{d}r|^2 + \mathrm{i}\sigma\,\mathrm{Re}\, p^r. \tag{5.5}$$

Again, we consider H_σ as an operator defined on $C_c^\infty(M)$ only.

Lemma 5.5. *There exists $\epsilon > 0$ such that, if $\nu \geqslant 1$ and $\sigma \geqslant 0$ are large, then, as quadratic forms on $C_c^\infty(M)$,*

$$\chi_\nu\big(2\,\mathrm{Im}(AH_\sigma)\big)\chi_\nu \geqslant \chi_\nu\big(\epsilon\sigma^2 r^\tau + 2\,\mathrm{Re}(\alpha H_\sigma)\big)\chi_\nu. \tag{5.6}$$

Proof. Similarly to (4.4), by (5.5) and Lemma 5.4 we obtain

$$2\,\mathrm{Im}(AH_\sigma) = 2\,\mathrm{Re}(\alpha H_\sigma) - 2\alpha V + p_i^*\left(\nabla^2 r^2 - \alpha g\right)^{ij} p_j - 2\,\mathrm{Im}\left(\beta^i p_i\right) + \gamma$$
$$+ \sigma^2\alpha|\mathrm{d}r|^2 + 2\sigma\,\mathrm{Im}\left(\alpha\,\mathrm{Re}\,p^r\right) + \sigma^2 r\left(\partial^r|\mathrm{d}r|^2\right) + 2\sigma\,\mathrm{Re}\left(A\,\mathrm{Re}\,p^r\right). \tag{5.7}$$

Let us estimate the right-hand side. We consider large $r \geqslant 1$, and omit the cutoff χ_ν for the moment. By (5.1a), the Cauchy–Schwarz inequality, Condition 5.2 and (5.4)

$$\begin{aligned}&-2\alpha V + p_i^*\left(\nabla^2 r^2 - \alpha g\right)^{ij} p_j - 2\,\mathrm{Im}\left(\beta^i p_i\right) + \gamma\\ &\quad\geqslant -2\alpha V - C\left(p^r\right)^* r^\tau p^r - \frac{1}{c_1 - c_2} r^{-\tau}|\beta|^2 + \gamma\\ &\quad\geqslant -C\left(p^r\right)^* r^\tau p^r - C r^\tau.\end{aligned} \tag{5.8}$$

On the other hand, by using (4.7) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}&2\sigma\,\mathrm{Re}\left(A\,\mathrm{Re}\,p^r\right) + 2\sigma\,\mathrm{Im}\left(\alpha\,\mathrm{Re}\,p^r\right)\\ &\quad= 4\sigma\left(\mathrm{Re}\,p^r\right) r\,\mathrm{Re}\,p^r + 2\sigma\,\mathrm{Im}\left(\alpha\,\mathrm{Re}\,p^r\right) - 2\sigma\,\mathrm{Im}\left(|\mathrm{d}r|^2\,\mathrm{Re}\,p^r\right)\\ &\quad\geqslant 2\sigma\left(\mathrm{Re}\,p^r\right) r^\tau\,\mathrm{Re}\,p^r - \sigma C r^\tau - \sigma\partial^r|\mathrm{d}r|^2.\end{aligned}$$

Whence, by using

$$\mathrm{Re}\,p^r = \frac{1}{2}\left(p^r + \left(p^r\right)^*\right) = p^r + \frac{1}{2\mathrm{i}}(\Delta r),$$

the Cauchy–Schwarz inequality and (3.4), we conclude that

$$\begin{aligned}&2\sigma\,\mathrm{Re}\left(A\,\mathrm{Re}\,p^r\right) + 2\sigma\,\mathrm{Im}\left(\alpha\,\mathrm{Re}\,p^r\right)\\ &\quad\geqslant \sigma\left(p^r\right)^* r^\tau p^r - C\sigma r^\tau - \sigma r^{\tau-1}\left(r^{1-\tau}\partial^r|\mathrm{d}r|^2 + \alpha r^{-\tau}|\mathrm{d}r|^2\right).\end{aligned} \tag{5.9}$$

Now we put (5.9) and (5.9) into (5.7) and then obtain that for all large $\nu \geqslant 1$ and $\sigma \geqslant 0$

$$\begin{aligned}&\chi_\nu\left(2\,\mathrm{Im}(AH_\sigma)\right)\chi_\nu\\ &\quad\geqslant \chi_\nu\left\{\sigma^2 r^\tau\left(1 - \sigma^{-1}r^{-1}\right)\left(r^{1-\tau}\partial^r|\mathrm{d}r|^2 + \alpha r^{-\tau}|\mathrm{d}r|^2\right) - C\sigma r^\tau + 2\,\mathrm{Re}(\alpha H_\sigma)\right\}\chi_\nu.\end{aligned}$$

Clearly (5.6) follows from this estimate and (5.1b). □

Proof of Proposition 5.3. Let $\phi \in \mathcal{D}(H)$ be a super-exponentially decaying eigenfunction as in Proposition 5.3. Then, by assumption, for any $\nu \geqslant 1$ and $\sigma \geqslant 0$

$$\phi_\sigma = \phi_{\sigma,\nu} := \chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\phi \in \mathcal{H}. \tag{5.10}$$

We will choose $\nu \geqslant 1$ large in agreement with Lemma 3.4 with $\psi = \phi$. In the following computations we actually have to first choose an approximate sequence for ϕ from $C_{\mathrm{c}}^\infty(M)$ and then take the limits in the last step as in the proof of Proposition 3.1. This can be done by using Lemma 3.4 and the closedness of H, but since the verification is rather straightforward we shall not elaborate on this point.

Put $H_\sigma = \mathrm{e}^{\sigma r} H \mathrm{e}^{-\sigma r}$ as in (4.2). Then by Lemma 5.5 for large $\nu \geqslant 1$ and $\sigma \geqslant 0$

$$\epsilon\sigma^2\langle r^\tau\rangle_{\phi_\sigma} \leqslant 2\,\mathrm{Im}\langle AH_\sigma\rangle_{\phi_\sigma} - 2\,\mathrm{Re}\,\langle\alpha H_\sigma\rangle_{\phi_\sigma}. \tag{5.11}$$

We note, putting $R_\nu = \mathrm{i}[H_0, \chi_\nu] = \mathrm{Re}\,(\chi_\nu' p^r)$ as in (3.6),

$$H_\sigma \phi_\sigma = E\phi_\sigma - \mathrm{i}\mathrm{e}^{\sigma(r-4\nu)} R_\nu \phi. \tag{5.12}$$

Hence we can write the right-hand side of (5.11) as

$$\begin{aligned} &2\,\mathrm{Im}\langle A H_\sigma\rangle_{\phi_\sigma} - 2\,\mathrm{Re}\langle \alpha H_\sigma\rangle_{\phi_\sigma} \\ &\quad = -2\,\mathrm{Re}\big\langle \mathrm{e}^{\sigma(r-4\nu)}\chi_\nu A \mathrm{e}^{\sigma(r-4\nu)} R_\nu\big\rangle_\phi - 2E\langle\alpha\rangle_{\phi_\sigma} - 2\,\mathrm{Im}\big\langle \alpha\chi_\nu \mathrm{e}^{2\sigma(r-4\nu)} R_\nu\big\rangle_\phi. \end{aligned} \tag{5.13}$$

As for the first term of (5.13) we estimate (recall the notation $\bar\chi_\nu = 1 - \chi_\nu$)

$$\begin{aligned} &-2\,\mathrm{Re}\big\langle \mathrm{e}^{\sigma(r-4\nu)}\chi_\nu A \mathrm{e}^{\sigma(r-4\nu)} R_\nu\big\rangle_\phi \\ &\quad \leqslant \big\|\bar\chi_{2\nu} A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\phi\big\|^2 + \big\|\mathrm{e}^{\sigma(r-4\nu)} R_\nu\phi\big\|^2 \\ &\quad \leqslant \Big\{\big\|2r\bar\chi_{2\nu}\chi_\nu \mathrm{e}^{\sigma(r-4\nu)} p^r\phi\big\| \\ &\qquad + \Big\|\bar\chi_{2\nu}\Big(2r|\mathrm{d}r|^2\chi_\nu' + 2\sigma r\chi_\nu|\mathrm{d}r|^2 + \frac{1}{2}(\Delta r^2)\chi_\nu\Big)\mathrm{e}^{\sigma(r-4\nu)}\phi\Big\|\Big\}^2 \\ &\qquad + \Big\{\big\|\chi_\nu' \mathrm{e}^{\sigma(r-4\nu)} p^r\phi\big\| + \frac{1}{2}\big\|\big(\chi_\nu''|\mathrm{d}r|^2 + \chi_\nu'(\Delta r)\big)\mathrm{e}^{\sigma(r-4\nu)}\phi\big\|\Big\}^2 \\ &\quad \leqslant C\nu^2\big\|\chi_{\nu/2}|p\phi|\big\|^2 + C\nu^2\langle\sigma\rangle^2\|\phi\|^2, \end{aligned}$$

where we have used (3.4). Note that $C > 0$ does not depend on ν or σ because $r \leqslant 2\nu$ on $\mathrm{supp}\,\chi_\nu'$. By using (3.2) and (3.3) (both with $\sigma = 0$) we then conclude

$$-2\,\mathrm{Re}\big\langle \mathrm{e}^{\sigma(r-4\nu)}\chi_\nu A \mathrm{e}^{\sigma(r-4\nu)} R_\nu\big\rangle_\phi \leqslant C\nu^2\langle\sigma\rangle^2\|\phi\|^2.$$

Next, we examine the third term of (5.13). This term is estimated similarly, and we obtain the rough bound

$$-2\,\mathrm{Im}\big\langle \alpha\chi_\nu \mathrm{e}^{2\sigma(r-4\nu)} R_\nu\big\rangle_\phi \leqslant C\nu^2\langle\sigma\rangle^2\|\phi\|^2.$$

We summarize

$$\big(\epsilon\sigma^2 - C\big)\big\langle r^\tau\big\rangle_{\phi_\sigma} \leqslant C\nu^2\langle\sigma\rangle^2\|\phi\|^2. \tag{5.14}$$

Now assume $\chi_{5\nu}\phi \not\equiv 0$. After division by $\langle\sigma\rangle^2$ on both sides of (5.14) the left-hand side grows exponentially as $\sigma \to \infty$ whereas the right-hand side is bounded, and hence we obtain a contradiction. Thus $\chi_{5\nu}\phi \equiv 0$, and then by Condition 1.4 we conclude that $\phi(x) = 0$ in M. □

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