# RENORMALIZED TWO-BODY LOW-ENERGY SCATTERING

*By*

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**Abstract.** For a class of long-range potentials, including ultra-strong perturbations of the attractive Coulomb potential in dimension $d \geq 3$, we introduce a stationary scattering theory for Schrödinger operators which is regular at zero energy. In particular, it is well-defined at this energy, and we use it to establish a characterization there of the set of generalized eigenfunctions in an appropriately adapted Besov space, generalizing parts of [DS1]. Principal tools include global solutions of the eikonal equation and strong radiation condition bounds.

## Contents





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# 1 Introduction

For a class of long-range potentials, we introduce a stationary scattering theory for Schrödinger operators $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ which is regular at zero energy. In particular, it is well-defined at this energy, and we use it to establish a characterization there of the set of generalized eigenfunctions in an appropriately adapted Besov space. The analogue of this characterization at positive energies for potentials obeying $\langle x\rangle^{\mu+|\alpha|}|\partial^\alpha V(x)| \leq C_\alpha$ for some $\mu > 0$ is well known [AH, Hö, GY]. It goes as follows.

For all $\lambda > 0$ and all generalized eigenfunctions, $(H-\lambda)u_\lambda = 0$ in the (dual) Besov space $B(|x|)^*$, there exist unique $\tau, \tilde\tau \in L^2(S^{d-1})$ such that

$$u_\lambda(x) - C|x|^{-(d-1)/2}\big(\mathrm{e}^{\mathrm{i}S(x,\lambda)}\tau(\omega) + \mathrm{e}^{-\mathrm{i}S(x,\lambda)}\tilde\tau(\omega)\big) \in B(|x|)^*_0. \tag{1.1}$$

Here, $S(\cdot,\lambda) = \sqrt{\lambda}|x| + o(|x|)$ is a solution of the eikonal equation, $\omega = x/|x|$, and $B(|x|)^*$ and $B(|x|)^*_0 \subset B(|x|)^*$ are specified by

$$\begin{aligned} &u \in B(|x|)^* \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \sup_{R>1} R^{-1/2}\|F(|x|<R)u\| < \infty,\\ &u \in B(|x|)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-1/2}\|F(|x|<R)u\| = 0, \end{aligned} \tag{1.2}$$

(we use the notation $F(\cdot < R) = 1_{(-\infty,R)}$ to designate a sharp cut-off function). Moreover, we can write $\tilde\tau(\omega) = (S(\lambda)^{-1}\tau)(-\omega)$, where the operator $S(\lambda)$ is a unitary operator on $L^2(S^{d-1})$ (called the scattering matrix at energy $\lambda$). The family of these operators is connected to scattering operators from time-dependent scattering theory. For one of the well-known time-dependent constructions, this connection is given in terms of a Legendre transformation; see [HS2, II].

The (inverse) scattering matrix at energy $\lambda$ is determined by (1.1): for all $\tau \in L^2(S^{d-1})$, there exist a unique $\tilde{\tau} \in L^2(S^{d-1})$ and a unique generalized eigenfunction $u_\lambda \in B(|x|)^*$ such that the asymptotics (1.1) is satisfied. Indeed, the set of generalized eigenfunctions in $B(|x|)^*$ at each positive energy $\lambda$ is characterized by (1.1). The variable $\omega$ may be thought of as the observable asymptotic normalized velocity; see [DS1] for discussion. We refer to [Me, Va] for a related approach to stationary scattering theory for a class of geometric models.

For a class of potentials negative at $\infty$ and to leading order spherically symmetric, the above constructions were extended down to (and including) zero energy [DS1]. We refer to [DS2, Fr] for explicit calculations of the scattering matrix at zero energy and to [Ya, SW] for related one-dimensional results on asymptotics of scattering quantities. The set of generalized zero energy eigenfunctions in an appropriately adapted Besov space is characterized in [DS1] for the restrictive class of potentials. Since this class of potentials is close to being optimal for the existence in classical mechanics of asymptotic normalized velocity at zero energy, the given characterization result may be viewed as "best possible." The purpose of this paper is to provide a similar characterization of generalized zero energy eigenfunctions for a larger class of potentials than that considered in [DS1]. Again we obtain a parametrization by $L^2(S^{d-1})$, however, the isomorphism is different. Rather than involving functions on a sphere of *asymptotic normalized velocities*, the parametrization is in terms of functions on a sphere of *initial velocities*; see Subsection 6.1 for a brief discussion of this difference. In this sense, our approach is in the spirit of [ACH], where a distorted Fourier transform is constructed for order zero potentials at high energies in terms of a family of initially controlled geodesics. Our main result is Theorem 6.3. We prove new low-energy radiation condition bounds of independent interest. These bounds are stated as Proposition 4.2.

The class of potentials studied in this paper is introduced in Section 2. In the remaining part of the present section, we review various background results for a somewhat larger class. The zero energy characterization problem makes sense for this larger class (at least to some degree); see Subsection 1.2. Although the class considered in the bulk of this paper may not be optimal for the characterization problem, a further extension would introduce difficult problems; see Subsection 1.3.

**1.1 A priori quantum bounds.** We give an account of some recent results [Sk]. These include Besov space bounds of the resolvent at low energies in any dimension for a class of potentials that are negative and obey a virial condition with these conditions imposed at $\infty$ only. There are two boundary values of the resolvent at zero energy which are characterized by radiation conditions. These radiation conditions are zero energy versions of the well-known Sommerfeld radiation condition.

We consider the Schrödinger operator $H = -\Delta + V$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \geq 1$, where the potential $V$ obeys the following condition. We set $\langle x \rangle = \sqrt{x^2+1}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for $\mu \in (0,2)$, $s_0 = 1/2 + \mu/4$.

**Condition 1.1.** Let $V = V_1 + V_2$ be a real-valued function defined on $\mathbb{R}^d$, $d \geq 1$. There exists $\mu \in (0,2)$ such that the following conditions hold.

(1) There exists $\epsilon_1 > 0$ such that $V_1(x) \leq -\epsilon_1 \langle x \rangle^{-\mu}$.
(2) $V_1 \in C^\infty(\mathbb{R}^d)$. For all $\alpha \in \mathbb{N}_0^d$, there exists $C_\alpha > 0$ such that

$$\langle x \rangle^{\mu+|\alpha|} |\partial^\alpha V_1(x)| \leq C_\alpha.$$

(3) There exists $\tilde{\epsilon}_1 > 0$ such that $-|x|^{-2} \left( x \cdot \nabla (|x|^2 V_1) \right) \geq -\tilde{\epsilon}_1 V_1$.
(4) There exist $\delta, C, R > 0$ such that $|V_2(x)| \leq C|x|^{-2s_0-\delta}$ for $|x| > R$.
(5) If $d = 1,2,3$, then $V_2 \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, where $p = 2$; if $d \geq 4$, then $p > d/2$.

Because of conditions (4) and (5), the operator $V_2(-\Delta + i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$, cf. [RS, Theorem X.20]. Hence $H$ is self-adjoint. The Schrödinger operator with an attractive Coulomb potential in dimension $d \geq 3$ is a particular example.

Let $\theta \in (0,\pi)$, $\lambda_0 > 0$, and define

$$\Gamma_{\theta,\lambda_0} = \{ z \in \mathbb{C} \setminus \{0\} \,|\, \arg z \in (0,\theta),\ |z| \leq \lambda_0 \}. \tag{1.3}$$

For a Hilbert space $\mathcal{H}$ (which in our case is $L^2(\mathbb{R}^d)$), we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$ (we use similar notation for Banach spaces). A $\mathcal{B}(\mathcal{H})$-valued function $T(\cdot)$ on $\Gamma_{\theta,\lambda_0}$ is said to be **uniformly Hölder continuous** in $\Gamma_{\theta,\lambda_0}$ if there exist $C, \gamma > 0$ such that

$$\|T(z_1) - T(z_2)\| \leq C|z_1 - z_2|^\gamma \text{ for all } z_1, z_2 \in \Gamma_{\theta,\lambda_0}.$$

We denote the resolvent of $H$ by $R(z) = (H-z)^{-1}$. The symbols $B(|x|)$ and $B(|x|)^*$ refer, respectively, to the Besov space for the operator of multiplication by $|x|$ and its dual space.

**Proposition 1.2** (Limiting absorption principle)**.** *Suppose Condition 1.1. For all $s > s_0$, the family of operators $T(z) = \langle x \rangle^{-s} R(z) \langle x \rangle^{-s}$ is uniformly Hölder* *continuous in $\Gamma_{\theta,\lambda_0}$. In particular, the limits*

$$T(0+\mathrm{i}0) = \langle x\rangle^{-s}R(0+\mathrm{i}0)\langle x\rangle^{-s} = \lim_{\Gamma_{\theta,\lambda_0}\ni z\to 0} T(z),$$
$$T(0-\mathrm{i}0) = \langle x\rangle^{-s}R(0-\mathrm{i}0)\langle x\rangle^{-s} = \lim_{\Gamma_{\theta,\lambda_0}\ni z\to 0} T(\bar{z})$$

*exist in $\mathcal{B}(L^2(\mathbb{R}^d))$. Also, there exists $C > 0$ such that for all $z \in \Gamma_{\theta,\lambda_0}$,*

$$\|(|z| + \langle x\rangle^{-\mu})^{1/4}R(z)(|z| + \langle x\rangle^{-\mu})^{1/4}\|_{\mathcal{B}(B(|x|),B(|x|)^*)} \le C. \tag{1.4}$$

**1.1.1 Zero energy Sommerfeld radiation condition.** We give an outline of some microlocal estimates and characterizations of solutions of the equation $Hu = v$. In particular, we estimate and characterize the "outgoing" solution, whose existence is provided by Proposition 1.2. This particular solution is given as follows in terms of Besov spaces. First note that the relevant Besov space at zero energy is $B^\mu := B(\langle x\rangle^{2s_0}) = \langle x\rangle^{-\mu/4}B(|x|)$; cf. (1.4). We have the following characterization of the corresponding dual space :

$$u \in (B^\mu)^* \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \sup_{R>1} R^{-s_0}\|F(|x| < R)u\| < \infty.$$

A slightly smaller space is given by

$$u \in (B^\mu)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-s_0}\|F(|x| < R)u\| = 0.$$

Now suppose $v \in B^\mu$. Then Proposition 1.2 guarantees the existence of the weak-star limit

$$u = R(0+\mathrm{i}0)v = \operatorname*{w^\star\text{–}lim}_{\Gamma_{\theta,\lambda_0}\ni z\to 0} R(z)v \in (B^\mu)^*.$$

Note that this $u$ is indeed a (distributional) solution of the equation $Hu = v$.

To state microlocal properties of this solution, we first introduce for all $\lambda \ge 0$ the function

$$f = f_\lambda(x) = (\lambda + K\langle x\rangle^{-\mu})^{1/2},\ x \in \mathbb{R}^d, \tag{1.5}$$

where $K := \epsilon_1\tilde{\epsilon}_1/(2-\mu)$ with the $\epsilon$'s given in Condition 1.1. In terms of $f_0$, we introduce symbols

$$a_0 = \frac{\xi^2}{f_0(x)^2},\quad b_0 = \frac{\xi}{f_0(x)}\cdot\frac{x}{\langle x\rangle}, \tag{1.6}$$

and have, using here Weyl quantization,

$$\mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in (B^\mu)^*_0 \tag{1.7a}$$
$$\text{for all } \chi_- \in C^\infty_{\mathrm{c}}(\mathbb{R}) \text{ and } \tilde{\chi}_- \in C^\infty_{\mathrm{c}}((-\infty,1)).$$

These estimates are accompanied by "high energy estimates," stated as follows. Let us note that

$$f_{|z|}^{-2}(x)\big|V_1(x)-z\big| \le C_0' := \max(C_0/K, 1),$$

where $C_0$ is given in Condition 1.1 (2) (i.e., the constant with $\alpha = 0$). Consider real-valued $\chi_- \in C_c^\infty(\mathbb{R})$ such that $\chi_-(t) = 1$ in a neighborhood of $[0, C_0']$, and let $\chi_+ := 1 - \chi_-$. For all such functions $\chi_+$,

$$\operatorname{Op}^{\mathrm{w}}(\chi_+(a_0))u \in (B^\mu)_0^*. \tag{1.7b}$$

The support property of $\tilde\chi_-$ in (1.7a) mirrors the fact that the particular solution studied is the outgoing one; hence we refer to (1.7a) as the **outgoing Sommerfeld radiation condition**. This condition (in fact a weaker version) suffices for a characterization as expressed in the following result. Here and henceforth, $L_m^2 := \langle x\rangle^{-m}L^2(\mathbb{R}^d)$.

**Proposition 1.3** (Uniqueness of outgoing solution, data in $B^\mu$)**.** *Suppose $v \in B^\mu$, $u$ is a distributional solution to the equation $Hu = v$ belonging to the space $L_m^2$ for some $m \in \mathbb{R}$, and that there there exists $\kappa \in (0, 1]$ such that*

$$\operatorname{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde\chi_-(b_0))u \in (B^\mu)_0^* \quad \textit{for all } \chi_- \in C_c^\infty(\mathbb{R}) \textit{ and } \tilde\chi_- \in C_c^\infty((-\infty, \kappa)). \tag{1.8}$$

*Then $u = R(0 + \mathrm{i}0)v$. In particular,* (1.7a) *and* (1.7b) *hold.*

The "incoming" solution $u = R(0 - \mathrm{i}0)v$ can be characterized similarly. These results generalize [DS1, Proposition 4.10] at zero energy.

**Remark 1.4.** There are similar results for positive energies. For $R(\lambda + \mathrm{i}0)$, we have the same conclusion $u = R(\lambda + \mathrm{i}0)v$ for an outgoing solution of $(H - \lambda)u = v$. More precisely, this means that if in the definition of the Besov spaces we replace $s_0$ with $s_0 = 1/2$ and change the localization symbols $a_0, b_0$ in (1.6) and (1.8) by replacing $f_0$ with $f_\lambda$, the solution $u$ is then given by $u = R(\lambda + \mathrm{i}0)v$. This result is known for larger classes of potentials; see [Hö, Theorem 30.2.10] and [GY].

## 1.2 Open problems.

Define under Condition 1.1 the operator

$$\delta(0) = (2\pi\mathrm{i})^{-1}\big(R(0 + \mathrm{i}0) - R(0 - \mathrm{i}0)\big) = \pi^{-1}\operatorname{Im}\big(R(0 + \mathrm{i}0)\big) \in \mathcal{B}(B^\mu, (B^\mu)^*)$$

and note that its range satisfies $\operatorname{Ran}(\delta(0)) \subseteq \mathcal{E}_0 := \{u \in (B^\mu)^* \mid Hu = 0\}$. It follows from [DS1, Theorem 8.2] that under some stronger conditions, $\operatorname{Ran}(\delta(0)) = \mathcal{E}_0$.

(This can be proved in terms of wave matrices at zero energy.) This identity and characterization of $\xi_0$ are not known under Condition 1.1; in fact, it is only known that $\delta_0 \neq 0$ (see [FS]). More specifically, "scattering theory at zero energy" in the spirit of [DS1, Theorem 8.2] has not been constructed under Condition 1.1. In this paper, we address these problems for an intermediate class of potentials, i.e., a class smaller than that defined by Condition 1.1, but larger than the one studied in [DS1].

**1.3 Ideas of procedure and results.** Let us give an outline of a possible procedure for solving the problems posed in the previous subsection. This procedure is implemented for the subclass of potentials introduced in Section 2. The corresponding (main) results are stated more precisely in Theorem 6.3. For simplicity, in the discussion below, we assume that $V$ is negative.

First, we need a global solution (or at least a solution valid outside a compact set) of the eikonal equation

$$|\nabla_x S(x,\lambda)|^2 = \lambda - V(x);\ \lambda \geq 0.$$

The existence for $\lambda = 0$ is not known under Condition 1.1. We could define $S(\cdot,\lambda)$ to be the distance in the metric $g_\lambda = (\lambda - V)\mathrm{d}x^2$ from a point in $\mathbb{R}^d$ to the origin, i.e. $S(x,\lambda) = d_{g_\lambda}(x,0)$. This is the so-called maximal solution of the eikonal equation. However, under Condition 1.1, it is a problematic choice; in fact, for $d \geq 2$, it might be expected that in some generic sense, $S(\cdot,\lambda) \notin C^1(\mathbb{R}^d \backslash \{0\})$. Nevertheless, for the subclass of potentials of Section 2, the above geometric construction is manageable. Thus we consider the corresponding geodesic flow

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi = (\lambda - V(\Phi))^{-1}\nabla_x S(\Phi,\lambda), \qquad \Phi(0,\sigma) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(s,\sigma)_{|s=0} = (\lambda - V(0))^{-1/2}\sigma; \qquad (s,\sigma) \in [0,\infty) \times S^{d-1}.$$

It turns out that this flow is a diffeomorphism $\Phi : \mathbb{R}_+ \times S^{d-1} \to \mathbb{R}^d \setminus \{0\}$.

Next, for an appropriate Jacobian type function $J$ (see (3.10a)) and $v$ in an appropriate dense subset of $L^2(\mathbb{R}^d)$ , we introduce

$$F^+(\lambda)v = \mathcal{G}\text{–}\lim_{s\to\infty} \left(J^{1/2}\mathrm{e}^{-\mathrm{i}S(\cdot,\lambda)}R(\lambda + \mathrm{i}0)v\right)\left(\Phi(s,\cdot)\right), \tag{1.9}$$

where $\mathcal{G} := L^2(S^{d-1}, \mathrm{d}\sigma)$; see Subsection 3.2 for a more detailed account of our procedure. Integrating by parts and using Stone's formula, we then derive the following formula for the orthogonal projection onto the continuous subspace of $H$:

$$\|P_\mathrm{c}v\|^2 = \int_0^\infty \|F^+(\lambda)v\|^2_{\mathcal{G}}\,\mathrm{d}\lambda.$$

This leads to the **distorted Fourier transform**

$$F^+ := \int_0^\infty \oplus F^+(\lambda)\,\mathrm{d}\lambda,$$

which is a partial isometry diagonalizing $H_c$, i.e., $F^+H_c = M_\lambda F^+$ (here, $M_\lambda$ is the operator of multiplication by $\lambda$). Using some low-energy radiation condition bounds valid under the conditions of Section 2, we show the existence of the limit (1.9). The reader may consult (5.10) for a somewhat cleaner definition.

Now we can address the problems of Subsection 1.2 (under these conditions). Indeed,

$$\mathrm{Ran}(\delta(0)) = \mathcal{E}_0 = \{u \in (B^\mu)^* \,|\, Hu = 0\}$$

follows from the fact that

$$\begin{aligned} &F^+(0) : B^\mu \to \mathcal{G} \text{ is onto},\\ &F^+(0)^* : \mathcal{G} \to \mathcal{E}_0 \text{ is a bi-continuous isomorphism},\\ &\delta(0) = F^+(0)^*F^+(0). \end{aligned} \tag{1.10}$$

Furthermore, (1.10) constitutes a parametrization of $\mathcal{E}_0$. The isomorphism $F^+(0)^*$ (called the **wave matrix at zero energy**) is given more explicitly as follows. For all $u = 2\pi\mathrm{i}F^+(0)^*\tau \in \mathcal{E}_0$,

$$u(x) - J^{-1/2}(x)\big(\mathrm{e}^{\mathrm{i}S(x,0)}\tau(\sigma) - \mathrm{e}^{-\mathrm{i}S(x,0)}\tilde{\tau}(\sigma)\big) \in (B^\mu)^*_0;\; x = \Phi(t,\sigma). \tag{1.11}$$

The function $\tilde{\tau} \in \mathcal{G}$ in (1.11) is uniquely determined from $u$ and is of the form $\tilde{\tau} = S(0)^{-1}\tau$, where $S(0)$ is a unitary operator on $\mathcal{G}$. This operator is called the **scattering matrix at zero energy**. Combined with similar constructions for $\lambda > 0$, the scattering matrix $S(\lambda)$ is strongly continuous in $\lambda \geq 0$. Hence this *renormalized* stationary scattering theory is *regular at zero energy*.

## 2 Class of potentials

We introduce the class of potentials studied in this paper. The zero energy dynamics for this class of potentials is generically qualitatively very different (unless $d = 1$) from that for potentials in the smaller class of [DS1]. We give an example to that effect.

### 2.1 Conditions.

**Condition 2.1** (Unperturbed potential)**.** Let $V = V_1 + V_2$ be a real-valued function defined on $\mathbb{R}^d$, $d \geq 1$. There exists $\mu \in (0,2)$ such that the following conditions (1)–(4) hold.

(1) There exists $\epsilon_1 > 0$ such that $V_1(x) \le -\epsilon_1 \langle x\rangle^{-\mu}$.
(2) $V_1 \in C^\infty(\mathbb{R}^d)$. For all $\alpha \in \mathbb{N}_0^d$ there exists $C_\alpha > 0$ such that

$$\langle x\rangle^{\mu+|\alpha|}|\partial^\alpha V_1(x)| \le C_\alpha.$$

(3) $V_1(x) = V_{\mathrm{rad}}(|x|)$ is spherically symmetric, and there exists $\tilde\epsilon_1 > 0$ such that

$$-2V_{\mathrm{rad}}(r) - rV'_{\mathrm{rad}}(r) \ge -\tilde\epsilon_1 V_{\mathrm{rad}}(r).$$

(4) $V_2$ is compactly supported and $V_2 \in L^p(\mathbb{R}^d)$, where $p = 2$ if $d = 1, 2, 3$ and $p > d/2$ if $d \ge 4$.

Note that the bound of (3) coincides with the bound Condition 1.1(3) for a spherically symmetric potential.

Given Condition 2.1, we consider the class $\mathcal{W}$ of real-valued smooth functions $W$ on $\mathbb{R}^d$ obeying, for all $\alpha \in \mathbb{N}_0^d$,

$$\sup_{x\in\mathbb{R}^d} \langle x\rangle^{\mu+|\alpha|}|\partial^\alpha W(x)| < \infty. \tag{2.1a}$$

Given $l \in \mathbb{N}$, we say that $W_\epsilon \in \mathcal{W}$ is **$\epsilon$-small** if for some $\epsilon > 0$,

$$\max_{|\alpha|\le l} \sup_{x\in\mathbb{R}^d} \langle x\rangle^{\mu+|\alpha|}|\partial^\alpha W_\epsilon(x)| \le \epsilon. \tag{2.1b}$$

Clearly, the left-hand side of (2.1b) depends on $l$; however, we prefer for the above terminology of $\epsilon$-smallness to suppress this dependence. If in a given context $l$ is not specified, it is tacitly understood that $l = 2$ (although, for example, $l = 1$ suffices for Proposition 4.1). We use $l = 4$ in Lemma 3.2 stated below. Similarly, we need $l \ge 4$ in Lemma 3.3 and Proposition 4.2 (a sufficient choice $l = l(\mu, d)$ can be calculated; however, we do not bother). Consequently, our main result Theorem 6.3 depends on some fixed $l \ge 4$ in the definition (2.1b) of $\epsilon$-small perturbations.

We study potentials of the form $V_\epsilon = V + W_\epsilon$, where $V = V_{\mathrm{rad}} + V_2$ satisfies Condition 2.1 and $W_\epsilon \in \mathcal{W}$ is an $\epsilon$-small perturbation. The class $\mathcal{V}_\epsilon$ of such potentials $V_\epsilon$ is a particular subclass of the one defined by Condition 1.1; here we require that $\epsilon$ be small. In fact, at various other points of the paper, we also need to take $\epsilon > 0$ small; however, this choice is expressible in terms of $V_{\mathrm{rad}}$ only, which henceforth is considered fixed. For convenience, we assume throughout the paper that

$$V_{\mathrm{rad}}(r) = V_{\mathrm{rad}}(0) \text{ for } r \le R := (-V_{\mathrm{rad}}(0))^{-1/2}, \tag{2.2a}$$

and, similarly for perturbations, that

$$W_\epsilon(x) = 0 \text{ for } |x| \le R. \tag{2.2b}$$

We can freely assume (2.2a) and (2.2b). As for $V_{\rm rad}$, we can assume (2.2a) by possibly choosing $\epsilon_1$ smaller (but not changing $\tilde\epsilon_1$) and changing $V_2$. This fact, while elementary, is not completely obvious. Thus, we give a proof.

Write $1 = \chi_+ + \chi_-$, where $\chi_+, \chi_- \in C^\infty(\mathbb{R}_+)$ are monotone, $\chi_+(t) = 1$ for $t \geq 2$, and $\chi_+(t) = 0$ for $t \leq 1$. Introduce

$$V_n(r) = V_{\rm rad}(r)\chi_+(r/n) - n^{-2}\chi_-(r/n);\ n \in \mathbb{N}. \tag{2.3}$$

We claim that for $n$ so large that $\epsilon_1\langle 2n\rangle^{-\mu} \geq n^{-2}$, Conditions 2.1(1)–(3) and (2.2a) hold with $\epsilon_1$ replaced by $n^{-2}$, new constants $C_\alpha$, the same constant $\tilde\epsilon_1$, and $R = n$. To see that indeed the same $\tilde\epsilon_1$ works, consider the estimates

$$\begin{aligned} -rV_n'(r) &\geq (2-\tilde\epsilon_1)V_n(r) + (2-\tilde\epsilon_1)n^{-2}\chi_-(r/n) - \frac{r}{n}\chi_-'(r/n)\big(-V_{\rm rad}(r) - n^{-2}\big)\\ &\geq (2-\tilde\epsilon_1)V_n(r) - \frac{r}{n}\chi_-'(r/n)\big(\epsilon_1\langle 2n\rangle^{-\mu} - n^{-2}\big)\\ &\geq (2-\tilde\epsilon_1)V_n(r), \end{aligned}$$

where we have assumed that $\tilde\epsilon_1 \leq 2$. The other statements are obvious.

Similarly, we may assume (2.2b) with different $V_2$ and possibly smaller $\epsilon$.

**2.1.1 Example.** Let $g \in C^\infty(\mathbb{R})$ be $2\pi$-periodic with $\max g' \geq 1 - \mu/2$. Let $\chi \in C^\infty(\mathbb{R}_+)$ satisfy $\chi(r) = 0$ for $r < 1$ and $\chi(r) = 1$ for $r > 2$. Similarly, introduce for $\mu \in (0, 2)$ and (large) $n \in \mathbb{N}$ a function $h = h_n \in C^\infty(\mathbb{R}_+)$ satisfying

$$\begin{cases} h(r) = r/n & \text{for } r \leq n,\\ h(r) = (1-\mu/2)^{-1}r^{1-\mu/2} + C_n & \text{for } r \geq 2n,\\ h(r) > \max(0, -rh''(r)) & \text{for } r > 0. \end{cases}$$

Note that the construction (2.3) with $V_{\rm rad}(r) = -r^{-\mu}$ leads to the particular example $h_n(r) = \int_0^r \sqrt{-V_n(t)}{\rm d}t$. We construct in dimension $d = 2$ the potential in terms of a parameter $\epsilon \geq 0$ and polar coordinates $(r, \theta)$ (i.e., $x = (r\cos\theta, r\sin\theta)$)

$$\begin{aligned} S_\epsilon(x, \lambda = 0) &= h_n(r)\exp\{\epsilon g(\theta - \epsilon \ln r)\chi(r/n)\},\\ V_\epsilon(x) &= -|\nabla S_\epsilon(x, \lambda = 0)|^2. \end{aligned}$$

Clearly, $V_{\epsilon=0}(x) = V_{\rm rad}(r)$ satisfies Condition 2.1 and (2.2a) (the latter with $R = n$). Moreover, clearly $W_\epsilon(x) := V_\epsilon(x) - V_{\rm rad}(r)$ satisfies (2.1a) and (2.2b). Also, for any $l \in \mathbb{N}$, there exists $C > 0$ sufficiently large and possibly depending on $n$ such that the potential $W_\epsilon$ is $(C\epsilon)$-small. So, up to a linear reparametrization, (2.1b) is also satisfied.

This example does not fit into the framework of [DS1]. In fact, for the class studied there, classical zero energy scattering orbits have asymptotic normalized velocities. Were this to be the case for the above example, $\lim_{t\to\pm\infty}\theta(t)$ would exist. However, as the following arguments show, this limit does not exist.

Consider the flow (in polar coordinates)

$$\begin{cases} \dot r(=\tfrac{\mathrm{d}}{\mathrm{d}s}r(s)) &= (-V_\epsilon(x))^{-1}\partial_r S_\epsilon(x,\lambda=0) \\ \dot\theta(=\tfrac{\mathrm{d}}{\mathrm{d}s}\theta(s)) &= (-V_\epsilon(x)r^2)^{-1}\partial_\theta S_\epsilon(x,\lambda=0) \\ (r,\theta)(s=1) &= (n,\sigma). \end{cases} \tag{2.4a}$$

Noticing that for $\epsilon > 0$ small, $\partial_r S_\epsilon > 0$, we can consider $\theta$ as a function of $r$ determined by the single equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = F(r,\theta) := \frac{\epsilon}{r}\frac{g'\chi}{rh'/h + \epsilon\frac{r}{n}g\chi' - \epsilon^2 g'\chi}. \tag{2.4b}$$

Here, of course, $g$ and $\chi$ are functions of $\psi := \theta - \epsilon\ln r$ and $r/n$, respectively. For $r \geq 2n$, (2.4b) reduces to

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{\epsilon}{r}\Big(\frac{g'}{1-\frac{\mu}{2}-\epsilon^2 g'} + O(r^{\mu/2-1}) - 1\Big) = \frac{\epsilon}{r}\Big(\frac{(1+\epsilon^2)g' - 1 + \frac{\mu}{2}}{1-\frac{\mu}{2}-\epsilon^2 g'} + O(r^{\mu/2-1})\Big).$$

Introducing a new time $\mathrm{d}\tau/\mathrm{d}r = \epsilon r^{-1}$, we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{(1+\epsilon^2)g'(\psi) - 1 + \frac{\mu}{2}}{1-\frac{\mu}{2}-\epsilon^2 g'(\psi)} + O\big(\mathrm{e}^{(\mu/2-1)\tau/\epsilon}\big). \tag{2.4c}$$

Note that to leading order, term (2.4c) is autonomous. Any solution $\psi$ to (2.4c) converges to a root of the corresponding fixed point equation

$$g'(\psi) = (1-\mu/2)(1+\epsilon^2)^{-1},$$

say $\psi \to \psi_0$. In particular, going back to the time $s$ of (2.4a), we conclude that $\theta - \epsilon\ln r \to \psi_0$ as $s \to \infty$; and, since $\ln r \to \infty$, also $\theta \to \infty$. So the asymptotic normalized velocity does not exist for the flow (2.4a). Noticing that (2.4a) defines a class of zero energy scattering orbits in a reparametrized time, we conclude that these orbits do not have asymptotic normalized velocity.

In Subsection 3.1, we study a flow of the type (2.4a) for general $\epsilon$-small perturbations in any dimension (extended as well to any non-negative energy).

## 3 Eikonal equation

One reason for considering only $\mathcal{V}_\epsilon$ with small $\epsilon$ is that classical mechanics is particularly nice for this class. In particular (cf. [CS]), there exists a global solution of the eikonal equation

$$\begin{aligned} &|\nabla S_\epsilon|^2 = K_\epsilon, \\ &K_\epsilon(x) = K_\epsilon(x, \lambda) := \lambda - V_{\mathrm{rad}}(|x|) - W_\epsilon(x), \ \lambda \ge 0. \end{aligned} \tag{3.1}$$

We also introduce

$$\begin{aligned} K_0(r) &= \lambda - V_{\mathrm{rad}}(r), \\ f(r, \lambda) &= \sqrt{K_0(r)} \\ S_0(x) &= S_0(|x|) = \int_0^{|x|} f(r, \lambda)\mathrm{d}r. \end{aligned} \tag{3.2}$$

As used in [CS], we have

$$crf(r, \lambda) \le S_0(r) \le Crf(r, \lambda)$$

uniformly in $r, \lambda \ge 0$. Notice also that $S_0$ is a solution to (3.1) if $W_\epsilon = 0$.

**Proposition 3.1** ([CS]). *Let $V_{\mathrm{rad}}$ be given as in Condition 2.1 (assuming also* (2.2a)*) and let $l \ge 2$. There exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ and all $\epsilon$-small perturbations $W_\epsilon$ (assuming* (2.2b)*), there exists a family of real-valued smooth functions $\{S_\epsilon \in C^\infty(\mathbb{R}^d \setminus \{0\}) | \lambda \ge 0\}$ with the following properties:*

(i) $|\nabla S_\epsilon(x)|^2 = K_\epsilon(x)$ *for* $x \in \mathbb{R}^d \setminus \{0\}$*;*

(ii) $S_\epsilon(x) = S_0(x) = f(0, \lambda)|x|$ *for* $r = |x| \le R = (-V_{\mathrm{rad}}(0))^{-1/2}$*;*

(iii) *for all* $r_0 > 0$*,*

$$\max_{|\alpha| \le l} \sup_{\lambda \ge 0} \sup_{|x| \ge r_0} \langle x \rangle^{|\alpha|} \left| S_0(x)^{-1} \partial_x^\alpha S_\epsilon(x) \right| < \infty$$

*uniformly in $W_\epsilon$ with $\epsilon \in (0, \epsilon_0]$;*

(iv) *uniformly in $W_\epsilon$, $\lambda \ge 0$, and $x \in \mathbb{R}^d \setminus \{0\}$,*

$$\begin{aligned} S_\epsilon(x) &= S_0(r)\big(1 + O(\epsilon)\big), \\ \nabla S_\epsilon(x) &= f(r, \lambda)\ \big(\langle \hat{x}| + O\big(\epsilon^{3/4}\big)\big);\ \hat{x} := x/r, \\ \nabla^2 S_\epsilon(x) &= \frac{f(r, \lambda)}{r}, \left(P_\perp + \frac{rf'(r, \lambda)}{f(r, \lambda)} P + O\big(\epsilon^{1/2}\big)\right), \\ P &= P(\hat{x}) := |\hat{x}\rangle\langle\hat{x}|, \quad P_\perp := I - P; \end{aligned}$$

(v) *for all* $\alpha \in \mathbb{N}_0^d$, $\partial_x^\alpha S_\epsilon \in C(\mathbb{R}^d \setminus \{0\} \times [0, \infty))$, $S_\epsilon = S_\epsilon(x, \lambda)$.

We remark that for $l = 2$, the bounds (iii) follow from (iv). Having $l > 2$ influences only (iii) and requires, according to the proposition, an $\epsilon_0 > 0$ possibly depending on $l$. It is tempting to conjecture that the proposition holds with $l = 2$

and the constraint $|\alpha| \leq l$ of (iii) replaced by $|\alpha| \leq k$, for arbitrary given $k$. The new bounds would be uniform in perturbations from any bounded family (bounded in terms of the seminorms (2.1a)). However, whether it does is an open problem. In fact, it is not known whether $\epsilon_0 > 0$ can be chosen independently of $l$, although there are known estimates weaker than (iii) which are independent of $l$; cf. [CS, Proposition 1.2]. The latter deficiency gives rise to a slight complication when dealing with $S_\epsilon$ in the context of pseudodifferential operators; see (4.8c).

**3.1 Geometric properties.** The construction of the function $S_\epsilon$ of Proposition 3.1 is given by a geometric procedure. We consider the metric $g_\epsilon = K_\epsilon \mathrm{d}x^2$ on the manifold $M = \mathbb{R}^d$ and let $o = 0 \in M$. Then for all $x \in M$, the number $S_\epsilon(x) = d_{g_\epsilon}(x, o)$ is the distance in this metric from $x$ to $o$, The function $S_\epsilon$ is called the **maximal** solution of the eikonal equation.

**3.1.1 Flow.** In the metric $g_\epsilon$, the unit-sphere in the tangent space $TM_o$ at the origin $o = 0$ is given by $f(0, \lambda)^{-1} S^{d-1}$, where $S^{d-1}$ is the standard unit-sphere in $\mathbb{R}^d$. We denote generic points of $S^{d-1}$ by $\sigma$ and let $\mathrm{d}\sigma$ denote the standard euclidean surface measure on $S^{d-1}$. The exponential mapping at the origin for the metric $g_\epsilon$ defines a diffeomorphism $\Phi : \mathbb{R}_+ \times S^{d-1} \to \mathbb{R}^d \setminus \{0\}$ by

$$\Phi(s, \sigma) = \exp_o(s f(0, \lambda)^{-1} \sigma),$$

and we have the flow property

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi = \big(K_\epsilon^{-1} \nabla S_\epsilon\big)(\Phi); \; s > 0, \sigma \in S^{d-1}. \tag{3.3a}$$

Since, by assumption (cf. (2.2a) and (2.2b)) the conformal factor $K_\epsilon$ is constant for $r = |x| \leq R = (-V_{\mathrm{rad}}(0))^{-1/2}$, we have explicitly

$$\Phi(s, \sigma) = s f(0, \lambda)^{-1} \sigma \text{ for } s \leq 1.$$

Hence we can supplement (3.3a) by the "initial condition"

$$\Phi(1, \sigma) = f(0, \lambda)^{-1} \sigma. \tag{3.3b}$$

The assertion above that $\Phi$ is a diffeomorphism can be proved taking (3.3a) and (3.3b) as a definition of the map. Note that a consequence of (3.1), (3.3a), and (3.3b) is that $d_{g_\epsilon}(x, o) = S_\epsilon(x) = s$. This point of view of definition is taken in the proof of an analogous statement [ACH, Proposition 2.2]. However, the mapping property can also be viewed as an independent part of the proof of Proposition

3.1 given in [CS]. The flow $\Phi$ constitutes a family of reparametrized Hamiltonian orbits for the Hamiltonian

$$h_\epsilon = \xi^2 + V_{\mathrm{rad}}(|x|) + W_\epsilon(x) \tag{3.4}$$

at energy $\lambda$. It is continuous in $\lambda$, i.e., $\Phi \in C(\mathbb{R}_+ \times S^{d-1} \times [0,\infty))$, $\Phi = \Phi(s,\sigma,\lambda)$.

**3.1.2 Surface measure.** The mapping

$$\Phi(s,\cdot) : S^{d-1} \to \mathcal{S}_\epsilon(s) := \{x \in \mathbb{R}^d \mid S_\epsilon(x) = s\}$$

induces a measure on $S^{d-1}$ by the pullback $\mathrm{d}\omega = \Phi(s,\cdot)^*\mathrm{d}A(x)$, where $\mathrm{d}A(x)$ refers to the euclidean surface measure on $\mathcal{S}_\epsilon(s)$. A computation using (3.3a) and (3.3b) shows that

$$\begin{aligned} &\mathrm{d}\omega = K_\epsilon^{1/2}(x) m_\epsilon(x)\,\mathrm{d}\sigma;\\ &m_\epsilon(x) = f(0,\lambda)^{2-d} K_\epsilon^{-1}(x) \exp\Big(\int_1^s \big(K_\epsilon^{-1} \triangle S_\epsilon\big)\big(\Phi(t,\sigma)\big)\,\mathrm{d}t\Big),\ x = \Phi(s,\sigma). \end{aligned} \tag{3.5}$$

To see this, take local coordinates $\theta_1, \dots, \theta_{d-1}$ on $S^{d-1}$, write (3.3a) as $\dot\eta = F(\eta)$, and let $A$ be the $d \times (d-1)$−matrix with entries $a_{ki} = \partial_{\theta_i}\eta^k$. The pullback $\mathrm{d}\omega$ is computed from the metric $g_{ij} = (A^TA)_{ij}$, noting that the determinant $|g|$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln|g| = \mathrm{tr}\big((A^TA)^{-1} \frac{\mathrm{d}}{\mathrm{d}s}(A^TA)\big) = \mathrm{tr}\big((B^T + B)P\big) = 2K_\epsilon^{-1}\triangle S_\epsilon - \frac{\mathrm{d}}{\mathrm{d}s} \ln K_\epsilon(\Phi),$$

where $B = F'$ (the Jacobian matrix) and $P_{kl} = \delta_{kl} - (\partial_k S_\epsilon)(\partial_l S_\epsilon)K_\epsilon^{-1}$. We integrate and obtain

$$\mathrm{d}\omega = |g|^{1/2}\mathrm{d}\theta = f(0,\lambda)K_\epsilon^{-1/2}(x)\exp\Big(\int_1^s \big(K_\epsilon^{-1}\triangle S_\epsilon\big)\big(\Phi(t,\sigma)\big)\mathrm{d}t\Big) f(0,\lambda)^{1-d}\,\mathrm{d}\sigma,$$

which yields (3.5).

**3.1.3 Volume measure.** In combination with (3.5), the co-area formula (cf. [Ev, Theorem C.5]) yields for (reasonable) functions $\phi$ on $\mathbb{R}^d$

$$\int \phi(x)\mathrm{d}x = \int_0^\infty \mathrm{d}s \int_{\mathcal{S}_\epsilon(s)} \phi K_\epsilon^{-1/2}\,\mathrm{d}A(x) = \int_0^\infty \mathrm{d}s \int_{S^{d-1}} \big(\phi m_\epsilon\big)\big(\Phi(s,\cdot)\big)\mathrm{d}\sigma. \tag{3.6}$$

Let

$$\mathcal{B}_\epsilon(s) = \{x \in \mathbb{R}^d | S_\epsilon(x) \le s\} \text{ for } s > 0. \tag{3.7}$$

Clearly, $\partial\mathcal{B}_\epsilon(s) = \mathcal{S}_\epsilon(s)$, so the Divergence Theorem (cf. [Ev, Theorem C.1]) yields for $j = 1, \dots, d$,

$$\begin{aligned}\int_{\mathcal{B}_\epsilon(s)} (\partial_j\phi)(x)dx &= \int_{\mathcal{S}_\epsilon(s)} \phi(\partial_j S_\epsilon)K_\epsilon^{-1/2}\,\mathrm{d}A(x)\\ &= \int_{S^{d-1}} \big(\phi(\partial_j S_\epsilon)m_\epsilon\big)\big(\Phi(s,\cdot)\big)\,\mathrm{d}\sigma.\end{aligned} \tag{3.8}$$

**3.2 Diagonalization.** We consider the Hamiltonian $H = -\Delta + V_\epsilon$ on $\mathcal{H} = L^2(\mathbb{R}^d)$ under the conditions of Section 2. Denoting the corresponding continuous part by $H_\mathrm{c}$, we aim to construct a diagonalizing transform $H_\mathrm{c} \to M_\lambda$, where $M_\lambda$ is multiplication by $\lambda$ in $\widetilde{\mathcal{H}} := L^2(\mathbb{R}_+, \mathrm{d}\lambda; \mathcal{G})$ with

$$\mathcal{G} := L^2(S^{d-1}, \mathrm{d}\sigma). \tag{3.9}$$

Here, we explain our procedure, leaving the details to the exposition in Section 5. Assuming below that $v \in L^2_3$ (recall $L^2_m := \langle x\rangle^{-m}L^2(\mathbb{R}^d)$), we have by Stone's formula (cf. [RS]),

$$\begin{aligned}\|P_\mathrm{c}v\|^2 &= \pi^{-1}\lim_{\lambda_0\to\infty}\int_0^{\lambda_0}\langle *\rangle v, (\operatorname{Im} R(\lambda+\mathrm{i}0))v\,\mathrm{d}\lambda\\ &= \pi^{-1}\int_0^\infty \langle *\rangle v, (\operatorname{Im} R(\lambda+\mathrm{i}0))v\,\mathrm{d}\lambda.\end{aligned}$$

Hence, writing $u = R(\lambda+\mathrm{i}0)v$, $p_j = -\mathrm{i}\partial_j$ and using (3.8), we obtain

$$\begin{aligned}\|P_\mathrm{c}v\|^2 &= \pi^{-1}\int_0^\infty \operatorname{Im}\langle *\rangle(H-\lambda)u, u\,\mathrm{d}\lambda\\ &= \pi^{-1}\int_0^\infty \lim_{s\to\infty}\operatorname{Re}\sum_{j=1}^d\int_{S^{d-1}} \big(\overline{(p_j u)}u(\partial_j S_\epsilon)m_\epsilon\big)\big(\Phi(s,\cdot)\big)\mathrm{d}\sigma\mathrm{d}\lambda.\end{aligned}$$

Next we substitute $p_j u = (p_j - \partial_j S_\epsilon)u + (\partial_j S_\epsilon)u$. It turns out that the contribution from the first term vanishes in the limit $s \to \infty$. Hence

$$\|P_\mathrm{c}v\|^2 = \pi^{-1}\int_0^\infty \lim_{s\to\infty}\int_{S^{d-1}} \big(|u|^2K_\epsilon m_\epsilon\big)\big(\Phi(s,\cdot)\big)\,\mathrm{d}\sigma\mathrm{d}\lambda.$$

We are led to define

$$F^+(\lambda)v = \mathcal{G}\text{–}\lim_{s\to\infty}\pi^{-1/2}\big(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}R(\lambda+\mathrm{i}0)v\big)\big(\Phi(s,\cdot)\big), \tag{3.10a}$$

yielding

$$\|P_\mathrm{c}v\|^2 = \int_0^\infty \|F^+(\lambda)v\|^2_{\mathcal{G}}\,\mathrm{d}\lambda.$$

Finally, we see that the "distorted Fourier transform"

$$F^+ := \int_0^\infty \oplus F^+(\lambda)\,\mathrm{d}\lambda$$

diagonalizes $H_c$, i.e., $F^+H_c = M_\lambda F^+$.

Similarly, we can define the "distorted Fourier transform"

$$F^- := \int_0^\infty \oplus F^-(\lambda)\,\mathrm{d}\lambda,$$

where

$$F^-(\lambda)v = \mathcal{G}\text{–}\lim_{s\to\infty} \pi^{-1/2}\big(\mathrm{e}^{\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} R(\lambda - \mathrm{i}0)v\big)\big(\Phi(s,\cdot)\big). \tag{3.10b}$$

**3.3 Outgoing approximate generalized eigenfunctions.** We conclude this section by stating and proving a technical result motivated by the formulas (3.10a) and (3.10b). This enables us to construct outgoing approximations of generalized eigenfunctions, which, in turn, are used to construct exact generalized eigenfunctions.

Let $\tau \in C^\infty(S^{d-1})$ and $\lambda \geq 0$ be given. Define the function $\tilde{u}$ by

$$\tilde{u}(x) = \pi^{1/2}\big(\chi \mathrm{e}^{\mathrm{i}S_\epsilon} K_\epsilon^{-1/2} m_\epsilon^{-1/2}\big)(x)\tau(\sigma);\ x = \Phi(s,\sigma),\ \ \chi(x) = \chi(|x|). \tag{3.11}$$

Here, $\chi$ is a smooth non-negative function satisfying $\chi(x) = 0$ for $|x| \leq 1$ and $\chi(x) = 1$ for $|x| \geq 2$. A short computation (using, for example, (4.10) stated below) shows that

$$(H - \lambda)\tilde{u} = -\pi^{1/2}\chi \mathrm{e}^{\mathrm{i}S_\epsilon(x)} \triangle_x\big((K_\epsilon^{-1/2} m_\epsilon^{-1/2})(x)\tau(\sigma)\big) + \text{ compactly supported term.}$$

It will be important for us that the first term on the right is also small at $\infty$.

**Lemma 3.2.** *Let $\varepsilon > 0$ be given, and suppose $l = 4$. Then for sufficiently small $\epsilon_0 > 0$, for all $\tau \in C^\infty(S^{d-1})$ and all $\lambda_0 > 0$, there exists $C > 0$ such that*

$$\forall |\alpha| \leq 2\ \forall |x| \geq 1 : \big|K_\epsilon^{1/2} m_\epsilon^{1/2} \partial_x^\alpha\big((K_\epsilon^{-1/2} m_\epsilon^{-1/2})(x)\tau(\sigma)\big)\big| \leq C\langle x\rangle^{\varepsilon - |\alpha|} \tag{3.12a}$$

*uniformly in $W_\epsilon$ with $\epsilon \in (0, \epsilon_0]$ and in $\lambda \in [0, \lambda_0]$. In particular, the function $\tilde{u}$ of* (3.11) *satisfies*

$$K_\epsilon^{1/2} m_\epsilon^{1/2}(H - \lambda)\tilde{u} = O\big(\langle x\rangle^{\varepsilon - 2}\big). \tag{3.12b}$$

**Proof.** Let

$$T_1(x) = \int_1^s \big(K_\epsilon^{-1} \triangle S_\epsilon\big)\big(\Phi(t,\sigma)\big)\mathrm{d}t,$$
$$T_2(x) = \tau(\sigma);\ x = \Phi(s,\sigma).$$

We need to show that

$$\forall|\alpha| \le 2\,\forall|x| \ge 1 : \left|\partial_x^\alpha T_j(x)\right| \le C\langle x\rangle^{\varepsilon-|\alpha|};\ j = 1, 2. \tag{3.13}$$

To that end, we use the diffeomorphism $\psi : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$y = \Psi(x) = S_0(x)\hat{x} = \int_0^{|x|} f(r, \lambda)\mathrm{d}r\,|x|^{-1}x$$

and invoke results of [CS] for the model metric $\tilde{g}_\epsilon = (\Psi^*)^{-1}g_\epsilon,\ g_\epsilon = K_\epsilon \mathrm{d}x^2$. This idea of changing framework is actually behind Proposition 3.1 too. Here, we use the bounds

$$\forall|\alpha| \le 2 : |\partial_x^\alpha \Psi(x)| \le C\langle x\rangle^{-|\alpha|}\langle\Psi(x)\rangle, \tag{3.14a}$$

$$\forall|\beta| \le 2 : |\partial_y^\beta \Psi^{-1}(y)| \le C\langle y\rangle^{-|\beta|}\langle\Psi^{-1}(y)\rangle, \tag{3.14b}$$

which are uniform in $\lambda \in [0, \lambda_0]$.

**Step I.** We note the representation

$$\Phi(t, \sigma) = \Psi^{-1}\big(\tilde{\gamma}_{\Psi(x)}\big(t/S_\epsilon(x)\big)\big),\ x = \Phi(s, \sigma) = \Phi(S_\epsilon(x), \sigma), \tag{3.15}$$

where, in the notation of [CS], $\tilde{\gamma}_y(t) = ty + \kappa_y(t)$ is the unique geodesic in the metric $\tilde{g}_\epsilon$ emanating from $0 \in \mathbb{R}^d$ with value $y$ at time 1. Thus we can rewrite $T_j(x)$ as

$$T_1(x) = \int_1^{S_\epsilon(x)} \phi\big(\tilde{\gamma}_{\Psi(x)}\big(t/S_\epsilon(x)\big)\big)\mathrm{d}t;\ \phi = (K_\epsilon^{-1}\triangle S_\epsilon)\circ\Psi^{-1},$$

$$T_2(x) = \tau\big(f(0, \lambda)\Psi^{-1}\big(\tilde{\gamma}_{\Psi(x)}\big(1/S_\epsilon(x)\big)\big)\big).$$

From Proposition 3.1(iii) and (3.14b), we have the bounds (since we have assumed that $l = 4$)

$$\forall|\beta| \le 2 : |\partial_y^\beta \phi| \le C\langle y\rangle^{-1-|\beta|}. \tag{3.16}$$

**Step II.** We prove Sobolev bounds of model geodesics. As in [CS, Section 6], introduce the Sobolev spaces $\mathcal{H}^p := W_0^{1,p}(0, 1)^d$, $1 < p < \infty$, consisting of absolutely continuous functions $h : [0, 1] \to \mathbb{R}^d$ vanishing at the endpoints and having $\dot{h} \in L^p(0, 1)^d = L^p(]0, 1[, \mathbb{R}^d)$ (we use the notation $L^p$ for this vector-valued $L^p$ space). The space $\mathcal{H}^p$ is equipped with the norm

$$\|h\|_{\mathcal{H}^p} = \|\dot{h}\|_p = \Big(\int_0^1 |\dot{h}(t)|^p dt\Big)^{1/p}.$$

By [CS, Proposition 6.8], with reference to the model geodesic $\tilde{\gamma}_y\big(t\big) = ty + \kappa_y(t)$, we have $\kappa_y \in \mathcal{H}^p$ for any prescribed $p \in [2, \infty)$; and for all sufficiently small $\epsilon > 0$,

$$\forall|\beta| \le 2 : \|\partial_y^\beta \kappa\|_{\mathcal{H}^p} \le C_p\langle y\rangle^{1-|\beta|}. \tag{3.17a}$$

We claim that for any such fixed $p$, the following generalization holds. For all $k \in \{0, 1, 2\}$,

$$\forall |\beta| \le 2 : \|t^{k-1}\partial_y^\beta \tilde{\gamma}_y^{(k)}(t)\|_p \le C_p\langle y\rangle^{1-|\beta|}. \tag{3.17b}$$

Here, $\tilde{\gamma}_y^{(k)}$ refers to the $k$-th time-derivative of $\gamma = \tilde{\gamma}_y$. In light of (3.17a) and the Hardy inequality [CS, Lemma 6.1], only the case $k = 2$ needs to be proved. But since $\gamma$ is a geodesic for the metric $\tilde{g}_\epsilon$, the second derivative $\gamma^{(2)}$ is a sum of expressions $\phi_{jk}(\gamma_y)(\dot{\gamma}_y)^j(\dot{\gamma}_y)^k$, where

$$\forall |\beta| \le 2 : |\partial_z^\beta \phi_{jk}| \le C\langle z\rangle^{-1-|\beta|}. \tag{3.18}$$

We use the product and chain rules to calculate derivatives $\partial_y^\beta$, $|\beta| \le 2$, of every such expression. Then, combining (3.17b) for $k = 0, 1$ (and some bigger values of $p$), (3.18), the a priori bounds

$$ct|y| \le |\gamma_y(t)| \le Ct|y| \tag{3.19}$$

(cf. [CS, Lemma 2.1]) and the generalized Hölder estimate, we can obtain the desired bound for any term in the resulting expansion. We omit the details. The reader may consult [CS, Section 6] for similar arguments.

**Step III.** We can treat $T_1(x)$ by combining Proposition 3.1(iii), (3.16), (3.17b), and the generalized Hölder estimate. The smaller the given $\varepsilon > 0$ is, the larger is the $p \ge 2$ (3.17b) required. The estimations are straightforward. For completeness, let us provide the details for $|\alpha| = 1$. We have

$$\begin{aligned}\partial^\alpha T_1 = (\partial^\alpha S_\epsilon)K_\epsilon^{-1}\triangle S_\epsilon + \int_1^{S_\epsilon} \nabla\phi \cdot \Big( &(\partial_y\gamma)_\Psi\big(t/S_\epsilon\big) \cdot \partial^\alpha \Psi \\ &- \dot{\gamma}_\Psi\big(t/S_\epsilon\big)\frac{t}{S_\epsilon^2}(\partial^\alpha S_\epsilon)\Big)\mathrm{d}t.\end{aligned} \tag{3.20}$$

The first term is $O\big(\langle x\rangle^{-1}\big)$. For the second term, we estimate for $\delta = \min(\varepsilon, 1)$

$$|(\nabla\phi)(\gamma_\Psi)| \le C\Big|\frac{t}{S_\epsilon}\Psi\Big|^{\delta-2}$$

and substitute $t \longrightarrow tS_\epsilon$. This leads to the upper bound for the integral

$$Cf\big|\Psi\big|^{\delta-2}\int_0^1 t^{\delta-1}\big(S_\epsilon t^{-1}|\partial_y\gamma)_\Psi(t)| + |\dot{\gamma}_\Psi(t)|\big)\mathrm{d}t.$$

We choose $p \ge 2$ so large that $(\delta - 1)/(1 - 1/p) > -1$ and use (3.17b) to obtain the upper bounds

$$C_1(\delta)fS_\epsilon^{\delta-1} \le C_2(\delta)S_\epsilon^\delta\langle x\rangle^{-1} = O\big(\langle x\rangle^{\varepsilon-1}\big).$$

The case $|\alpha| = 2$ is treated similarly by differentiating (3.20), except that now there is one term involving $\dot\gamma_\Psi(1)$. For this term, we use the formula

$$\dot\gamma_y(t) = 2\int_{1/2}^{1}\Big(\gamma_y^{(1)}(s) + \int_s^t \gamma_y^{(2)}(t')\mathrm{d}t'\Big)\mathrm{d}s \tag{3.21a}$$

with $t = 1$ and again invoke (3.17b).

**Step IV.** We need to treat $T_2(x)$. In addition to (3.21a), we use

$$\gamma_y(t) = \int_0^t \gamma_y^{(1)}(s)\mathrm{d}s. \tag{3.21b}$$

The case $|\alpha| = 0$ is trivial. We treat the case $|\alpha| = 1$ in detail, leaving the remaining case $|\alpha| = 2$ to the reader (it is very similar apart from an application of (3.18) for one term arising after yet another differentiation). We begin with

$$\partial^\alpha T_2 = \nabla\big(\tau\circ\big(f(0,\lambda)\Psi^{-1}\big)\big)\cdot\partial^\alpha\gamma_\Psi\big(1/S_\epsilon\big). \tag{3.22}$$

Here, the first factor is evaluated in $\gamma_\Psi\big(1/S_\epsilon\big)\in S^{d-1}$ and whence (cf. (3.14b)) is bounded (uniformly in $\lambda$). For the second factor of (3.22), we compute

$$\partial^\alpha\gamma_\Psi\big(1/S_\epsilon\big) = (\partial_y\gamma)_\Psi\big(1/S_\epsilon\big)\cdot\partial^\alpha\Psi - \dot\gamma_\Psi\big(1/S_\epsilon\big)S_\epsilon^{-2}\partial^\alpha S_\epsilon. \tag{3.23}$$

Consider the first term. Using Hölder's inequality, (3.17b), and (3.21b), we obtain the estimate

$$|(\partial_y\gamma)_\Psi(t)| \le C_p t^{1/p'},\quad \frac{1}{p'}+\frac{1}{p} = 1,$$

which is used with $t = 1/S_\epsilon$. Moreover, because of (3.14a), we have $|\partial^\alpha\Psi| \le C\langle x\rangle^{-1}\langle\Psi\rangle$, so altogether

$$|(\partial_y\gamma)_\Psi\big(1/S_\epsilon\big)\cdot\partial^\alpha\Psi| \le C_p S_\epsilon^{1-1/p'}\langle x\rangle^{-1};\, |x|\ge 1.$$

If $p \ge 2$ is chosen large enough, $1 - 1/p' = 1/p \le \varepsilon$, so the first term of (3.23) conforms with (3.13) with $j = 2$ and $|\alpha| = 1$.

Consider now the second term. Using Hölder's inequality, (3.17b), and (3.21a), we obtain the estimate

$$|\dot\gamma_\Psi\big(t\big)| \le C_p t^{-1/p}\langle\Psi\rangle,$$

which again is used with $t = 1/S_\epsilon$, yielding

$$|\dot\gamma_\Psi\big(1/S_\epsilon\big)| \le C_p S_\epsilon^{1+1/p},\ |x|\ge 1.$$

Moreover, $|S_\epsilon^{-2}\partial^\alpha S_\epsilon| \le C S_\epsilon^{-1}\langle x\rangle^{-1}$, so altogether

$$|\dot\gamma_\Psi(1/S_\epsilon)S_\epsilon^{-2}\partial^\alpha S_\epsilon| \le C_p S_\epsilon^{1/p}\langle x\rangle^{-1},$$

which again conforms with (3.13) with $j = 2$ and $|\alpha| = 1$, provided $p \ge 2$ is chosen as above. □

**Remark.** The similar result [ACH, Proposition 2.5] also contains a loss of decay (in Lemma 3.2 expressed by the power $\langle x\rangle^{\varepsilon}$). In general, such loss cannot be avoided. This can seen from the example in Subsection 2.1.1.

**3.3.1 Generalized eigenfunctions.** We learn from (3.6) and (3.12b) that

$$(H-\lambda)\tilde u \in f^{1/2}L^2_\delta, \quad \delta < \frac{3}{2} - \frac{\mu}{2} - \varepsilon. \tag{3.24}$$

In particular, we can choose $\delta > 1/2$ in (3.24), provided $\varepsilon > 0$ is small enough. With such $\delta$, we can define the generalized eigenfunctions

$$u^\pm = u^\pm(\cdot,\lambda) = \tilde u - R(\lambda \pm \mathrm{i}0)(H-\lambda)\tilde u. \tag{3.25}$$

Since, intuitively, $u^+$ is a purely outgoing exact eigenfunction, it should be 0. This is the content of the following result.

**Lemma 3.3.** *There exist $l \ge 4$ and $\epsilon_0 > 0$ such that for all $\epsilon$-small perturbations $W_\epsilon$ with $\epsilon \in (0,\epsilon_0]$, the generalized eigenfunction $u^+$ of* (3.25) *vanishes for any $\tau \in C^\infty(S^{d-1})$ and any $\lambda \ge 0$.*

**Proof.** By Proposition 1.2, the second term of (3.25) is in $f^{-1/2}B(|x|)^*$. The first term is also in this space, as can be seen from an explicit calculation using (3.6) and the Besov space bound (5.13c) (stated below); see (5.17) for a more general statement. So we conclude that $f^{1/2}u^\pm \in B(|x|)^*$.

Let us argue for $\lambda = 0$ only. The case $\lambda > 0$ can be treated similarly using Remark 1.4. In light of Proposition 1.3, to conclude that $u^+ = 0$ for $\lambda = 0$, it suffices to show (with reference to the notation (1.6)), that for some small $\kappa > 0$,

$$\mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))u^+ \in (B^\mu)^*_0 \quad \text{for all } \chi_- \in C_c^\infty(\mathbb{R}) \text{ and } \tilde\chi_- \in C_c^\infty((-\infty,\kappa)). \tag{3.26}$$

The contribution from the second term of (3.25), $R(\lambda+\mathrm{i}0)(H-\lambda)\tilde u$, is treated by (1.7a) (here we may have $\kappa = 1$).

As for the contribution from the first term $\tilde u$, a computation using (3.12a) shows that $f^{-1}(p - \nabla S_\epsilon)\tilde u \in (B^\mu)^*_0$. On the other hand, because of Proposition 3.1(iv), as in (3.26), for small $\kappa, \epsilon > 0$, the symbol $f^{-1}(\xi - \nabla S_\epsilon)$ is elliptic on the support of any symbol $\chi_-(a_0)\tilde{\chi}_-(b_0)$. This intuitively yields the desired bound. However, at this point, some care must be taken since $\partial_j S_\epsilon$ is singular at 0 and the good bounds of Proposition 3.1(iii) are only valid for $|\alpha| \leq l$ (which consequently must be chosen sufficiently large). A similar difficulty arises in Section 4; see (4.8c) and the discussion there.

Let us elaborate. First, it is more convenient to use $S_0$ (modified by a cutoff near $\infty$) rather than $S_\epsilon$. Then we have good bounds on all derivatives, well suited for the calculus of pseudodifferential operators; see Section 4 for some details. Using this calculus, we can invoke the ellipticity property (with the above replacement and for small $\kappa > 0$) to write (abbreviating $T = \mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde{\chi}_-(b_0))$)

$$T = T\sum_{j=1}^{d} T_j\big(f(r,0)^{-1}p_j - \langle x\rangle^{-1}x_j\big) + \widetilde{T}\langle x\rangle^{\mu/2-1},$$

where $T_j$ and $\widetilde{T}$ are bounded pseudodifferential operators. We apply this identity to $\tilde{u}$. The last term contributes by a term in $(B^\mu)^*_0$. As for the first term, we use (3.12a) (with $\varepsilon < 1 - \mu/2$) and get a similar contribution and the term

$$T\sum_{j=1}^{d} T_j\phi_j\tilde{u};\ \phi_j = f(r,0)^{-1}\chi(r > 1)\partial_j S_\epsilon - \langle x\rangle^{-1}x_j.$$

Next, using a statement similar to (4.8c) (see the discussion there), we can write

$$T\sum_{j=1}^{d} T_j\phi_j = \sum_{j=1}^{d}\phi_j T_j T + \widetilde{T}\langle x\rangle^{\mu/2-1}$$

for some $\widetilde{T} \in \mathcal{B}\big((B^\mu)^*_0\big)$. Note that at this point, we need Proposition 3.1(iii) for an appropriate $l = l(\mu, d)$. By Proposition 3.1(iv), we then conclude that

$$T\tilde{u} - O(\epsilon^{3/4})T\tilde{u} \in (B^\mu)^*_0.$$

Consequently (for small $\epsilon > 0$), also $T\tilde{u} \in (B^\mu)^*_0$. □

# 4 Quantum bounds

In this section, we collect various microlocal resolvent bounds that are used in Section 5 for establishing the existence of the limit (3.10a) for $v \in L^2_3$ as well as for proving some continuity properties. Our main result, Proposition 4.2, is of independent interest and is new, even for spherically symmetric potentials.

**4.1 Microlocalization for $\epsilon$-small perturbation.** Let $\tilde{r}$ be a smooth convex function of $r \geq 0$ equal to $1/2$ for $r \leq 1/4$ and equal to $r$ for $r \geq 1$. For $\lambda \geq 0$, we introduce the symbols

$$a_\lambda = a_\lambda(x,\xi) = \frac{\xi^2}{f_\lambda(r)^2}, \quad b_\lambda = b_\lambda(x,\xi) = \frac{\xi}{f_\lambda(r)} \cdot \frac{x}{\tilde{r}}, \tag{4.1}$$

given in terms of the function $\tilde{r}$ of $r = |x|$ and the function $f = f_\lambda = f_\lambda(x) = f_\lambda(r) = f(r,\lambda)$ of Section 3. Note that $b_\lambda^2 \leq a_\lambda$. We state microlocal properties in terms of these observables. For zero energy, the resolvent bounds of this subsection are stronger than similar estimates obtainable using the observables (1.6); see [Sk, Proposition 3.5 ii)]. They are in the spirit of [DS1, Proposition 4.1] and [Sk, Lemmas 3.2 and 3.3]. We use Weyl quantization of symbols in a uniform symbol class $S_{\mathrm{unif}}(m_{|z|}, g_{|z|})$,

$$g = g_\lambda = \langle x\rangle^{-2}\mathrm{d}x^2 + f_\lambda(x)^{-2}\mathrm{d}\xi^2.$$

The word "uniform" refers to the requirement that bounds of derivatives are uniform in $z$ in the closure $\Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$ of $\Gamma_{\theta,\lambda_0} \subset \mathbb{C}$. To be precise, a symbol $c = c_z$ satisfies $c \in S_{\mathrm{unif}}(m_{|z|}, g_{|z|})$ with $z$ in this set if and only if $c$ satisfies the bounds

$$|\partial_x^\gamma \partial_\xi^\beta c_z(x,\xi)| \leq C_{\gamma,\beta} m_{|z|}(x,\xi)\langle x\rangle^{-|\gamma|} f_{|z|}^{-|\beta|}(x). \tag{4.2}$$

The symbol $m = m_{|z|}$ is usually referred to as a **weight function**. For example, $a_{|z|}$ is defined for $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$ (obviously) and $a_{|z|} \in S_{\mathrm{unif}}(a_{|z|}+1, g_{|z|})$; similarly for the symbol $h_\epsilon$ defined in (3.4), $h_\epsilon, h_\epsilon - z \in S_{\mathrm{unif}}(f_{|z|}^2(a_{|z|}+1), g_{|z|})$. For the corresponding calculus, the quantity $\langle x\rangle^{\mu/2-1}$ plays the role of a "uniform Planck constant." We refer to [Sk] for a more elaborate discussion. The corresponding class of Weyl quantized operators is denoted by $\Psi_{\mathrm{unif}}(m_{|z|}, g_{|z|})$.

Consider a real-valued $\chi_- \in C_c^\infty(\mathbb{R})$ such that $\chi_-(t) = 1$ in a neighborhood of $[0,1]$ and $\chi_-'(t) \leq 0$ for $t > 0$. Correspondingly, let $\chi_+ = 1 - \chi_-$. Consider $\tilde{\chi}_- \in C^\infty(\mathbb{R})$ with $\tilde{\chi}_-' \in C_c^\infty((-1,1))$, $\tilde{\chi}_-(-1) = 1$ and $\tilde{\chi}_-(1) = 0$. Let $\tilde{\chi}_+ = 1 - \tilde{\chi}_-$. Clearly, the bound (4.3) below (involving the function $\chi_+$) is an energy bound. The bound (4.4) (involving the functions $\chi_-$ and $\tilde{\chi}_-$) is a microlocal bound, whose classical analogue is partly explained after Proposition 4.1.

**Proposition 4.1.** *Let functions $\chi_-$, $\chi_+$ and $\tilde{\chi}_-$ be given as above. There exists $\epsilon_0 > 0$ such that for all $\theta \in (0,\pi)$, $\lambda_0 > 0$, $\delta > 1/2$, $t \geq 0$, and $\epsilon$-small perturbations $W_\epsilon$ with $\epsilon \in (0,\epsilon_0]$, there exists $C > 0$ such that*

(1) *with $T_+(z) := \langle x\rangle^{t-\delta} f_{|z|}^{1/2}\,\mathrm{Op}^{\mathrm{w}}(a_{|z|}\chi_+(a_{|z|}))R(z) f_{|z|}^{1/2}\langle x\rangle^{-t-\delta}$ for $z \in \Gamma_{\theta,\lambda_0}$,*

$$\|T_+(z)\| \leq C; \tag{4.3}$$

(2) *with* $T_-(z) := \langle x\rangle^{t-\delta} f_{|z|}^{1/2} \operatorname{Op^w}(\chi_-(a_{|z|})\tilde\chi_-(b_{|z|}))R(z) f_{|z|}^{1/2}\langle x\rangle^{-t-\delta}$ *for* $z \in \Gamma_{\theta,\lambda_0}$,

$$\|T_-(z)\| \le C; \tag{4.4}$$

(3) *the limit*

$$T_\pm(\lambda + \mathrm{i}0) := \lim_{\Gamma_{\theta,\lambda_0}\ni z\to\lambda} T_\pm(z) \text{ in } \mathcal{B}(L^2) \tag{4.5}$$

*exists uniformly in* $\lambda \in [0, \lambda_0]$.

There are analogous properties for $\bar z \in \Gamma_{\theta,\lambda_0}$. By the calculus (or the same proof), we can replace the symbol $a_{|z|}\chi_+(a_{|z|})$ in (1) by $\chi_+(a_{|z|})$. In particular, the combination of (1) and (2) yields the effective microlocalization

$$R(z)v \approx \operatorname{Op^w}(\chi_-(a_{|z|})\tilde\chi_+(b_{|z|}))R(z)v.$$

For later applications, let us choose the localization more concretely. Let $\chi$ be a decreasing smooth function on $\mathbb{R}$ satisfying $\chi(t) = 1$ for $t \le 1/2$ and $\chi(t) = 0$ for $t \ge 1$. For $\kappa > 0$ (small), introduce the functions $\chi_{t<\kappa}(t) := \chi(t/\kappa)$ and $\chi_{t>\kappa} := 1 - \chi_{t<\kappa}$. Choose

$$\chi_- = \chi_\kappa^- = \chi_{t<\kappa}(\cdot - 1), \quad \tilde\chi_+ = \tilde\chi_\kappa^+ = \chi_{t<\kappa}(1 - \cdot).$$

This leads to the introduction of the symbols

$$\chi_\kappa = \chi_{\kappa,|z|} = \chi_\kappa^-(a_{|z|})\tilde\chi_\kappa^+(b_{|z|}) \in S_{\mathrm{unif}}(1, g_{|z|});\ \kappa > 0. \tag{4.6}$$

The proof of Proposition 4.1 (not given here in detail) is similar to those of [Sk, Lemmas 3.2 and 3.3], using instead of [Sk, (3.12)] the following computation; cf. [DS1, (4.30)]. Let $h_{\mathrm{rad}} = \xi^2 + V_{\mathrm{rad}}(r)$. The Poisson bracket with $b = b_\lambda$ (i.e., the derivative of $b$ along the flow generated by $h_{\mathrm{rad}}$) is given by

$$\{h_{\mathrm{rad}}, b\} = \frac{2f}{\tilde r}\Big(1 - \frac{rV'_{\mathrm{rad}}}{2f^2}\Big)\big(1 - b^2\big) + \frac{2}{f\tilde r}(h_{\mathrm{rad}} - \lambda). \tag{4.7a}$$

Because of Condition 2.1(3), $1 - rV'_{\mathrm{rad}}/2f^2 \ge \tilde\epsilon_1/2$. For the Hamiltonian $h_\epsilon$ of (3.4), we have, uniformly for $x, \xi \in \mathbb{R}^d$ and $\lambda \ge 0$,

$$\{h_\epsilon, b\} = \frac{2f}{\tilde r}\left(1 - \frac{rV'_{\mathrm{rad}}}{2f^2}\right)(1 - b^2)\frac{2}{f\tilde r}(h_\epsilon - \lambda) + O(\epsilon)\frac{f}{\tilde r}. \tag{4.7b}$$

Hence, uniformly in a set of the form $\{b^2 \le 1 - \delta\}$, $\delta > 0$, and $\lambda \ge 0$,

$$\begin{aligned}\{h_\epsilon, b\} &\ge \tilde\epsilon_1\frac{f}{\tilde r}\big(1 - b^2\big) + \frac{2}{f\tilde r}(h_\epsilon - \lambda) - \epsilon C\frac{f}{\tilde r}\\ &\ge \frac{f}{\tilde r}\big(\tilde\epsilon_1\delta - C\epsilon\big) + \frac{2}{f\tilde r}(h_\epsilon - \lambda).\end{aligned} \tag{4.7c}$$

We learn from (4.7c) that provided $\epsilon$ is taken small, the observable $b$ grows along the flow generated by $h_\epsilon$ on any set $\{b^2 \leq 1-\delta,\ h_\epsilon = \lambda\}$. This is part of the classical analogue of (4.4). For $\kappa$-dependent symbols used as input in Proposition 4.1, the bounds (4.7c) indicate an optimal choice, $\epsilon_0 \approx \kappa$. In fact, and more precisely, the proof of Proposition 4.1 shows that we can choose $\epsilon_0 = \kappa/C$ for some $C > 0$ in the regime $\kappa > 0$ small, thus allowing us to write $R(z)v \approx \mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})R(z)v$ for all $(\kappa/C)$-small perturbations. This is one reason for considering only perturbations of a spherically symmetric potential. Another reason originates in the construction of $S_\epsilon$, viz., Proposition 3.1.

**4.2 Preliminary considerations.** In this subsection, we assume that $V_2 = 0$. Hence we consider the quantization of (3.4), $H = H_\epsilon$.

**4.2.1 Calculus considerations.** By the calculus, the family of symbols (4.6) has the property that for all $n \in \mathbb{N}$ and all $c = c_z \in S_{\mathrm{unif}}(a_{|z|}+1, g_{|z|})$,

$$\big(I - \mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|})\big)\,\mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|}) \in \Psi_{\mathrm{unif}}(\langle x\rangle^{-n}\langle\xi\rangle^{-2}, g_{|z|}), \tag{4.8a}$$

$$[\mathrm{Op}^{\mathrm{w}}(c_z), \mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})]\,\mathrm{Op}^{\mathrm{w}}(\chi_{\kappa/2,|z|}) \in \Psi_{\mathrm{unif}}(\langle x\rangle^{-n}\langle\xi\rangle^{-2}, g_{|z|}). \tag{4.8b}$$

Note that, in particular, (4.8b) applies to $c_z = h - z$ and any function $c_z = c_z(x,\xi) = \phi_z(x) \in S_{\mathrm{unif}}(1, g_{|z|})$. We need the following modification of the latter statement. Consider $\phi_z \in C^N(\mathbb{R}^d)$, with $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$, which satisfies the uniform bounds

$$|\partial_x^\alpha \phi_z(x)| \leq C_\alpha f_{|z|}(x)\langle x\rangle^{-|\alpha|},\ |\alpha| \leq N.$$

Now, for any given $n \in \mathbb{N}$, we can find $N = N(n,\mu,d)$ such that for all $\phi_z \in C^N(\mathbb{R}^d)$ satisfying these bounds,

$$[\phi_z(x), \mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|})]\,\mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|}) = \langle x\rangle^{-n} B \langle x\rangle^{-n},\ \|B\| \leq C \tag{4.8c}$$

(uniformly in $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$). This statement can be proved by the symbolic calculus and an explicit estimation of an associated oscillatory integral. Note that the constant $C$ in (4.8c) can be chosen proportional to a natural norm of $\phi_z$, and so the bound is an example of a familiar continuity property of the calculus of pseudodifferential operators. We apply (4.8c) to $\phi_z = \chi(r > 2)\partial_j S_\epsilon(x,|z|)$, $j = 1,\dots,d$. Note that here we need $l = N+1$ in Proposition 3.1(iii). If $n$ in (4.8c) is taken large, $\epsilon_0 > 0$ in Proposition 3.1 may then be small, since in practice, for given large $n$, large $N = l-1$ is needed for (4.8c) to hold.

The last preliminary property we discuss is an application of the Fefferman-Phong inequality [Hö, Theorem 18.6.8] (uniform version), which provides concrete bounds for the symbols $b_{|z|}$ and $\chi_{2\kappa,|z|}$. For all $\kappa > 0$, there exists $C = C_\kappa > 0$

such that for all $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$,

$$\mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|})\,\mathrm{Op}^{\mathrm{w}}(b_{|z|}-1+2\kappa)\,\mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|}) \geq -C(\langle x\rangle f_{|z|})^{-2}, \tag{4.9a}$$

$$\mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|})\,\mathrm{Op}^{\mathrm{w}}(b_{|z|}-1-2\kappa)\,\mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|}) \leq C(\langle x\rangle f_{|z|})^{-2}, \tag{4.9b}$$

$$\mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|})\,\mathrm{Op}^{\mathrm{w}}\left((b_{|z|}-1)^2-(2\kappa)^2\right)\mathrm{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|}) \leq C(\langle x\rangle f_{|z|})^{-2}. \tag{4.9c}$$

**4.2.2 Radiation operators.** We combine Propositions 3.1 and 4.1 to obtain radiation condition bounds similar to some of [HS1, HS2] for positive energies; see also [Sa1, Sa2]. Our method is different in that it is purely stationary, whereas [HS1, HS2] rely on propagation estimates. We introduce for $\lambda \geq 0$ and any given $\epsilon$-small perturbation $W_\epsilon$, **radiation operators**, defined in terms of the function $S_\epsilon = S_\epsilon(x,\lambda)$ from Proposition 3.1 as

$$\gamma = p - \nabla S_\epsilon,\ \ \gamma_j = p_j - \partial_j S_\epsilon,\ \ j = 1,\dots,d,\ \text{ and } \gamma_\parallel = \mathrm{Re}\left(\nabla S_\epsilon \cdot \gamma\right).$$

Using (3.1), we obtain (cf. [HS1, HS2])

$$2\gamma_\parallel = (H-\lambda) - \gamma^2. \tag{4.10}$$

Next we compute the Heisenberg derivative $\mathrm{D} = \mathrm{i}[H,\cdot]$ of $\gamma$. The operators involved are local, and we only need the computation for $r \geq 1$:

$$\begin{aligned}
\mathrm{D}\gamma &= -2\nabla^2 S_\epsilon \gamma + \mathrm{i}2\nabla\triangle S_\epsilon\\
&= -\frac{2f}{r}\Big(\gamma - \big(1+\frac{rV'_{\mathrm{rad}}}{2f^2}\big)f^{-1}|\hat{x}\rangle\nabla S_\epsilon\cdot\gamma + O(\epsilon^{1/2})\gamma\Big) + \mathrm{i}2\nabla\triangle S_\epsilon\\
&= -\frac{2f}{r}\Big(\gamma + Ff^{-1}\big(2\gamma_\parallel + \mathrm{i}\triangle S_\epsilon\big) + O(\epsilon^{1/2})\gamma\Big) + \mathrm{i}2\nabla\triangle S_\epsilon\\
&= -\frac{2f}{r}\Big(\gamma + Ff^{-1}\big((H-\lambda)-\gamma^2\big) + O(\epsilon^{1/2})\gamma\Big) - 2\mathrm{i}\big(\frac{F}{r}\triangle S_\epsilon - \nabla\triangle S_\epsilon\big);\\
&\qquad F := \frac{\tilde f}{\tilde r}x,\ \ \tilde f = -\frac{1}{2}\big(1+\frac{rV'_{\mathrm{rad}}}{2f^2}\big).
\end{aligned} \tag{4.11}$$

Here, we have used Proposition 3.1(iv) and (4.10). The meaning of $O(\epsilon^{1/2})$ is the same as in the Proposition, viz., it is a uniform bound. We can simplify the right hand side using Proposition 3.1(iii) (assuming $l \geq 3$) and conclude that

$$\mathrm{D}\gamma = -\frac{2f}{r}\Big(\big(I+O(\epsilon^{1/2})\big)\gamma + Ff^{-1}(H-\lambda) - Ff^{-1}\gamma^2 - \mathrm{i}r^{-1}O(\epsilon^0)\Big), \tag{4.12}$$

where, as above, the estimates are uniform in $W_\epsilon$, $\lambda \geq 0$, and $x$ with $r = |x| \geq 1$.

Next we compute

$$\mathrm{Re}\left(f^{-1}F\cdot\gamma\right) = \mathrm{Re}\left(\tilde f\left(\mathrm{Op}^{\mathrm{w}}(b) - 1 + O(\epsilon^{3/4})\right)\right). \tag{4.13}$$

Effectively, the right hand side is "small"; we use it to treat the third term in (4.12). Here, it is also useful to note that

$$[\gamma_i, \gamma_j] = 0 \text{ for } 1 \le i, j \le d. \tag{4.14}$$

**4.3 Strong radiation condition bounds.** For $k \in \mathbb{N}$, we introduce

$$X = \langle x \rangle = (1 + r^2)^{1/2}, \quad X_k = X(1 + r^2/k)^{-1/2},$$

and "propagation observables"

$$P_1 = \sum_i Q_i^* Q_i, \qquad Q_i = X_k^{1-\epsilon'} \chi(r)\gamma_i \operatorname{Op^w}(\chi_{\kappa,|z|}); \tag{4.15a}$$

$$P_2 = \sum_{i,j} Q_{ij}^* Q_{ij}, \qquad Q_{ij} = X_k^{2(1-\epsilon')} \chi(r)\gamma_i\gamma_j \operatorname{Op^w}(\chi_{\kappa,|z|}), \tag{4.15b}$$

where $\gamma = p - \nabla S_\epsilon(x, |z|)$, $\chi(r) = \chi(r > 2)$ and $\epsilon' \in (0, 1]$ needs to be specified. Note that the powers of $X_k$ are bounded factors and that $X_k \uparrow X$ pointwise for $k \to \infty$.

We compute the Poisson brackets

$$\{h, \chi(r)\} = 2\chi'(r)\hat{x} \cdot \xi = 2f\chi'(r)\, b, \tag{4.16a}$$

$$\{h, X\} = 2X^{-1}x \cdot \xi = \frac{2f}{r}\phi\, bX; \qquad \phi = \frac{r\tilde{r}}{X^2}, \tag{4.16b}$$

$$\{h, X_k\} = \frac{2f}{r}(\phi - \phi_k)\, bX_k; \qquad \phi_k = \frac{r\tilde{r}}{k + r^2}. \tag{4.16c}$$

For (4.16c), note that

$$0 \le \phi - \phi_k = \frac{(k-1)r\tilde{r}}{(k + r^2)X^2} \le 1. \tag{4.17}$$

We also use (tacitly) the uniform bounds

$$|\partial_x^\alpha f_\lambda^s| \le C_{\alpha,s} f_\lambda^s \langle x \rangle^{-|\alpha|}, \ \lambda \ge 0, \ x \in \mathbb{R}^d,$$
$$|\partial_x^\alpha X_k^s| \le C_{\alpha,s} X_k^s \langle x \rangle^{-|\alpha|}, \ k \in \mathbb{N}, \ x \in \mathbb{R}^d.$$

We now state the main result of this section.

**Proposition 4.2.** *There exist $l = l(\mu, d) \in \mathbb{N}$ and $\epsilon_0, C_0 > 0$ with $\sqrt{C_0}\epsilon_0 \le 1$ such that for all $\epsilon$-small perturbations $W_\epsilon$ with $\epsilon \in (0, \epsilon_0]$ and $\epsilon' = C_0\sqrt{\epsilon}$,*

$$\left\| X^{1-\epsilon'} \left(\frac{f_\lambda}{r}\right)^{1/2} \chi(r)\gamma_i R(\lambda + \mathrm{i}0) f_\lambda^{1/2} X^{-3/2} \right\| \le C, \tag{4.18a}$$

$$\left\| X^{2(1-\epsilon')} \left(\frac{f_\lambda}{r}\right)^{1/2} \chi(r)\gamma_i\gamma_j R(\lambda + \mathrm{i}0) f_\lambda^{1/2} X^{-5/2} \right\| \le C \tag{4.18b}$$

*hold uniformly in* $\lambda$ *in intervals of the form* $I = [0, \lambda_0]$. *Here,* $(\gamma_1, \dots, \gamma_d) = \gamma = p - \nabla S_\epsilon(x, \lambda)$.

**Proof.** Because of the resolvent equations, we can assume that $V_2 = 0$. Throughout the proof, $H$ and $R(z)$ refer to this case. Fix $\theta \in (\pi/2, \pi)$ and $\lambda_0 > 0$. We prove microlocal bounds of states $u = R(z)v$ in terms of quantities related to $P_1$ of (4.15a) and $P_2$ of (4.15b), where $z \in \Gamma_{\theta,\lambda_0}$. In particular, we consider below $\gamma_i = p_i - \partial_i S_\epsilon(x, |z|)$ with $z \in \Gamma_{\theta,\lambda_0}$. We could take $\kappa > 0$ in the definition of the factors $\mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})$ to be proportional to $\epsilon$ with a sufficiently large constant of proportionality; cf. the discussion at the end of Subsection 4.1. However, the larger choice $\kappa = \epsilon^{3/4}$ suffices and is the one used below. In any case, for the corresponding localization operators $B = \mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})$ and $B = \mathrm{Op}^{\mathrm{w}}(\chi_{\kappa/2,|z|})$, we can use the bounds of Proposition 4.1 for $\epsilon$-small perturbations (more precisely, we have such bounds upon replacing the pseudodifferential operators there by $I - B$). This is done in (4.20b) and (4.23b) below (for (4.18a)). We choose $\epsilon_0 > 0$ in agreement with any such application as well as with Proposition 3.1 with $l$ (the one to be used in the proposition) chosen sufficiently large. How large $l$ must be depends for (4.18a) partly on applications below of Proposition 3.1 and the symbolic calculus property (4.8c), used in (4.26) and (4.29); see (4.35) for the case of (4.18b) (used in (4.34) and (4.37)). Of course, it is legitimate to take $\epsilon_0$ smaller, if needed. The choice $\epsilon' = C_0\sqrt{\epsilon}$ for some (large) constant $C_0$ (rather than $\epsilon'$ being proportional to $\epsilon$) is needed (and best possible) in our treatment of the contribution from the term $O(\epsilon^{1/2})\gamma$ in (4.12) in the computation and estimation of a commutator; see (4.24) below. We fix an applicable $C_0$ for (4.18a) at the end of Step I (this constant also works for (4.18b); see the end of Step II).

**Step I.** We show (4.18a) by first establishing the bound

$$\langle P_1' \rangle_u \le C_1 \| f_{|z|}^{-1/2} X^{3/2} v \|^2 + C_2 \frac{|(z - |z|)|^2}{\operatorname{Im} z} \| X^{-1} X_k^{1-\epsilon'} u \|^2; \tag{4.19}$$
$$P_1' = \sum_i Q_i^* \frac{f_{|z|}}{r} Q_i.$$

Here (and henceforth), $\langle \cdot \rangle_u$ is the expectation in the state $u$; we have suppressed the dependence of $|z|$ in $Q_i$ (as above). The constants are independent of $z \in \Gamma_{\theta,\lambda_0}$ and $k \in \mathbb{N}$; however, they depend on $\epsilon$ and possibly also on $W_\epsilon$. We conclude by first letting $\operatorname{Im} z \to 0$ (for fixed $\lambda = \operatorname{Re} z \ge 0$) and then letting $k \to \infty$, that at all energies $\lambda \in [0, \lambda_0]$

$$\sum_i \langle Q_i^* \frac{f_\lambda}{r} Q_i \rangle_u \le C_1 \| f_\lambda^{-1/2} X^{3/2} v \|^2,$$

and hence

$$\|X^{1-\epsilon'}\big(\frac{f_\lambda}{r}\big)^{1/2}\chi(r)\gamma_i \operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,\lambda})R(\lambda+\mathrm{i}0)v\| \leq \sqrt{C_1}\|f_\lambda^{-1/2}X^{3/2}v\|. \tag{4.20a}$$

On the other hand, we have (cf. Proposition 4.1)

$$\|X^{1-\epsilon'}\big(\frac{f_\lambda}{r}\big)^{1/2}\chi(r)\gamma_i\big(I-\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,\lambda})\big)R(\lambda+\mathrm{i}0)v\| \leq C_3\|f_\lambda^{-1/2}X^{3/2}v\|. \tag{4.20b}$$

Clearly, (4.18a) follows from (4.20a) and (4.20b).

To show (4.19), we calculate the expectation

$$\langle \mathrm{i}[H,P_1]\rangle_u = 2\operatorname{Im} z\,\langle P_1\rangle_u + 2\operatorname{Im}\langle P_1 u, v\rangle, \tag{4.21a}$$

$$\langle \mathrm{i}[H,P_1]\rangle_u = 2\sum_i \operatorname{Re}\langle Q_i u, \mathrm{i}[H,Q_i]u\rangle, \tag{4.21b}$$

$$\begin{aligned} \mathrm{i}[H,Q_i] &= T_i^1 + T_i^2;\\ T_i^1 &= \mathrm{i}[H, X_k^{1-\epsilon'}\chi(r)\gamma_i]\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,|z|}),\\ T_i^2 &= X_k^{1-\epsilon'}\chi(r)\gamma_i \mathrm{i}[H,\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})]. \end{aligned} \tag{4.21c}$$

The idea of the proof is to show that (4.21a) "tends" to be non-negative while (4.21b) "tends" to be non-positive. To keep notation to a minimum, we abbreviate $f_{|z|} = f$ in the remaining part of the proof of this proposition.

Clearly, the first term on the right-hand side of (4.21a) is non-negative (a fact that is used in (4.25) stated below). The second term on the right-hand side of (4.21a) can be estimated as

$$\begin{aligned} |2\operatorname{Im}\langle P_1 u, v\rangle| &\leq \delta\langle P_1'\rangle_u + \delta^{-1}\sum_i \langle r/f\rangle_{Q_i v}\\ &\leq \delta\langle P_1'\rangle_u + \delta^{-1}C_1\|X^{3/2-\epsilon'}f^{1/2}v\|^2\\ &\leq \delta\langle P_1'\rangle_u + \delta^{-1}C_2\|f^{-1/2}X^m v\|^2;\ m\geq 3/2-\epsilon'. \end{aligned} \tag{4.22}$$

We choose $\delta > 0$ suitably small later.

As for (4.21b), we substitute (4.21c). The contribution from the terms $T_i^2$ is estimated similarly using (4.8b) (for suitable $n$) and (1.4):

$$\begin{aligned} 2\sum_i |\langle Q_i u, T_i^2\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa/2,|z|})u\rangle| 2\sum_i |\langle Q_i u, T_i^2\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa/2,|z|})u\rangle|\\ \leq \frac{\delta}{2}\langle P_1'\rangle_u + \delta^{-1}C_1\|f^{-1/2}X^m v\|^2;\ m > 1/2, \end{aligned} \tag{4.23a}$$

and from Proposition 4.1,

$$\begin{aligned} 2\sum_i |\langle Q_i u, T_i^2\big(I-\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa/2,|z|})\big)u\rangle|\\ \leq \frac{\delta}{2}\langle P_1'\rangle_u + \delta^{-1}C_2\|f^{-1/2}X^m v\|^2;\ m > 3/2-\epsilon'. \end{aligned} \tag{4.23b}$$

It remains to consider the contribution from $T_i^1$. We write $T_i^1 = S_i^1 + S_i^2 + S_i^3$, where

$$S_i^1 = X_k^{1-\epsilon'}\chi(r)\big(\mathrm{D}\gamma_i\big)\,\mathrm{Op^w}(\chi_{\kappa,|z|}),$$
$$S_i^2 = X_k^{1-\epsilon'}\big(\mathrm{D}\chi(r)\big)\gamma_i\,\mathrm{Op^w}(\chi_{\kappa,|z|}),$$
$$S_i^3 = \big(\mathrm{D}X_k^{1-\epsilon'}\big)\chi(r)\gamma_i\,\mathrm{Op^w}(\chi_{\kappa,|z|}).$$

We intend to use (4.12), (4.16a) and (4.16c) to treat the contribution to (4.21b) from the three terms, respectively.

The sought after negativity comes from the terms $S_i^1$, more precisely, from the contribution from the first term on the right-hand side of (4.12). Thus, by the Cauchy Schwarz inequality,

$$2\sum_i \mathrm{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\frac{-2f}{r}\big(\gamma_i + \big(O(\epsilon^{1/2})\gamma\big)_i\big)\,\mathrm{Op^w}(\chi_{\kappa,|z|})u\rangle \le \big(-4+\tilde C\sqrt{\epsilon}\big)\langle P_1'\rangle_u. \tag{4.24}$$

To bound the contribution from the second term on the right-hand side of (4.12), we use the fact that $F$ is (uniformly) bounded and the estimate

$$\begin{aligned}
&2\sum_i \mathrm{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\frac{-2F^i}{r}(H-|z|)\,\mathrm{Op^w}(\chi_{\kappa,|z|})u\rangle\\
&\quad\le -4(z-|z|)\sum_i \mathrm{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\frac{F^i}{r}\,\mathrm{Op^w}(\chi_{\kappa,|z|})u\rangle\\
&\qquad + \delta\langle P_1'\rangle_u + \delta^{-1}C_1\|f^{-1/2}X^m v\|^2\\
&\quad\le 2\,\mathrm{Im}\,z\,\langle P_1\rangle_u + \tilde C_2\frac{|(z-|z|)|^2}{\mathrm{Im}\,z}\|X^{-1}X_k^{1-\epsilon'}u\|^2\\
&\qquad + \delta\langle P_1'\rangle_u + \delta^{-1}C_1\|f^{-1/2}X^m v\|^2;\quad m>1/2.
\end{aligned} \tag{4.25}$$

To bound the contribution from the third term on the right-hand side of (4.12) we "redistribute" the factors of components in $\gamma^2$ and use (1.4), (4.8a), (4.8c), (4.9c), (4.13), and (4.14), estimating with $u_j := \mathrm{Op^w}(\chi_{2\kappa,|z|})(2f/r)^{1/2}Q_j u$ to obtain

$$\begin{aligned}
&2\sum_i \mathrm{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\frac{2F^i}{r}\gamma^2\,\mathrm{Op^w}(\chi_{\kappa,|z|})u\rangle\\
&\quad\le 2\sum_j \mathrm{Re}\langle \frac{2}{r}F\cdot\gamma Q_j u, Q_j u\rangle + C_1\|f^{-1/2}X^m v\|^2\\
&\quad= 2\sum_j \mathrm{Re}\langle \tilde f\big(\mathrm{Op^w}(b)-1+O(\epsilon^{3/4})\big)\rangle_{u_j} + C_2\|f^{-1/2}X^m v\|^2\\
&\quad\le \big(2\kappa(\sup|\tilde f|^2+1)+C_1\epsilon^{3/4}\big)\langle 2P_1'\rangle_u + C_3\|f^{-1/2}X^m v\|^2\\
&\quad= C_4\epsilon^{3/4}\langle P_1'\rangle_u + C_3\|f^{-1/2}X^m v\|^2,\quad m>1/2.
\end{aligned} \tag{4.26}$$

To bound the contribution from the fourth term on the right side of (4.12), it is convenient (although not necessary) to symmetrize (assuming then $l \geq 4$). This gives, with $\tilde{u} := X_k^{1-\epsilon'} \operatorname{Op^w}(\chi_{\kappa,|z|})u$,

$$
\begin{aligned}
&2\sum_i \operatorname{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\mathrm{i}\frac{2f}{r^2}O_i(\epsilon^0)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\
&\qquad \leq C_1\langle \frac{2f}{\tilde{r}^3}\rangle_{\tilde{u}} \leq C_2\|f^{-1/2}X^m v\|^2; \quad m > 1/2.
\end{aligned} \tag{4.27}
$$

Clearly, the contributions from the terms $S_i^2$ are bounded similarly (cf. (4.16a)):

$$
2\sum_i \operatorname{Re}\langle Q_i u, S_i^2 u\rangle \leq C\|f^{-1/2}X^m v\|^2,\ m > 1/2. \tag{4.28}
$$

It remains to examine the contribution from the terms $S_i^3$. Using (4.8a), (4.8c), (4.9b), (4.16c), and (4.17), estimating with $u_i := \operatorname{Op^w}(\chi_{2\kappa,|z|})((\phi-\phi_k)2f/r)^{1/2}Q_i u$, we obtain

$$
\begin{aligned}
&2\sum_i \operatorname{Re}\langle Q_i u, S_i^3 \operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\
&\quad\leq 2(1-\epsilon')\sum_i \langle \operatorname{Op^w}(b)\rangle_{u_i} + C_1\|f^{-1/2}X^m v\|^2 \\
&\quad\leq 2(1-\epsilon')(1+2\kappa)\sum_i \|u_i\|^2 + C_2\|f^{-1/2}X^m v\|^2 \\
&\quad\leq 4(1-\epsilon')(1+2\epsilon^{3/4})\langle P_1'\rangle_u + C_3\|f^{-1/2}X^m v\|^2, \quad m > 1/2.
\end{aligned} \tag{4.29}
$$

Now, by combining (4.22)–(4.29) with (4.21a)–(4.21c), we obtain

$$
\begin{aligned}
&\big(4 - \tilde{C}\sqrt{\epsilon} - 3\delta - C_4\epsilon^{3/4} - 4(1-\epsilon')(1+2\epsilon^{3/4})\big)\langle P_1'\rangle_u \\
&\qquad \leq C_\delta\|f^{-1/2}X^m v\|^2 + \tilde{C}_2\frac{|(z-|z|)|^2}{\operatorname{Im} z}\|X^{-1}X_k^{1-\epsilon'}u\|^2, \quad m > 3/2-\epsilon'.
\end{aligned} \tag{4.30}
$$

We choose $\delta = \sqrt{\epsilon}$ and fix $C_0 = \frac{1}{4}(\tilde{C}+5)$. Then (possibly by taking $\epsilon_0 > 0$ smaller), we conclude the bound

$$
\sqrt{\epsilon}\langle P_1'\rangle_u \leq C_\delta\|f^{-1/2}X^m v\|^2 + \tilde{C}_2\frac{|(z-|z|)|^2}{\operatorname{Im} z}\|X^{-1}X_k^{1-\epsilon'}u\|^2, \quad m = 3/2,
$$

whence (4.19) follows.

**Step II.** We show (4.18b) by establishing the bound

$$
\begin{aligned}
&\langle P_2'\rangle_u \leq C_1\|f^{-1/2}X^{5/2}v\|^2 + C_2\frac{|(z-|z|)|^2}{\operatorname{Im} z}\big(\|X^{-2}X_k^{2(1-\epsilon')}u\|^2 \\
&\qquad\qquad + k^2\langle P_1'\rangle_u\big) + C_3\langle P_1'\rangle_u; \\
&P_2' = \sum_i Q_{ij}^*\frac{f}{r}Q_{ij}.
\end{aligned} \tag{4.31}
$$

Here, $P_1'$ is given as in (4.19). Because of (4.19), we can proceed as above, letting first $\operatorname{Im} z \to 0$ (for fixed $\lambda = \operatorname{Re} z \geq 0$) and then letting $k \to \infty$. Then again we invoke Proposition 4.1. Hence it suffices for (4.18b) to show (4.31).

For (4.31), we proceed as we did in Step I, only now giving fewer details. We replace $P_1$ by $P_2$ in (4.21a)–(4.21c) and need to show "essential positivity" and "essential negativity" of the expressions on the right hand sides of the analogous (4.21a) and (4.21b), respectively. The most interesting contribution to the analogous commutator (4.21b) comes from an expression similar to $S_i^1$. More precisely, this term is now replaced by $S_{ij}^1 = X_k^{2(1-\epsilon')}\chi(r)\big(\mathrm{D}(\gamma_i\gamma_j)\big)\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})$.

We write $\mathrm{D}(\gamma_i\gamma_j) = (\mathrm{D}\gamma_i)\gamma_j + \gamma_i(\mathrm{D}\gamma_j)$ and again invoke (4.12), which contains four terms.

As for the first term, the analogue of (4.24) reads (using the constant $\tilde{C}$ from (4.24))

$$
\begin{aligned}
&2\sum_{i,j}\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\\
&\qquad\cdot\Big(\frac{-2f}{r}\big(\gamma_i + (O(\epsilon^{1/2})\gamma)_i\big)\gamma_j + \gamma_i\frac{-2f}{r}\big(\gamma_j + (O(\epsilon^{1/2})\gamma)_j\big)\Big)\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})u\rangle\\
&\leq \big(-8 + 2\tilde{C}\sqrt{\epsilon} + \delta\big)\langle P_2'\rangle_u + \delta^{-1}C_1\langle P_1'\rangle_u. \qquad (4.32)
\end{aligned}
$$

As for the analogue of (4.25), we have, using the bound $X_k^2 \leq k$ and the first identity of (4.11) and arguing as in (4.23a)–(4.23b),

$$
\begin{aligned}
&2\sum_{i,j}\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\frac{-2F^i}{r}(H-|z|)\gamma_j + \gamma_i\frac{-2F^j}{r}(H-|z|)\Big)\\
&\qquad\cdot\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})u\rangle\\
&\leq 2\operatorname{Im} z\,\langle P_2\rangle_u + \tilde{C}_1\frac{|(z-|z|)|^2}{\operatorname{Im} z}\big(\|X^{-2}X_k^{2(1-\epsilon')}u\|^2 + k^2\langle P_1'\rangle_u\big)\\
&\qquad + \delta\langle P_2'\rangle_u + \delta^{-1}C_2\big(\|f^{-1/2}X^{3/2}v\|^2 + \langle P_1'\rangle_u\big). \qquad (4.33)
\end{aligned}
$$

As for the analogue of (4.26), redistributing components of $\gamma^2$ and writing $u_{mn} = \operatorname{Op}^{\mathrm{w}}(\chi_{2\kappa,|z|})(4f/r)^{1/2}Q_{mn}$, we obtain

$$
\begin{aligned}
&2\sum_{i,j}\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\frac{2F^i}{r}\gamma^2\gamma_j + \gamma_i\frac{2F^j}{r}\gamma^2\Big)\operatorname{Op}^{\mathrm{w}}(\chi_{\kappa,|z|})u\rangle\\
&\leq 2\sum_{m,n}\operatorname{Re}\langle\tilde{f}\big(\operatorname{Op}^{\mathrm{w}}(b) - 1 + O(\epsilon^{3/4})\big)\rangle_{u_{mn}} + \frac{\delta}{2}\langle P_2'\rangle_u + C_{1,\delta}\|f^{-1/2}Xv\|^2\\
&\leq \big(2\kappa(\sup|\tilde{f}|^2 + 1) + C_1\epsilon^{3/4}\big)\langle 4P_2'\rangle_u + \delta\langle P_2'\rangle_u + C_{2,\delta}\|f^{-1/2}Xv\|^2\\
&= \tilde{C}_3\epsilon^{3/4}\langle P_2'\rangle_u + \delta\langle P_2'\rangle_u + C_{2,\delta}\|f^{-1/2}Xv\|^2. \qquad (4.34)
\end{aligned}
$$

Here, we have twice used the fact that $(rf)^{-1} = O\big(r^{\mu/2-1}\big)$ (so that, up to a compactly supported term, this function is bounded by $\delta/C$) and a uniform bound similar to (4.8c), for example

$$[(4f/r)^{1/2}X_k^{2(1-\epsilon')}\chi(r)\gamma_m\gamma_n, \operatorname{Op^w}(\chi_{2\kappa,|z|})]\operatorname{Op^w}(\chi_{\kappa,|z|}) = X^{-2}BX^{-1}f^{1/2}, \quad \|B\| \le C. \tag{4.35}$$

As for the analogue of (4.27), we have (using $l \ge 4$)

$$\begin{aligned} &2\sum_{i,j}\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\mathrm{i}\frac{2f}{r^2}O_i(\epsilon^0)\gamma_j + \gamma_i\mathrm{i}\frac{2f}{r^2}O_j(\epsilon^0)\Big)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\quad = 4\sum_{i,j}\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\mathrm{i}\frac{2f}{r^2}O_i(\epsilon^0)\gamma_j + \frac{f}{r^3}O_{ij}(\epsilon^0)\Big)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\quad \le \delta\langle P_2'\rangle_u + \delta^{-1}C_1\big(\|f^{-1/2}Xv\|^2 + \langle P_1'\rangle_u\big).. \end{aligned} \tag{4.36}$$

The analogue of (4.28) is obvious.

The analogue of (4.29) reads, with $u_{ij} := \operatorname{Op^w}(\chi_{2\kappa,|z|})((\phi-\phi_k)2f/r)^{1/2}Q_{ij}u$,

$$\begin{aligned} &2\sum_{i,j}\operatorname{Re}\langle Q_{ij}u, \big(\mathrm{D}X_k^{2(1-\epsilon')}\big)\chi(r)\gamma_i\gamma_j\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\qquad \le 4(1-\epsilon')\sum_{i,j}\langle\operatorname{Op^w}(b)\rangle_{u_{ij}} + C_1\|f^{-1/2}Xv\|^2 \\ &\qquad \le 4(1-\epsilon')(1+2\kappa)\sum_{i,j}\|u_{ij}\|^2 + \delta\langle P_2'\rangle_u + C_2\|f^{-1/2}Xv\|^2 \\ &\qquad \le \big(8(1-\epsilon')(1+2\epsilon^{3/4})+\delta\big)\langle P_2'\rangle_u + C_3\|f^{-1/2}Xv\|^2. \end{aligned} \tag{4.37}$$

Here, we have used the bound (4.35) trivially modified by insertion of a factor $(\phi-\phi_k)$.

Collecting bounds, we get (similarly to (4.30))

$$\begin{aligned} &\big(8 - 2\tilde{C}\sqrt{\epsilon} - 7\delta - \tilde{C}_3\epsilon^{3/4} - 8(1-\epsilon')(1+2\epsilon^{3/4})\big)\langle P_2'\rangle_u \\ &\le C_1(\delta)\|f^{-1/2}X^{5/2}v\|^2 + \tilde{C}_1\frac{|(z-|z|)|^2}{\operatorname{Im} z}\big(\|X^{-2}X_k^{2(1-\epsilon')}u\|^2 + k^2\langle P_1'\rangle_u\big) + C_2(\delta)\langle P_1'\rangle_u. \end{aligned} \tag{4.38}$$

Letting $\delta = \sqrt{\epsilon}$ and $C_0 = \frac{1}{4}(\tilde{C}+5)$ in (4.38) (as in Step I), we discover that the left hand side bounds $\sqrt{\epsilon}\langle P_2'\rangle_u$ from above, and (4.31) follows. □

## 5 Distorted Fourier transform

We prove the existence of the limit (3.10a) for all $\lambda \ge 0$ and all $v \in L_3^2$. To that end, we first compute the derivative

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}R(\lambda+\mathrm{i}0)v\big) = \mathrm{i}\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{-1/2}m_\epsilon^{1/2}\gamma_\parallel(\lambda)R(\lambda+\mathrm{i}0)v. \tag{5.1}$$

along the flow $\Phi(s,\cdot)$. It suffices to show that the right hand side is integrable as a $\mathcal{G}$-valued function. For this, we use (3.6), the identity $S_\epsilon(\Phi(s,\cdot)) = s$, and the Cauchy Schwarz inequality to conclude that, in turn, it suffices to find $\delta > 0$ such that

$$\|S_\epsilon^{1/2+\delta} K_\epsilon^{-1/2} \gamma_\parallel(\lambda) R(\lambda + \mathrm{i}0) v\| \leq C < \infty. \tag{5.2}$$

We plug in (4.10). Since

$$c r f(r,\lambda) \leq S_\epsilon(x) \leq C r f(r,\lambda), \tag{5.3}$$

the contribution from the first term $(H - V_2 - \lambda)$ is in $\mathcal{H}$ for any $\delta \leq 2$ (then we have $(fr)^{1/2+\delta} K_\epsilon^{-1/2} \leq C X^3$ and, by assumption, $X^3 v \in \mathcal{H}$). As for the contribution from the second term $-\gamma^2$, we use (4.18b) with $i = j$. Again, since $X^3 v \in \mathcal{H}$, we need only examine the weight $X^{2(1-\epsilon')} f^{1/2} r^{-1/2}$ on the left hand side of (4.18b); in particular, we need to specify applicable $\delta$ and $\epsilon'$. More precisely, we need to specify these parameters in such way that

$$S_\epsilon^{1/2+\delta} K_\epsilon^{-1/2} X^{-2(1-\epsilon')} f^{-1/2} r^{1/2} \text{ is bounded.}$$

Because of (5.3), this follows from boundedness of $X^{2\epsilon'+\delta-1} f^{\delta-1}$, which in turn, for $\delta \leq 1$, follows from boundedness of $X^{2\epsilon'+(\delta-1)(1-\mu/2)}$. This latter expression is bounded for all $\delta \in (0,1)$ (henceforth taken fixed) and for all sufficiently small $\epsilon, \epsilon' = C_0\sqrt{\epsilon} > 0$. Hence we have established the bound (5.2) for all $\lambda \geq 0$ and all $v \in L^2_3$. Since it is uniform in $\lambda \in [0,\lambda_0]$ for any $\lambda_0 > 0$, and since the function $[0,\lambda_0] \ni \lambda \to \big(\mathrm{e}^{-\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} R(\lambda+\mathrm{i}0)v\big)\big(\Phi(s,\cdot)\big) \in \mathcal{G}$ is continuous for any (large) fixed $s > 1$, we conclude that also

$$[0,\infty) \ni \lambda \to F^+(\lambda)v \in \mathcal{G} \text{ is continuous.} \tag{5.4}$$

Clearly (by time-reversal invariance), the limit in (3.10b) exists for all $v \in L^2_3$ also. Similarly, $F^-(\lambda)v$ is continuous in $\lambda \geq 0$.

There are other assertions in Subsection 3.2. As for the formula

$$\|P_\mathrm{c} v\|^2 = \lim_{\lambda_0\to\infty} \int_0^{\lambda_0} \|F^+(\lambda)v\|^2_{\mathcal{G}} \,\mathrm{d}\lambda,$$

it suffices to show that for all $\lambda \geq 0$,

$$\pi^{-1}\langle v, (\operatorname{Im} R(\lambda + \mathrm{i}0))v\rangle = \|F^+(\lambda)v\|^2_{\mathcal{G}}. \tag{5.5}$$

We begin with the estimate (recall $u := R(\lambda + \mathrm{i}0)v$)

$$(5.6)\quad \Big|\lim_{s\to\infty} \mathrm{Re} \sum_{j=1}^{d} \int_{S^{d-1}} \big(\overline{(\gamma_j(\lambda)u)}u(\partial_j S_\epsilon)m_\epsilon\big)\big(\Phi(s,\cdot)\big)\, \mathrm{d}\sigma\Big| \le \liminf_{s\to\infty} \sum_{j=1}^{d} \int_{S^{d-1}} |\big(\overline{(\gamma_j(\lambda)u)}u(\partial_j S_\epsilon)m_\epsilon\big)\big(\Phi(s,\cdot)\big)|\, \mathrm{d}\sigma.$$

By (3.6) and the Cauchy Schwarz inequality, for large enough $s_0 > 0$,

$$\begin{aligned}
&\int_{s_0}^{\infty} \mathrm{d}s\, s^{-1} \sum_{j=1}^{d} \int_{S^{d-1}} |\big(\overline{(\gamma_j(\lambda)u)}u(\partial_j S_\epsilon)m_\epsilon\big)\big(\Phi(s,\cdot)\big)|\, \mathrm{d}\sigma \\
&\quad = \sum_{j=1}^{d} \int_{s_0}^{\infty} \mathrm{d}s \int_{S^{d-1}} \Big| m_\epsilon^{1/2} X^{1-\epsilon'} \Big(\frac{f_\lambda}{r}\Big)^{1/2} \gamma_j(\lambda)u \Big| \\
&\qquad \cdot \Big| m_\epsilon^{1/2} X^{\epsilon'-1} \Big(\frac{f_\lambda}{r}\Big)^{-1/2} (\partial_j \ln S_\epsilon)u \Big|\, \mathrm{d}\sigma \\
&\quad \le \sum_{j=1}^{d} \Big\| X^{1-\epsilon'} \Big(\frac{f_\lambda}{r}\Big)^{1/2} \chi(r)\gamma_j(\lambda)u \Big\| \, \Big\| X^{\epsilon'-1} \Big(\frac{f_\lambda}{r}\Big)^{-1/2} \chi(r)(\partial_j \ln S_\epsilon)u \Big\|.
\end{aligned}$$

Since $|X^{\epsilon'-1}(f_\lambda/r)^{-1/2}\chi(r)(\partial_j \ln S_\epsilon)| \le CX^{\epsilon'+\mu/2-1}(f_\lambda/r)^{1/2}\chi(r)$ (cf. (5.3)), we conclude, using (1.4) and (4.18a), that for all $\epsilon' = C_0\sqrt{\epsilon} \in (0, 1-\mu/2)$, the latter integral is finite. Hence for all small $\epsilon > 0$, the right hand side of (5.6) vanishes. We have shown (5.5).

Throughout the rest of the paper, we use the abbreviations $B = B(|x|)$ and $B^* = B(|x|)^*$. Note that because of (1.4) and (5.5),

$$(5.7)\qquad \forall \lambda \ge 0 :\ F^+(\lambda) f^{1/2} \in \mathcal{B}(B, \mathcal{G}).$$

Next we introduce $F^+ = \int_0^\infty \oplus F^+(\lambda)\mathrm{d}\lambda$, which, because of (5.5), satisfies $(F^+)^* F^+ = P_\mathrm{c}$. Notice that here we consider $F^+ \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{H}})$. A short argument shows that for all $v \in (H-\lambda)C_\mathrm{c}^\infty(\mathbb{R}^d)$, the function $F^+(\lambda)v$ vanishes. Thus $F^+ H_\mathrm{c} \subset M_\lambda F^+$. We claim that $F^+$ diagonalizes $H_\mathrm{c}$. This stronger statement is part of the following proposition.

**Proposition 5.1.** *The map* $F^+ : \mathrm{Ran}\, P_\mathrm{c} \to \widetilde{\mathcal{H}}$ *is a unitary diagonalizing transform. In particular,*

$$(5.8)\qquad \mathrm{Ran}\, F^+ = \widetilde{\mathcal{H}} \text{ and } F^+ H_\mathrm{c} = M_\lambda F^+.$$

**Proof.** It suffices to show the first identity of (5.8), since then the restricted map $F^+ : \mathcal{H}_\mathrm{c}(H) = \mathrm{Ran}\, P_\mathrm{c} \to \widetilde{\mathcal{H}}$ is unitary and the second identity of (5.8) holds.

**Step I.** Let $\tau \in C^\infty(S^{d-1})$ and consider the function $\tilde u$ of (3.11). We claim that

$$\tau = F^+(\lambda)(H-\lambda)\tilde u. \tag{5.9}$$

Note that because of Lemma 3.3, this is formally true; however, since we do not know that $(H-\lambda)\tilde u \in L^2_3$, a continuity argument is required. This motivates the claim that for all $v \in B$,

$$F^+(\lambda)f^{1/2}v = \mathcal{G}\text{–}\lim_{S\to\infty} S^{-1}\int_0^S \pi^{-1/2}\big(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}R(\lambda+\mathrm{i}0)f^{1/2}v\big)\big(\Phi(s,\cdot)\big)\,\mathrm{d}s. \tag{5.10}$$

Clearly, this is consistent with (5.7) if $v \in f^{-1/2}L^2_3$. To show that the right hand side of (5.10) makes sense for $v \in B$, we need to show the Cauchy property. After $C_\mathrm{c}(\mathbb{R}^d) \ni v_n \to v \in B$ has been approximated, it suffices to show the bound

$$\sup_{S>1}\|S^{-1}\int_0^S \pi^{-1/2}\big(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}R(\lambda+\mathrm{i}0)f^{1/2}(v-v_n)\big)\big(\Phi(s,\cdot)\big)\,\mathrm{d}s\|_{\mathcal{G}} \le C\|v-v_n\|_B. \tag{5.11}$$

Proceeding slightly more generally, we show that for all $w \in B(|x|)^*$,

$$\sup_{S>1}\|S^{-1}\int_0^S \pi^{-1/2}\big(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}f^{-1/2}w\big)\big(\Phi(s,\cdot)\big)\,\mathrm{d}s\|_{\mathcal{G}} \le C\|w\|_{B^*}. \tag{5.12}$$

In fact, given (5.12), the bound (5.11) follows from (1.4); then (5.10) follows and in turn, (5.9) is justified.

To show (5.12), we first recall the Besov space bound

$$\sup_{\rho>1}\rho^{-1}\int_{|x|\le\rho}|w|^2\,\mathrm{d}x \le C\|w\|^2_{B^*} \tag{5.13a}$$

and its proof. Let $R_0=0$ and $R_j=2^{j-1}$ for $j\in\mathbb{N}$. Then

$$\begin{aligned}\rho^{-1}\int_{|x|\le\rho}|w|^2\,\mathrm{d}x &\le \sum_{j\le J;\,R_{J-1}\le\rho<R_J}\big(\rho^{-1}R_j\big)R_j^{-1}\int_{R_{j-1}\le|x|<R_j}|w|^2\,\mathrm{d}x\\ &\le \sum_{j\le J;\,R_{J-1}\le\rho<R_J}\big(\rho^{-1}R_j\big)\|w\|^2_{B^*}\\ &\le 4\|w\|^2_{B^*}.\end{aligned}$$

Now, using the Cauchy Schwarz inequality, notation from Subsection 3.1, (3.6) and (5.13b) (stated below), we estimate (5.12) by

$$\begin{aligned}
&\Big\| S^{-1}\int_0^S \pi^{-1/2}\big(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}f^{-1/2}w\big)\big(\Phi(s,\cdot)\big)\,\mathrm{d}s\Big\|_{\mathcal{G}}\\
&\quad\le C_1 S^{-1}\int_0^S \|\big(K_\epsilon^{1/2}m_\epsilon^{1/2}f^{-1/2}w\big)\big(\Phi(s,\cdot)\big)\|_{\mathcal{G}}\,\mathrm{d}s\\
&\quad\le C_1 S^{-1/2}\Big(\int_0^S \|\big(K_\epsilon^{1/2}m_\epsilon^{1/2}f^{-1/2}w\big)\big(\Phi(s,\cdot)\big)\|_{\mathcal{G}}^2\,\mathrm{d}s\Big)^{1/2}\\
&\quad= C_1 S^{-1/2}\Big(\int_{\mathcal{B}_\epsilon(S)} |K_\epsilon^{1/2}f^{-1/2}w|^2\,\mathrm{d}x\Big)^{1/2}\\
&\quad\le C_2 S^{-1/2}\Big(\int_{\mathcal{B}_\epsilon(S)} |f^{1/2}w|^2\,\mathrm{d}x\Big)^{1/2}\\
&\quad\le C_3\|w\|_{B^*}.
\end{aligned}$$

In the last step, we have used the following analogue of (5.13a):

$$\sup_{s>1} s^{-1}\int_{\mathcal{B}_\epsilon(s)} |f^{1/2}w|^2\,\mathrm{d}x \le C(\lambda)\|w\|_{B^*}^2. \tag{5.13b}$$

Note that for $\lambda > 0$, we can bound $S_\epsilon(x) \ge c|x|$ and $f^{1/2}(x) \le C$, which yields (5.13b) in this case because of (5.13a). For $\lambda = 0$, we have the bound $S_\epsilon(x) \ge c|x|^{1-\mu/2}$ and $f^{1/2}(x) \le C|x|^{-\mu/4}$ for $|x| \ge 1$, which yield (5.13b) in that case also. This can be seen by arguing as in the above proof of (5.13a). Consequently, (5.13b) holds for all $\lambda \ge 0$, and (5.12) is proved.

For a later application, let us note the following inverse of (5.13b) (proved similarly):

$$\|w\|_{B^*}^2 \le C(\lambda)\sup_{s>1} s^{-1}\int_{\mathcal{B}_\epsilon(s)} |f^{1/2}w|^2\,\mathrm{d}x. \tag{5.13c}$$

For an abstract version of (5.13b) and (5.13c), see [Sk, Lemma 2.4].

**Step II.** Using (5.9), we can mimic the proof of [ACH, Theorem 1.1]. Notice that we only need (5.9) for $\lambda > 0$ (which is the analogue of [ACH, Theorem 3.3 iv)]). Details are omitted. □

**Corollary 5.2.** *For all $\tau \in C^\infty(S^{d-1}) \subset \mathcal{G}$, the generalized eigenfunction $u^- = u^-(\lambda)$, $\lambda \ge 0$, defined by* (3.11) *and* (3.25) *is also given by*

$$u^-(\lambda) = 2\pi\mathrm{i}F^+(\lambda)^*\tau. \tag{5.14}$$

**Proof.** By Lemma 3.3, (5.5), and (5.9)

$$\begin{aligned} u^- &= \big(R(\lambda+\mathrm{i}0) - R(\lambda-\mathrm{i}0)\big)(H-\lambda)\tilde{u} \\ &= 2\pi\mathrm{i}F^+(\lambda)^*F^+(\lambda)(H-\lambda)\tilde{u} = 2\pi\mathrm{i}F^+(\lambda)^*\tau. \end{aligned}$$

□

**Definition 5.3.** For $\lambda \geq 0$, we define the **scattering matrix** $S(\lambda) \in \mathcal{B}(\mathcal{G})$ by the identity

$$F^+(\lambda)v = S(\lambda)F^-(\lambda)v,\ v \in f_\lambda^{1/2}B(|x|). \tag{5.15}$$

**Proposition 5.4.** *The operator $S(\lambda)$ is a well-defined unitary operator on $\mathcal{G}$. It is strongly continuous as a function of $\lambda \geq 0$. In particular, the scattering matrix at zero energy $S(0)$ is uniquely determined by the diagonalizing transforms $F^\pm$.*

**Proof.** We apply (5.5), (5.7), (5.9) and their analogues for change of superscript $+ \to -$. This yields the well-definedness and the unitarity. For all $v \in L_3^2$, the functions $\{F^\pm(\lambda)v \,|\, \lambda \geq 0\} \in \widetilde{\mathcal{H}}$ are continuous in $\lambda$; cf. (5.4). Since $F^-(\lambda)L_3^2$ is dense in $\mathcal{G}$, the continuity property for any fixed $\lambda$ follows by a density argument.

□

**5.1 Asymptotics of generalized eigenfunctions.** We complete this section with a discussion of the asymptotics of the generalized eigenfunctions

$$u_\tau^-(\cdot,\lambda) := 2\pi\mathrm{i}F^+(\lambda)^*\tau;\ \tau \in \mathcal{G}. \tag{5.16}$$

Notice that Corollary 5.2 provides a representation for $\tau \in C^\infty(S^{d-1})$.

Let $B_0^* \subset B^*$ be the closure of $C_c(\mathbb{R}^d)$ in $B^*$. For all $\lambda \geq 0$ and $\tau \in \mathcal{G}$, the function

$$w(x) = \big(K_\epsilon^{-1/2}m_\epsilon^{-1/2}f^{1/2}\big)(x)\tau(\sigma),\ x = \Phi(t,\sigma),$$

belongs to $B^*$ with

$$\|w\|_{B^*} \leq C\|\tau\|_{\mathcal{G}}. \tag{5.17}$$

This is because of (3.6) and (5.13c).

Next, using (5.9) and (5.15), for all $\tau \in C^\infty(S^{d-1})$, we consider the decomposition

$$\begin{aligned} w_\tau^-(x) :&= \pi^{1/2}\big(K_\epsilon^{-1/2}m_\epsilon^{-1/2}f^{1/2}\big)(x)\big(S(\lambda)^{-1}\tau\big)(\sigma) \\ &= \pi^{1/2}\big(K_\epsilon^{-1/2}m_\epsilon^{-1/2}f^{1/2}\big)(x)\big(F^-(\lambda)(H-\lambda)\tilde{u}\big)(\sigma) \\ &= w_1^-(x) + w_2^-(x); \\ &\qquad w_1^- := \mathrm{e}^{\mathrm{i}S_\epsilon}f^{1/2}R(\lambda-\mathrm{i}0)(H-\lambda)\tilde{u}. \end{aligned}$$

While $w_\tau^-$, $w_1^- \in B^*$, we assert the stronger statement for the second term

$$w_2^- \in B_0^*. \tag{5.18}$$

To prove (5.18), we introduce the quantity

$$w_n = \pi^{1/2}\big(K_\epsilon^{-1/2} m_\epsilon^{-1/2} f^{1/2}\big)(x)\big(F^-(\lambda)\big((H-\lambda)\tilde{u} - f^{1/2}v_n\big)\big)(\sigma),$$

where $C_c(\mathbb{R}^d) \ni v_n \to v := f^{-1/2}(H-\lambda)\tilde{u} \in B$. Then $\|w_n\|_{B^*} \le C\|v_n - v\|_B$ (cf. (5.17)), which shows that $\|w_n\|_{B^*} \to 0$ for $n \to \infty$. Similarly,

$$\mathrm{e}^{\mathrm{i}S_\epsilon} f^{1/2} R(\lambda - \mathrm{i}0) f^{1/2} v_n \to w_1^- \text{ in } B^*.$$

We are led to consider the quantity (for fixed $n \in \mathbb{N}$)

$$\begin{aligned}\widetilde{w}_n(x) = \pi^{1/2}\big(K_\epsilon^{-1/2} m_\epsilon^{-1/2} f^{1/2}\big)(x)\Big(&\big(F^-(\lambda) f^{1/2} v_n\big)(\sigma)\\ &- \pi^{-1/2}\big(\mathrm{e}^{\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} R(\lambda - \mathrm{i}0) f^{1/2} v_n\big)(x)\Big),\end{aligned}$$

It follows from (3.6), (5.1), (5.2), and (5.13c) by yet another approximation that $\widetilde{w}_n \in B_0^*$. Note that

$$\|\widetilde{w}_n - 1_{\mathcal{B}_\epsilon(s)}\widetilde{w}_n\|_{B^*} \to 0 \text{ for } s \to \infty,$$

while (obviously) $1_{\mathcal{B}_\epsilon(s)}\widetilde{w}_n \in B_0^*$ for all $s > 1$. Thus (5.18) is proved.

Now, combining (3.25) and (5.18), we conclude that for all $\tau \in C^\infty(S^{d-1})$,

$$u_\tau^-(\cdot, \lambda) - \big(\tilde{u} - \mathrm{e}^{-\mathrm{i}S_\epsilon} f^{-1/2} w_\tau^-\big) \in f^{-1/2} B_0^*.$$

This formula extends to $\mathcal{G}$ and implies an injectivity property.

**Corollary 5.5.** *Let $\lambda \ge 0$ and $\tau \in \mathcal{G}$ be given. Let*

$$u_{0,\tau}^-(x,\lambda) = \pi^{1/2}\big(K_\epsilon^{-1/2} m_\epsilon^{-1/2}\big)(x)\big(\mathrm{e}^{\mathrm{i}S_\epsilon(x)}\tau(\sigma) - \mathrm{e}^{-\mathrm{i}S_\epsilon(x)}\big(S(\lambda)^{-1}\tau\big)(\sigma)\big),$$

*where $x = \Phi(t, \sigma)$, Then*

$$u_\tau^-(\cdot, \lambda) - u_{0,\tau}^-(\cdot, \lambda) \in f_\lambda^{-1/2} B_0^*. \tag{5.19a}$$

*In particular,*

$$2\pi\|\tau\|_{\mathcal{G}}^2 = \lim_{s\to\infty} s^{-1}\int_{\mathcal{B}_\epsilon(s)} |K_\epsilon^{1/2} u_\tau^-(x,\lambda)|^2\,\mathrm{d}x. \tag{5.19b}$$

**Proof.** Since (5.19a) holds for $\tau \in C^\infty(S^{d-1})$, the statement for $\tau \in \mathcal{G}$ follows by approximation; cf. (5.17). As for (5.19b), we can replace $u_\tau^-$ on the right hand side by $u_{0,\tau}^-$, because of (5.19a). We then compute, using (3.6) and the unitarity of the scattering matrix. Note that because of oscillatory behavior, cross terms do not contribute to the limit. □

# 6 Characterization of generalized eigenfunctions

We introduce the following class of generalized eigenfunctions.

**Definition 6.1.** For $\lambda \geq 0$, let

$$\mathcal{E}_\lambda = \{u \in f_\lambda^{-1/2} B^* | (H - \lambda)u = 0\}. \tag{6.1}$$

Notice that it follows from (5.7) and the definition (5.16) that for all $\tau \in \mathcal{G}$,

$$u_\tau^-(\cdot, \lambda) \in \mathcal{E}_\lambda.$$

In fact, it follows from (5.7), (5.13b), and (5.19b) that the map

$$\mathcal{G} \ni \tau \to u_\tau^-(\cdot, \lambda) \in \mathcal{E}_\lambda$$

is a bi-continuous linear isomorphism onto its range. The latter is identified in the following proposition.

**Proposition 6.2.** *For all $\lambda \geq 0$,*

$$\mathcal{E}_\lambda = \{u_\tau^-(\cdot, \lambda) | \tau \in \mathcal{G}\}. \tag{6.2}$$

**Proof.** Let arbitrary $u_\lambda \in \mathcal{E}_\lambda$ be given. We need to show that it must have the form $u_\lambda = u_\tau^-(\cdot, \lambda)$ for some $\tau \in \mathcal{G}$. For that, we partly mimic [DS1, Section 8]. In particular, with reference to the symbols (4.1) and the corresponding localization symbols appearing in Proposition 4.1, let us introduce the functions

$$\chi^\pm = \chi_-(a_\lambda)\tilde{\chi}_\pm(b_\lambda) + \frac{1}{2}\chi_+(a_\lambda). \tag{6.3}$$

In what follows, exactly as in Proposition 4.1, we treat these functions as fixed and consider $\epsilon$-small perturbations $W_\epsilon \in \mathcal{W}$ with $\epsilon > 0$ small. Note the properties

$$\mathrm{Op^w}\left(a_\lambda \chi_+(a_\lambda)\right)u_\lambda,\ \mathrm{Op^w}\left(\chi_+(a_\lambda)\right)u_\lambda \in f_\lambda^{-1/2} B_0^*; \tag{6.4}$$

cf. [Sk, Lemma 3.1]. In fact, the functions in (6.4) are in every weighted $L^2$-space $L^2_m$. Hence the quantization of the second term of (6.3) contributes a small term when applied to $u_\lambda$. The quantization of the first term localizes to an outgoing (incoming) region of phase space. A priori, we have only

$$\mathrm{Op^w}\left(\chi^\pm\right)u_\lambda,\ \mathrm{Op^w}\left(\chi_-(a_\lambda)\tilde{\chi}_\pm(b_\lambda)\right)u_\lambda \in f_\lambda^{-1/2} B^*.$$

**Step I.** We construct a candidate $\tau$. Pick non-negative $g \in C_c^\infty(\mathbb{R}_+)$ with $\int_0^\infty g(t)\mathrm{d}t = 1$, and let $G_n(s) = 1 - \int_0^{s/n} g(t)\mathrm{d}t$, $n \in \mathbb{N}$. Define

$$\tau_n = F^+(\lambda)G_n(S_\epsilon)(H-\lambda)\operatorname{Op}^{\mathrm{w}}\big(\chi^+\big)u_\lambda,\ n \in \mathbb{N}.$$

This family $\{\tau_n\}$ is a bounded subset of $\mathcal{G}$; in fact,

$$\begin{aligned}\tau_n &= \mathrm{i}F^+(\lambda)\mathrm{i}[H, G_n(S_\epsilon)]\operatorname{Op}^{\mathrm{w}}\big(\chi^+\big)u_\lambda\\ &= -\mathrm{i}F^+(\lambda)\big(\operatorname{Re}(p\cdot\nabla S_\epsilon)\frac{2}{n}g(S_\epsilon/n) + \mathrm{i}|\nabla S_\epsilon|^2 n^{-2}g'(S_\epsilon/n)\big)\operatorname{Op}^{\mathrm{w}}\big(\chi^+\big)u_\lambda\\ &= \tau_n^1 + \tau_n^2.\end{aligned}$$

For each $\tilde\tau \in \mathcal{G}$,

$$\|f_\lambda^{1/2}u_{\tilde\tau}^-(\cdot,\lambda)\|_{B^*} \le C\|\tilde\tau\|_{\mathcal{G}}, \tag{6.5a}$$

$$\|f_\lambda^{-3/2}\chi(|x|>2)\operatorname{Re}(p\cdot\nabla S_\epsilon)u_{\tilde\tau}^-(\cdot,\lambda)\|_{B^*} \le C\|\tilde\tau\|_{\mathcal{G}}; \tag{6.5b}$$

cf. (6.4). We aim to prove the uniform bounds

$$|\langle\tilde\tau, \tau_n^j\rangle| \le C\|\tilde\tau\|_{\mathcal{G}}, \quad j = 1, 2, \tag{6.6}$$

which suffices for the boundedness.

For $j = 2$, we write

$$\begin{aligned}-2\pi\mathrm{i}\langle\tilde\tau, \tau_n^2\rangle &= \langle f_\lambda^{1/2}u_{\tilde\tau}^-(\cdot,\lambda), h_n^2 w\rangle;\\ h_n^2 &= f_\lambda^{-1/2}|\nabla S_\epsilon|^2 n^{-2}g'(S_\epsilon/n)f_\lambda^{-1/2},\ w = f_\lambda^{1/2}\operatorname{Op}^{\mathrm{w}}\big(\chi^+\big)u_\lambda.\end{aligned}$$

For $\lambda > 0$, we have $|h_n^2(x)| \le C_\lambda\langle x\rangle^{-2}$, while for $\lambda = 0$, we have the bound $|h_n^2(x)| \le C\langle x\rangle^{-2+\mu/2}$. Since $B^*$ is continuously imbedded in $L^2_{-\delta}$ for each $\delta > 1/2$, we obtain the bound (6.6) for $j = 2$ using (6.5a).

Decomposing similarly for $j = 1$, we obtain

$$\langle\tilde\tau, \tau_n^1\rangle = \langle\tilde w, h_n^1 w\rangle, \quad h_n^1 = f_\lambda^{3/2}n^{-1}g(S_\epsilon/n)f_\lambda^{-1/2}, \quad w = f_\lambda^{1/2}\operatorname{Op}^{\mathrm{w}}\big(\chi^+\big)u_\lambda;$$

and using (6.5b), we need to show the bound

$$|\langle\tilde w, h_n^1 w\rangle| \le C\|\tilde w\|_{B^*}\,\|w\|_{B^*}.$$

For that it suffices, for each $\lambda \ge 0$, to find $C > 1$ and a bounded interval $I$ such that for each $n \in \mathbb{N}$, there exists $R \ge 1$ such that $|h_n^1(x)| \le CR^{-1}1_I(|x|/R)$ for all $x \in \mathbb{R}^d$. We recall the bounds $crf_\lambda \le S_\epsilon \le Crf_\lambda$. In particular, the assertion is immediate for $\lambda > 0$ with $R = n$. For $\lambda = 0$, we choose $R = n^{1/(1-\mu/2)}$ and obtain the same conclusion. Thus (6.6) holds, and the sequence $\{\tau_n\} \subset \mathcal{G}$ is bounded.

Take $\tau \in \mathcal{G}$ as the weak limit of some subsequence of $\{\tau_n\}$; cf. [Yo, Theorem 1, p. 126]. With a change of notation, we can assume that

$$\tau = \text{w--}\mathcal{G}\text{--}\lim_{n\to\infty} F^+(\lambda)G_n(S_\epsilon)[H, \mathrm{Op}^{\mathrm{w}}\left(\chi^+\right)]u_\lambda. \tag{6.7}$$

**Step II.** We show that this $\tau$ works. We compute using (5.5) in the third step, Propositions 3.1 and 4.1 in the last step, and taking $m = -3$, thus obtaining

$$\begin{aligned}
f^{1/2}u_\tau^-(\cdot,\lambda) &= 2\pi\mathrm{i} f^{1/2}F^+(\lambda)^*\tau \\
&= 2\pi\mathrm{i}\,\text{w}^\star\text{--}\mathrm{B}^*\text{--}\lim_{n\to\infty} f^{1/2}F^+(\lambda)^*F^+(\lambda)G_n(S_\epsilon)[H, \mathrm{Op}^{\mathrm{w}}\left(\chi^+\right)]u_\lambda \\
&= \text{w}^\star\text{--}\mathrm{B}^*\text{--}\lim_{n\to\infty} f^{1/2}\big(R(\lambda-\mathrm{i}0)-R(\lambda+\mathrm{i}0)\big)G_n(S_\epsilon)[\mathrm{Op}^{\mathrm{w}}\left(\chi^+\right), H-\lambda]u_\lambda \\
&= \text{w}^\star\text{--}L^2_m\text{--}\lim_{n\to\infty} f^{1/2}R(\lambda-\mathrm{i}0)G_n(S_\epsilon)[H-\lambda, \mathrm{Op}^{\mathrm{w}}\left(\chi^-\right)]u_\lambda \\
&\quad + \text{w}^\star\text{--}L^2_m\text{--}\lim_{n\to\infty} f^{1/2}R(\lambda+\mathrm{i}0)G_n(S_\epsilon)[H-\lambda, \mathrm{Op}^{\mathrm{w}}\left(\chi^+\right)]u_\lambda \\
&= w^- + w^+;\ w^\mp = f^{1/2}R(\lambda\mp\mathrm{i}0)(H-\lambda)\,\mathrm{Op}^{\mathrm{w}}\left(\chi^\mp\right)u_\lambda.
\end{aligned}$$

Using Proposition 4.1 again, we obtain

$$\begin{aligned}
w^\mp &= \text{w}^\star\text{--}L^2_m\text{--}\lim_{\epsilon\searrow 0} f^{1/2}R(\lambda\mp\mathrm{i}\epsilon)(H-\lambda)\,\mathrm{Op}^{\mathrm{w}}\left(\chi^\mp\right)u_\lambda \\
&= f^{1/2}\,\mathrm{Op}^{\mathrm{w}}\left(\chi^\mp\right)u_\lambda \mp \text{w}^\star\text{--}L^2_m\text{--}\lim_{\epsilon\searrow 0} \mathrm{i}\epsilon f^{1/2}R(\lambda\mp\mathrm{i}\epsilon)\,\mathrm{Op}^{\mathrm{w}}\left(\chi^\mp\right)u_\lambda \\
&= f^{1/2}\,\mathrm{Op}^{\mathrm{w}}\left(\chi^\mp\right)u_\lambda.
\end{aligned}$$

Thus

$$u_\tau^-(\cdot,\lambda) = f^{-1/2}\left(w^- + w^+\right) = \Big(\mathrm{Op}^{\mathrm{w}}\left(\chi^-\right) + \mathrm{Op}^{\mathrm{w}}\left(\chi^+\right)\Big)u_\lambda = u_\lambda. \qquad \square$$

For the reader's convenience, before summarizing and stating our main result, let us provide a list of references to previously introduced notation used for stating the result.

For the $\epsilon$-small perturbation $W_\epsilon$ see (2.1b); for the Hilbert space $\mathcal{G}$, see (3.9); for the solution $S_\epsilon$ of the eikonal equation, the weight function $f_\lambda = f(|x|, \lambda)$, the "Jacobian type functions" $K_\epsilon$ and $m_\epsilon$, and the flow $\Phi(t, \sigma)$, see Section 3 (more precisely , (3.1), (3.2), (3.3a), and (3.5)); for the (dual) Besov spaces $B^* = B(|x|)^*$ (corresponding to the Besov space $B = B(|x|)$) and $B_0^* = B(|x|)_0^*$, see (1.2); for the space of generalized eigenfunctions $\mathcal{E}_\lambda$, see (6.1); for the geodesic ball $\mathcal{B}_\epsilon(s)$, see (3.7); for (the fiber of) the distorted Fourier transform $F^+(\lambda)$, see (5.10) (see also (3.10a) and (3.10b)); for the scattering matrix $S(\lambda)$, see (5.15).

**Theorem 6.3.** *Assume Condition 2.1 (and* (2.2a)*). Then there exist* $\epsilon_0 > 0$ *and* $l = l(\mu, d) \in \mathbb{N}$ *such that for all* $\epsilon$*-small perturbations* $W_\epsilon$ *with* $\epsilon \in (0, \epsilon_0]$ *(assuming also* (2.2b)*), the following statements hold for all* $\lambda \geq 0$.

(i) *For each* $\tau \in \mathcal{G}$*, there exist unique* $\tilde{\tau} \in \mathcal{G}$ *and* $u_\lambda \in \mathcal{E}_\lambda$ *such that (with* $x = \Phi(t, \sigma)$*),*

$$u_\lambda(x) - \pi^{1/2}\big(K_\epsilon^{-1/2} m_\epsilon^{-1/2}\big)(x)\big(\mathrm{e}^{\mathrm{i}S_\epsilon(x)}\tau(\sigma) - \mathrm{e}^{-\mathrm{i}S_\epsilon(x)}\tilde{\tau}(\sigma)\big) \in f_\lambda^{-1/2} B_0^*. \tag{6.8a}$$

(ii) *For all* $u_\lambda \in \mathcal{E}_\lambda$*, there exist unique* $\tau, \tilde{\tau} \in \mathcal{G}$ *such that* (6.8a) *holds. In particular, the map* $\mathcal{G} \ni \tau \to u_\lambda \in \mathcal{E}_\lambda$ *is a linear isomorphism. It is bi-continuous; in fact,*

$$2\pi \|\tau\|_{\mathcal{G}}^2 = \lim_{s \to \infty} s^{-1} \int_{\mathcal{B}_\epsilon(s)} |K_\epsilon^{1/2} u_\lambda|^2 \,\mathrm{d}x. \tag{6.8b}$$

(iii) *There are formulas for quantities in* (6.8a)*:*

$$u_\lambda = 2\pi \mathrm{i} F^+(\lambda)^* \tau \text{ and } \tilde{\tau} = S(\lambda)^{-1}\tau. \tag{6.8c}$$

*In particular, the wave matrix* $F^+(\lambda)^* : \mathcal{G} \to \mathcal{E}_\lambda$ *is a bi-continuous linear isomorphism.*

(iv) *The maps* $F^+(\lambda) f_\lambda^{1/2} : B \to \mathcal{G}$ *and* $\delta(\lambda) = \pi^{-1} \operatorname{Im}\big(R(\lambda + \mathrm{i}0)\big) : f_\lambda^{1/2} B \to \mathcal{E}_\lambda$ *are onto.*

**Proof.** The uniqueness of $\tilde{\tau}$ and $u_\lambda$ in (6.8a) follows from the proof of Lemma 3.3; and the existence part, agreeing with (6.8c), follows from (5.19a). The assertions (ii) are consequences of Proposition 6.2 and (5.19b). The assertions (iv) are consequences of (iii) and Banach's Closed Range Theorem [Yo, Theorem p. 205]. □

**6.1 Concluding remarks.** With some additional effort, one should be able to show that the operator $F^+(\lambda)^*$ has a somewhat regular kernel, formally given by $(F^+(\lambda)^*\delta_\sigma)(x)$. More precisely,

$$(F^+(\lambda)^*\tau)(x) = \int_{S^{d-1}} \phi^+(x, \sigma, \lambda)\tau(\sigma)\,\mathrm{d}\sigma$$

should hold, where the plane wave type eigenfunction $\phi^+$ has a degree of regularity. In particular, it should be continuous in all variables for $x \notin \operatorname{supp} V_2$, provided the perturbation $W_\epsilon \in \mathcal{W}$ is $\epsilon$-small for small enough $\epsilon > 0$. More regularity in $\sigma \in S^{d-1}$ would require smaller $\epsilon$. The validity of these assertions would depend on generalizations of Proposition 4.2; cf. [HS2]. We do not elaborate further on this issue. Note that smoothness in the angular variable of analogous plane wave type eigenfunctions was obtained in [DS1].

Another remark concerns the relationship between the scattering theory developed here and [DS1] in case of overlapping conditions (which means under the conditions of [DS1]). In the case of a spherically symmetric potential, one has $\sigma = \eta(\sigma)$ for all $\lambda \geq 0$, where $\eta(\sigma) := \lim_{s\to\infty} \Phi(s,\sigma)/|\Phi(s,\sigma)|$, and the two involved solutions of the eikonal equation are identical up to a trivial explicit term. In particular, the two families of $S$-matrices are explicitly connected as follows. For all $\lambda \geq 0$, the operator $S(\lambda)$ of this paper and the scattering matrix of [DS1], say $S_{\mathrm{DS}}(\lambda)$, are related up to an explicit phase factor as $S(\lambda) = S_{\mathrm{DS}}(\lambda)R$, where $(R\tau)(\omega) = \tau(-\omega)$; cf. the discussion at the beginning of Section 1.

More generally, under the conditions of [DS1], the asymptotic normalized velocity $\eta(\sigma)$ exists for all $\lambda \geq 0$. Moreover, as a map, it is a diffeomorphism on $S^{d-1}$. This property, Theorem 6.3, and [DS1, Theorem 8.2] yield the connection formula

$$S_{\mathrm{DS}}(\lambda)^{-1} = R\mathrm{e}^{-\mathrm{i}\phi(\cdot,\lambda)}D_\eta S(\lambda)^{-1}D_{\eta^{-1}}\mathrm{e}^{-\mathrm{i}\phi(\cdot,\lambda)}, \tag{6.9}$$

where $\phi(\omega,\lambda)$ is real; and for any diffeomorphism $\psi$ on $S^{d-1}$, the operator $D_\psi$ is the unitary map on $L^2(S^{d-1})$ implemented by the classical map

$$S^{d-1} \ni \omega \to \psi(\omega) \in S^{d-1},$$

viz., $(D_\psi\tau)(\omega) = J^{1/2}(\omega)\tau(\psi^{-1}(\omega))$.

Although we do not elaborate, the formula (6.9) suggests a criterion for regularity at zero energy of a family of (inverse) scattering matrices under the conditions of Section 2; namely, it should suffice that the families of diffeomorphisms $\eta = \eta_\lambda$ and $\eta_\lambda^{-1}$ on $S^{d-1}$ as well as the family of phases $\phi(\cdot,\lambda)$ be regular at zero energy. Indeed, in this case, the right hand side of (6.9) has a limit as $\lambda \to 0$, because of Proposition 5.4. This criterion is of course not applicable to the example in Subsection 2.1. Note that under the conditions of Section 2, the form of the right hand side of (6.9) makes sense for positive energies, and the expression coincides with the (inverse) scattering matrix discussed at the beginning of Section 1.

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