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Two-body threshold spectral analysis, the critical case

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Abstract

We study in dimension $d \geqslant 2$ low-energy spectral and scattering asymptotics for two-body d-dimensional Schrödinger operators with a radially symmetric potential falling off like $-\gamma r^{-2}$, $\gamma > 0$. We consider angular momentum sectors, labelled by $l = 0, 1, \ldots,$ for which $\gamma > (l + d/2 - 1)^2$. In each such sector the reduced Schrödinger operator has infinitely many negative eigenvalues accumulating at zero. We show that the resolvent has a non-trivial oscillatory behaviour as the spectral parameter approaches zero in cones bounded away from the negative half-axis, and we derive an asymptotic formula for the phase shift.

Keywords: Threshold spectral analysis; Schrödinger operator; Critical potential; Phase shift

Contents

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1. Introduction

The low-energy spectral and scattering asymptotics for two-body Schrödinger operators depend heavily on the decay of the potential at infinity. The most well-studied class is given by potentials decaying faster than r^{-2} (see for example [6] and references there). The expansion of the resolvent is in this case in terms of powers of dimension-dependent modifications of the spectral parameter and it depends on possible existence of zero-energy bound states and/or zero-energy resonance states. Classes of negative potentials decaying slower than r^{-2} were studied in [5,7,16]. In that case the resolvent is more regular at zero energy. It has an expansion in integer powers of the spectral parameter and there are no zero-energy bound states nor resonance states. Moreover, the nature of the expansion is "semi-classical". For general perturbations of critical decay of the order r^{-2} and with an assumption related to the Hardy inequality, the threshold spectral analysis is carried out in [13,14]. It is shown that for this class of potentials the zero resonance may appear in any space dimension with arbitrary multiplicity. Recall that for potentials decaying faster than r^{-2}, the zero resonance is absent if the space dimension d is bigger than or equal to five and its multiplicity is at most one when d is equal to three or four. The goal of this paper is to treat a class of radially symmetric potentials decaying like $-\gamma r^{-2}$ at infinity, where $\gamma>0$ is big such that the condition used in [13,14] is not satisfied. In this case, there exist infinitely many negative eigenvalues (see (1.3) for a precise condition). We will give a resolvent expansion as well as an asymptotic formula for the phase shift. These expansions are to our knowledge not semi-classical even though there are common features with the more slowly decaying case.

Consider for $d\geqslant 2$ the d-dimensional Schrödinger operator

$$Hv=(-\Delta+W)v=0,$$

for a radial potential $W=W(|x|)$ obeying

Condition 1.1.

1) $W(r)=W_1(r)+W_2(r)$; $W_1(r)=-\frac{\gamma}{r^2}\chi(r>1)$ for some $\gamma>0$,
2) $W_2\in C(]0,\infty[,\mathbb{R})$,
3) $\exists\epsilon_1,C_1>0$: $|W_2(r)|\leqslant C_1r^{-2-\epsilon_1}$ for $r>1$,
4) $\exists\epsilon_2,C_2>0$: $|W_2(r)|\leqslant C_2r^{\epsilon_2-2}$ for $r\leqslant 1$.

Here the function $\chi(r>1)$ is a smooth cutoff function taken to be 1 for $r\geqslant 2$ and 0 for $r\leqslant 1$ (see the end of this introduction for the precise definition). Under Condition 1.1 H is self-adjoint as defined in terms of the Dirichlet form on $H^1(\mathbb{R}^d)$. Let H_l, $l=0,1,\dots$, be the corresponding reduced Hamiltonian on $L^2(\mathbb{R}_+)$ corresponding to the eigenvalue $l(l+d-2)$ of the Laplace–Beltrami operator on S^{d-1}

$$H_l u = -u'' + (V_\infty + V)u. \tag{1.1}$$

Here

$$V_\infty(r) = \frac{\nu^2 - 1/4}{r^2}\chi(r>1); \quad \nu^2 = \left(l + \frac{d}{2} - 1\right)^2 - \gamma, \tag{1.2a}$$

$$V(r) = W_2(r) + \frac{(l+\frac{d}{2}-1)^2 - 1/4}{r^2}\bigl(1 - \chi(r>1)\bigr). \tag{1.2b}$$

Notice that V is small at infinity compared to V_∞. We are interested in spectral and scattering properties of H_l at zero energy in the case

$$\gamma > \left(l + \frac{d}{2} - 1\right)^2. \tag{1.3}$$

This condition is equivalent to having ν in (1.2a) purely imaginary (for convenience we fix it in this case as $\nu = -\mathrm{i}\sigma$, $\sigma > 0$), and it implies the existence of a sequence of negative eigenvalues of H_l accumulating at zero energy.

Our first main result is on the expansion of the resolvent

$$R_l(k) := \bigl(H_l - k^2\bigr)^{-1} \quad \text{for } k \in \Gamma_\theta^\pm,$$

where here (for any $\theta \in \,]0, \pi/2[$)

$$\Gamma_\theta^+ = \{k \neq 0 \mid 0 < \arg k \leqslant \theta\}, \tag{1.4a}$$

$$\Gamma_\theta^- = \{k \neq 0 \mid \pi - \theta \leqslant \arg k < \pi\}. \tag{1.4b}$$

We say that a solution u to the equation

$$-u''(r) + \bigl(V_\infty(r) + V(r)\bigr)u(r) = 0 \tag{1.5}$$

is *regular* if the function $r \to \chi(r<1)u(r)$ belongs to $\mathcal{D}(H_l)$. For any $t \in \mathbb{R}$ we introduce the weighted L^2-space $\mathcal{H}_t := \langle r\rangle^{-t} L^2(\mathbb{R}_+)$; $\langle r\rangle = (1+r^2)^{1/2}$.

Theorem 1.2. *Suppose Condition* 1.1 *and* (1.3) *for some* (*fixed*) $l \in \mathbb{N} \cup \{0\}$. *Let* $\theta \in \,]0, \pi/2[$. *There exist* (*finite*) *rational functions* $f^\pm$ *in the variable* $k^{2\nu}$ *for* $k \in \Gamma_\theta^\pm$ *for which*

$$\lim_{\Gamma_\theta^\pm \ni k \to 0} \operatorname{Im} f^\pm\bigl(k^{2\nu}\bigr) \textit{ do not exist}, \tag{1.6}$$

there exist Green's functions for H_l at zero energy, denoted by $R_0^\pm$, and there exists a real nonzero regular solution to (1.5)*, denoted by u, such that the following asymptotics hold. For all $s > s' > 1$, $s \leqslant 1 + \epsilon_1/2$, $s' \leqslant 3$:*

$$\limsup_{\Gamma_\theta^\pm \ni k \to 0} |k|^{1-s'} \left\| R_l(k) - R_0^\pm - f^\pm\left(k^{2\nu}\right)|u\rangle\langle u| \right\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} < \infty. \tag{1.7}$$

Due to (1.6) the rank-one operators $f^\pm(k^{2\nu})|u\rangle\langle u|$ in (1.7) are non-trivially oscillatory. This phenomenon does not occur for low-energy resolvent expansions for potentials either decaying faster or slower than r^{-2} (cf. [6] and [5,16], respectively), nor for sectors where (1.3) is not fulfilled (cf. [14]). Combining Theorem 1.2 and the results of [14], we can deduce the resolvent asymptotics near threshold for d-dimensional Schrödinger operators with critically decaying, spherically symmetric potentials, see Theorem 3.7. An advantage to work with spherically symmetric potentials is that we can diagonalise the operator in spherical harmonics and explicitly calculate some subtle quantities. For example, one can easily show that if zero is a resonance of H, then its multiplicity is equal to

$$\frac{(m+d-3)!}{(d-2)!(m-1)!} + \frac{(m+d-2)!}{(d-2)!m!}$$

where $m \in \mathbb{N} \cup \{0\}$ is such that $(m + \frac{d}{2} - 1)^2 - \gamma \in]0, 1]$. This shows that multiplicity of zero resonance grows like $\gamma^{\frac{d-2}{2}}$ when γ is big and $d \geqslant 3$. To study the resolvent asymptotics for non-spherically symmetric potential $W(x)$ behaving like $\frac{q(\theta)}{r^2}$ at infinity ($x = r\theta$ with $r = |x|$), one is led to analyse the interactions between different oscillations and resonant states. This is not carried out in the present work.

Our second main result is on the asymptotics of the phase shift. Let u_l be a regular solution to the reduced Schrödinger equation

$$-u'' + (V_\infty + V)u = \lambda u; \quad \lambda > 0.$$

Write

$$\lim_{r\to\infty} \left(u_l(r) - C\sin(\sqrt{\lambda} r + D_l)\right) = 0.$$

The standard definition of the phase shift (coinciding with the time-depending definition) is

$$\sigma_l^{\mathrm{phy}}(\lambda) = D_l + \frac{d - 3 + 2l}{4}\pi.$$

The notation $\sigma^{\mathrm{per}} = \sigma^{\mathrm{per}}(t)$ signifies below the continuous real-valued 2π-periodic function determined by

$$\begin{cases} \sigma^{\mathrm{per}}(0) = 0, \\ \mathrm{e}^{\pi\sigma}\mathrm{e}^{-\mathrm{i}t} - \mathrm{e}^{\mathrm{i}t} = r(t)\mathrm{e}^{\mathrm{i}(\sigma^{\mathrm{per}}(t) - t)}; \quad r(t) > 0,\ t \in \mathbb{R}. \end{cases}$$

Theorem 1.3. *Suppose Condition* 1.1 *and* (1.3) *for some* $l \in \mathbb{N} \cup \{0\}$. *Let*

$$\sigma = \sqrt{\gamma - \left(l + \frac{d}{2} - 1\right)^2}$$

(*recall* $\nu = -\mathrm{i}\sigma$). *There exist* $C_1, C_2 \in \mathbb{R}$ *such that*

$$\sigma_l^{\mathrm{phy}}(\lambda) + \sigma \ln \sqrt{\lambda} - \sigma^{\mathrm{per}}(\sigma \ln \sqrt{\lambda} + C_1) \to C_2 \quad \text{for } \lambda \downarrow 0. \tag{1.8}$$

Whence the leading term in the asymptotics of the phase shift is linear in $\ln \sqrt{\lambda}$ while the next term is oscillatory in the same quantity. The (positive) sign agrees with the well-known Levinson theorem (cf. [8, (12.95) and (12.156)]) valid for potentials decaying faster than r^{-2}. Also the qualitative behaviour of these terms as $\sigma \to 0$ (i.e. finiteness in the limit) is agreeable to the case where (1.3) is not fulfilled (studied in [1] from a different point of view).

The bulk of this paper concerns somewhat more general one-dimensional problems than discussed above. In particular we consider for $(d, l) \neq (2, 0)$ a model with a local singularity at $r = 0$ that is more general than specified by Condition 1.1 4) and (1.2b). This extension does not contribute by any complication and is therefore naturally included. It would be possible to extend our methods to certain types of more general local singularities, however this would add some extra complication that we will not pursue. Our methods rely heavily on explicit properties of solutions to the Bessel equation as well as ODE techniques. These properties compensate for the fact that, at least to our knowledge, semi-classical analysis is not doable in the present context (for instance the semi-classical formula (6.8) for the asymptotics of the phase shift for slowly decaying potentials is not correct under Condition 1.1). See however [2] in the case the potential is positive.

One of our motivations for studying a potential with critical fall off comes from an N-body problem: Consider a 2-cluster N-body threshold under the assumption of Coulomb pair interactions, this could be given by two atoms each one being confined in a bound state. Suppose one atom is charged while the other one is neutral. The effective intercluster potential will in this case in a typical situation (given by nonzero moment of charge of the bound state of the neutral atom) have r^{-2} decay although with some angular dependence (the so-called dipole approximation). Whence we expect (due to the present work) that the N-body resolvent will have some oscillatory behaviour near the threshold in question. Proving this (and related spectral and scattering properties) would, in addition to material from the present paper, rely on a reduction scheme not to be discussed here. We plan to study this problem in a separate future publication.

In this paper we consider parameters $\pm\nu, z \in \mathbb{C}$ satisfying $\nu = -\mathrm{i}\sigma$ where $\sigma > 0$ and $z \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} z \geqslant 0$. Powers of z are throughout the paper defined in terms of the argument function fixed by the condition $\arg z \in [0, \pi]$. We shall use the standard notation $\langle z \rangle := (1 + |z|^2)^{1/2}$. For any given $c > 0$ we shall use the notation $\chi(r > c)$ to denote a given real-valued function $\chi \in C^\infty(\mathbb{R}_+)$ with $\chi(r) = 0$ for $r \leqslant c$ and $\chi(r) = 1$ for $r \geqslant 2c$. We take it such that there exists a real-valued function $\chi_< \in C^\infty(\mathbb{R}_+)$, denoted by $\chi_< = \chi(\cdot < c)$, such that $\chi^2 + \chi_<^2 = 1$. Let for $\theta \in [0, \pi/2[$ and $\epsilon > 0$

$$\Gamma_{\theta,\epsilon} = \{k \neq 0 \mid 0 \leqslant \arg k \leqslant \theta \text{ or } \pi - \theta \leqslant \arg k \leqslant \pi\} \cap \{|k| \leqslant \epsilon\}, \tag{1.9}$$

$$\Gamma_{\theta,\epsilon}^{\pm} = \Gamma_{\theta,\epsilon} \cap \{\pm \operatorname{Re} k > 0\}. \tag{1.10}$$

2. Model asymptotics

In this section, we give the resolvent asymptotics at zero for a model operator under the condition (1.3). See [13] when (1.3) is not satisfied. Recall firstly some basic formulas for Bessel and Hankel functions from [11, pp. 228–230] and [12, pp. 126–127, 204] (or see [15]):

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} \left(1-t^2\right)^{\nu-1/2} \mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t, \tag{2.1a}$$

$$\int_{-1}^{1} \left(1-t^2\right)^{\nu-1/2}\,\mathrm{d}t = \frac{\Gamma(1/2)\Gamma(\nu+1/2)}{\Gamma(\nu+1)}, \tag{2.1b}$$

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - \mathrm{e}^{-\mathrm{i}\nu\pi} J_\nu(z)}{\mathrm{i}\sin(\nu\pi)}, \tag{2.1c}$$

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \frac{\mathrm{e}^{\mathrm{i}(z-\nu\pi/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty \mathrm{e}^{-t} t^{\nu-1/2} \left(1 - \frac{t}{2\mathrm{i}z}\right)^{\nu-1/2} \mathrm{d}t. \tag{2.1d}$$

The functions J_ν and $H_\nu^{(1)}$ solve the Bessel equation

$$z^{-1/2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\nu^2-1/4}{z^2} - 1\right) z^{1/2} u(z) = 0. \tag{2.2}$$

We have

$$J_\nu(z) = \mathrm{e}^{\mathrm{i}\nu\pi}\, \overline{J_{\bar\nu}(-\bar z)}, \tag{2.3a}$$

$$H_\nu^{(1)}(z) = \mathrm{e}^{-\mathrm{i}\nu\pi} H_{-\nu}^{(1)}(z) = -\overline{H_{-\bar\nu}^{(1)}(-\bar z)}. \tag{2.3b}$$

2.1. Model operator and construction of model resolvent

Consider

$$H^D = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\nu^2-1/4}{r^2} \quad \text{on } \mathcal{H}^D = L^2\big([1,\infty[\big) \tag{2.4}$$

with Dirichlet boundary condition at $r=1$. Let for any $\zeta \in \mathbb{C}$, $\phi = \phi_\zeta$ be the (unique) solution to

$$\begin{cases} -\phi''(r) + \dfrac{\nu^2-1/4}{r^2}\phi(r) = \zeta\phi(r), \\ \phi(1) = 0, \\ \phi'(1) = 1. \end{cases} \tag{2.5}$$

This solution ϕ_ζ is entire in ζ, and

$$\phi_0(r) = \frac{r^{1/2+\nu} - r^{1/2-\nu}}{2\nu}. \tag{2.6}$$

In fact, cf. [11, (3.6.27)],

$$\phi_{k^2}(r) = \frac{\pi}{2\sin(\nu\pi)} r^{1/2}\big(J_{\bar\nu}(k)J_\nu(kr) - J_\nu(k)J_{\bar\nu}(kr)\big). \tag{2.7}$$

Let for $k \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} k \geqslant 0$ and $H_\nu^{(1)}(k) \neq 0$

$$\phi_k^+(r) = r^{1/2}\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(k)}. \tag{2.8}$$

Due to (2.3b) the dependence of ν in ϕ_k^+ is through ν^2 only, i.e. replacing $\nu \to \bar\nu$ yields the same expression (obviously this is also true for ϕ_{k^2}). Notice also that ϕ_{k^2} and ϕ_k^+ solve the equation

$$-\phi''(r) + \frac{\nu^2 - 1/4}{r^2}\phi(r) = k^2\phi(r). \tag{2.9}$$

The kernel $R_k^D(r,r')$ of $(H^D - k^2)^{-1}$ for k with $\operatorname{Im} k > 0$ and $H_\nu^{(1)}(k) \neq 0$ is given by

$$R_k^D\big(r,r'\big) = \phi_{k^2}(r_<)\phi_k^+(r_>); \tag{2.10}$$

here and henceforth $r_< := \min(r,r')$ and $r_> := \max(r,r')$. (The fact that the right-hand side of (2.10) defines a bounded operator on $\mathcal{H}^D$ follows from the Schur test and the bounds (2.15) and (2.21) given below.) The condition $H_\nu^{(1)}(k) \neq 0$ is fulfilled for $k \in \{\operatorname{Im} k > 0\} \setminus \mathrm{i}\mathbb{R}_+$ since otherwise k^2 would be a non-real eigenvalue of H^D. The zeros in $\mathrm{i}\mathbb{R}_+$ correspond to the negative eigenvalues of H^D. They constitute a sequence accumulating at zero.

We have the properties, cf. (2.3b),

$$R_k^D\big(r,r'\big) = \overline{R_{-\bar k}^D\big(r,r'\big)} = R_k^D\big(r',r\big). \tag{2.11}$$

In the regime where $|k|$ is very small and stays away from the imaginary axis, more precisely in $\Gamma_{\theta,\epsilon}$ for any $\theta \in [0,\pi/2[$ and $\epsilon > 0$, we can derive a lower bound of $|H_\nu^{(1)}(k)|$ as follows: From (2.1a) and (2.1b) we obtain that

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\big(1 + O\big(z^2\big)\big). \tag{2.12}$$

Whence (recall that $\nu = -\mathrm{i}\sigma$ where $\sigma > 0$) we obtain with $C_\nu := |\Gamma(\nu+1)\sin(\nu\pi)|$

$$\begin{aligned}\big|H_\nu^{(1)}(k)\big| &\geqslant \big(\big|\mathrm{e}^{-\sigma\arg k} - \mathrm{e}^{-\sigma\pi}\mathrm{e}^{\sigma\arg k}\big| - O\big(|k|^2\big)\big)/C_\nu \\ &\geqslant \mathrm{e}^{-\sigma\theta}\big(1 - \mathrm{e}^{-\sigma(\pi-2\theta)}\big)/C_\nu - O\big(|k|^2\big) \quad \text{for all } k \in \Gamma_{\theta,\epsilon}.\end{aligned} \tag{2.13}$$

In particular for $\epsilon > 0$ small enough (depending on θ)

$$\forall k \in \Gamma_{\theta,\epsilon}: \quad \left|H_\nu^{(1)}(k)\right| \geqslant \mathrm{e}^{-\sigma\pi/2}\bigl(1 - \mathrm{e}^{-\sigma(\pi-2\theta)}\bigr)/C_\nu. \tag{2.14}$$

Note that the bound (2.14) implies that there is a limiting absorption principle at all real $E = k^2$ with $k \in \Gamma_{\theta,\epsilon}$. In particular H^D does not have small positive eigenvalues.

2.2. Asymptotics of model resolvent

Let us note the following global bound (cf. (2.1d))

$$\left|\phi_k^+(r)\right| \leqslant C\left(\frac{r}{\langle kr\rangle}\right)^{1/2} \mathrm{e}^{-(\operatorname{Im} k)r} \quad \text{for all } k \in \Gamma_{\theta,\epsilon} \text{ and } r \geqslant 1. \tag{2.15}$$

Let

$$D_\nu = 2^{-\nu}/\Gamma(\nu+1). \tag{2.16}$$

Notice that $\bar{D}_\nu = D_{-\nu}$. By (2.1c) and (2.12) we obtain the following asymptotics of ϕ_k^+ as $k \to 0$ in $\Gamma_{\theta,\epsilon}$:

$$\phi_k^+(r) = r^{1/2}\frac{\bar{D}_\nu r^{-\nu}k^{-\nu} - \mathrm{e}^{-\sigma\pi} D_\nu r^\nu k^\nu + O((kr)^2)}{\bar{D}_\nu k^{-\nu} - \mathrm{e}^{-\sigma\pi} D_\nu k^\nu + O(k^2)}. \tag{2.17}$$

Introducing

$$\zeta(k) = \frac{2\mathrm{i}\sigma\mathrm{e}^{-\sigma\pi} D_\nu k^{2\nu}}{\bar{D}_\nu - D_\nu \mathrm{e}^{-\sigma\pi}k^{2\nu}}, \tag{2.18}$$

we can slightly modify (2.17) (in terms of (2.6) and by using (2.15)) as

$$\phi_k^+(r) = r^{1/2-\nu} + \zeta(k)\phi_0(r) + r^{1/2}O\bigl((kr)^2\bigr) + \left(\frac{r}{\langle kr\rangle}\right)^{1/2}\mathrm{e}^{-(\operatorname{Im} k)r}O\bigl(k^2\bigr). \tag{2.19}$$

There is a "global" bound of the third term (due to (2.15)):

$$\left|r^{1/2}O\bigl((kr)^2\bigr)\right| \leqslant Cr^{1/2}\frac{|kr|^2}{\langle kr\rangle^2} \quad \text{for all } k \in \Gamma_{\theta,\epsilon} \text{ and } r \geqslant 1. \tag{2.20}$$

As for ϕ_{k^2} we first note the following global bound (cf. (2.1a), (2.7) and [9, Theorem 4.6.1])

$$\left|\phi_{k^2}(r)\right| \leqslant C\left(\frac{r}{\langle kr\rangle}\right)^{1/2}\mathrm{e}^{(\operatorname{Im} k)r} \quad \text{for all } k \in \Gamma_{\theta,\epsilon} \text{ and } r \geqslant 1. \tag{2.21}$$

Using (2.21) we obtain similarly

$$\phi_{k^2}(r) = \phi_0(r) + r^{1/2}O\bigl((kr)^2\bigr) + \left(\frac{r}{\langle kr\rangle}\right)^{1/2}\mathrm{e}^{(\operatorname{Im} k)r}O\bigl(k^2\bigr). \tag{2.22}$$

There is a global bound of the second term:

$$\left|r^{1/2}O\big((kr)^2\big)\right| \leqslant Cr^{1/2}\frac{|kr|^2}{\langle kr\rangle^2}\mathrm{e}^{(\operatorname{Im} k)r} \quad \text{for all } k \in \Gamma_{\theta,\epsilon} \text{ and } r \geqslant 1. \tag{2.23}$$

Whence in combination with (2.10) we obtain uniformly in $k \in \Gamma_{\theta,\epsilon}$ and $r, r' \geqslant 1$

$$R_k^D\big(r,r'\big) = R_0^D\big(r,r'\big) + \zeta(k)T\big(r,r'\big) + r^{1/2}\big(r'\big)^{1/2}E_k\big(r,r'\big); \tag{2.24a}$$

$$R_0^D\big(r,r'\big) = \phi_0(r_<)r_>^{1/2-\nu}, \tag{2.24b}$$

$$T\big(r,r'\big) = \phi_0(r)\phi_0\big(r'\big), \tag{2.24c}$$

$$\left|E_k\big(r,r'\big)\right| \leqslant C\left(\frac{|k|r_>}{\langle kr_>\rangle}\right)^2. \tag{2.24d}$$

Clearly $T = |\phi_0\rangle\langle\phi_0|$ is a rank-one operator and the function ζ has a non-trivial oscillatory behaviour. The error estimate can be replaced by:

$$\exists C > 0\ \forall \delta \in [0,2]\colon \quad \left|E_k\big(r,r'\big)\right| \leqslant C|kr_>|^\delta \quad \text{for all } k \in \Gamma_{\theta,\epsilon} \text{ and } r, r' \geqslant 1. \tag{2.25}$$

In particular introducing weighted spaces

$$\mathcal{H}_s^D = \langle r\rangle^{-s}\mathcal{H}^D,$$

we obtain

$$\forall s > 1\colon \quad \lim_{\Gamma_{\theta,\epsilon}\ni k\to 0}\left\|R_k^D - R_0^D - \zeta(k)T\right\|_{\mathcal{B}(\mathcal{H}_s^D,\mathcal{H}_{-s}^D)} = 0. \tag{2.26}$$

In fact we deduce from (2.24a)–(2.24d) the following more precise result:

Lemma 2.1. *For all $s > s' > 1$, $s' \leqslant 3$, there exists $C > 0$:*

$$\left\|\big(H^D - \mathrm{i}\big)\big(R_k^D - R_0^D - \zeta(k)T\big)\right\|_{\mathcal{B}(\mathcal{H}_s^D,\mathcal{H}_{-s}^D)} \leqslant C|k|^{s'-1} \quad \textit{for all } k \in \Gamma_{\theta,\epsilon}. \tag{2.27}$$

3. Asymptotics for full Hamiltonian, compactly supported perturbation

Consider with $V_\infty(r) := \frac{\nu^2-1/4}{r^2}\chi(r>1)$

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_\infty + V \quad \text{on } \mathcal{H} := L^2\big(]0,\infty[\big) \tag{3.1}$$

with Dirichlet boundary condition at $r = 0$. As for the potential V we impose in this section

Condition 3.1.

1) $V \in C(]0,\infty[,\mathbb{R})$,
2) $\exists R > 3$: $V(r) = 0$ for $r \geqslant R$,
3) $\exists C_1, C_2 > 0\ \exists \kappa > 0$: $C_1(r^{-2}+1) \geqslant V(r) \geqslant (\kappa^2 - 1/4)r^{-2} - C_2$.

Notice that the operator H is defined in terms of the (closed) Dirichlet form on the Sobolev space $H_0^1(\mathbb{R}_+)$ (i.e. H is the Friedrichs extension), cf. [3, Lemma 5.3.1]. For the limiting cases $C_1 = \infty$ and/or $\kappa = 0$ in 3) it is still possible to define H as the Friedrichs extension of the action on $C_c^\infty(\mathbb{R}_+)$ however the form domain of the extension might be different from $H_0^1(\mathbb{R}_+)$ and some arguments of this paper would be more complicated. An example of this type (with $\kappa = 0$) is discussed in Appendix B. If $V(r) \geqslant 3/4r^{-2} - C$ the operator H is essentially self-adjoint on $C_c^\infty(\mathbb{R}_+)$, cf. [10, Theorem X.10].

In terms of the resolvent R_k^D considered in Section 2 and cutoffs $\chi_1 = \chi_1(r < 7)$ and $\chi_2 = \chi_2(r > 7)$ we introduce for $k \in \Gamma_{\theta,\epsilon}$

$$G_k = \chi_1\left(H - \frac{\operatorname{Re} k}{|\operatorname{Re} k|}\mathrm{i}\right)^{-1}\chi_1 + \chi_2 R_k^D \chi_2. \tag{3.2}$$

Let

$$G_0^\pm = \chi_1(H \mp \mathrm{i})^{-1}\chi_1 + \chi_2 R_0^D \chi_2 \tag{3.3}$$

and

$$K^\pm = H G_0^\pm - I. \tag{3.4}$$

Notice that the operators $K^\pm$ are compact on $\mathcal{H}_s := \langle r\rangle^{-s}\mathcal{H}$ for $s > 1$.

Due to Lemma 2.1 we have the following expansions in $\mathcal{B}(\mathcal{H}_s)$ (with $s > s' > 1$ and $s' \leqslant 3$)

$$\forall k \in \Gamma_{\theta,\epsilon}^\pm\colon \quad \left(H - k^2\right)G_k = I + K^\pm + \zeta(k)|\psi_0\rangle\langle\chi_2\phi_0| + O\left(|k|^{s'-1}\right); \quad \psi_0 := H\chi_2\phi_0. \tag{3.5}$$

Lemma 3.2. *For all $k \in \Gamma_{\theta,\epsilon}^\pm$ the following form inequality holds* (*on $\mathcal{H}_s$ for any $s > 1$*)

$$\pm \operatorname{Im} G_k \geqslant \chi_1(H \pm \mathrm{i})^{-1}(H \mp \mathrm{i})^{-1}\chi_1. \tag{3.6}$$

Proof. This is obvious from the fact that $\pm \operatorname{Im} R_k^D \geqslant 0$. □

Proposition 3.3. *For all $s > 1$ the operators $I + K^\pm \in \mathcal{B}(\mathcal{H}_s)$ have zero null space, i.e.*

$$\operatorname{Ker}\left(I + K^\pm\right) = \{0\}. \tag{3.7}$$

Proof. We prove only (3.7) for the superscript "+ case". The "− case" is similar. Suppose $0 = HG_0^+ f$ for some $f \in \mathcal{H}_s$. We shall show that $f = 0$. Let $u_0 = G_0^+ f$. Integrating by parts yields

$$\begin{aligned} 0 = \operatorname{Im}\langle u_0, -Hu_0\rangle &= \lim_{r\to\infty} \operatorname{Im}\left(\bar u_0 u_0'\right)(r) \\ &= \lim_{r\to\infty} \operatorname{Im}\left((1/2-\nu)|u_0|^2(r)/r\right) = \sigma\left|\langle\chi_2\phi_0, f\rangle\right|^2. \end{aligned} \tag{3.8}$$

So

$$\langle\chi_2\phi_0, f\rangle = 0, \tag{3.9}$$

and therefore (seen again by using the explicit kernel of R_0^D and by estimating by the Cauchy–Schwarz inequality)

$$u_0 = O\big(r^{3/2-s}\big) \quad \text{and} \quad u_0' = O\big(r^{1/2-s}\big) \quad \text{for } r \to \infty. \tag{3.10}$$

From (3.10) we can conclude that

$$u_0 = 0; \tag{3.11}$$

this can be seen by writing u_0 as a linear combination of $r^{1/2+\nu}$ and $r^{1/2-\nu}$ at infinity, deduce that u_0 vanishes at infinity and then invoke unique continuation. For a more general result (with detailed proof) see Lemma 4.2.

Using Lemmas 2.1 and 3.2, (3.9) and (3.11) we compute

$$0 = \operatorname{Im}\langle f, u_0\rangle = \lim_{\Gamma^+_{\theta,\epsilon} \ni k \to 0} \operatorname{Im}\langle f, G_k f\rangle \geqslant \big\|(H-\mathrm{i})^{-1}\chi_1 f\big\|^2. \tag{3.12}$$

We conclude that

$$\chi_1 f = 0. \tag{3.13}$$

So $0 = G_0^+ f = \chi_2 R_0^D \chi_2 f$, and therefore

$$R_0^D \chi_2 f = 0 \quad \text{on } \operatorname{supp}(\chi_2). \tag{3.14}$$

We apply H^D to (3.14) and conclude that

$$\chi_2 f = 0, \tag{3.15}$$

so indeed $f = 0$. □

3.1. Construction of resolvent

Due to Proposition 3.3 we can write, cf. (3.5),

$$\big(H - k^2\big) G_k \big(I + K^\pm\big)^{-1} = I + \zeta(k)|\psi_0\rangle\langle\phi^\pm| + O\big(|k|^{s'-1}\big), \tag{3.16}$$

for $k \in \Gamma^\pm_{\theta,\epsilon}$, where

$$\phi^\pm := \big(\big(I + K^\pm\big)^{-1}\big)^* \chi_2 \phi_0. \tag{3.17}$$

We have

$$\big(I + \zeta(k)|\psi_0\rangle\langle\phi^\pm|\big)^{-1} = I - \frac{\zeta(k)}{\eta^\pm(k)}|\psi_0\rangle\langle\phi^\pm|; \quad \eta^\pm(k) := 1 + \zeta(k)\langle\phi^\pm, \psi_0\rangle. \tag{3.18}$$

Of course this is under the condition that

$$\eta^{\pm}(k) \neq 0. \tag{3.19}$$

Lemma 3.4. *For all $k \in \Gamma^{\pm}_{\theta,\epsilon}$ the condition* (3.19) *is fulfilled.*

Proof. Let us prove (3.19) for the superscript "+ case". The "− case" is similar.

Suppose on the contrary that $\eta^{+}(k) = 0$ for some $k \in \Gamma^{+}_{\theta,\epsilon}$. Then

$$k^{2\nu} = \frac{\bar{D}_\nu}{D_\nu} \frac{\mathrm{e}^{\sigma\pi}}{1 - 2\mathrm{i}\sigma\langle\phi^{+}, \psi_0\rangle}. \tag{3.20}$$

$k^{2\nu}$ is oscillatory, the set of all solutions of (3.20) in $\Gamma^{+}_{\theta,\epsilon}$ constitutes a sequence converging to zero. In particular we can pick a sequence $\Gamma^{+}_{\theta,\epsilon} \ni k_n \to 0$ with $0 \neq \eta^{+}(k_n) \to 0$. We apply (3.16) and (3.18) to this sequence $\{k_n\}$. Substituting (3.18) into (3.16) and multiplying the equation obtained by $\eta(k_n)$, we get

$$\big(H - k_n^2\big) G_{k_n} \big(I + K^{+}\big)^{-1} \big(\eta^{+}(k_n) - \zeta(k_n)|\psi_0\rangle\langle\phi^{+}|\big) = \eta^{+}(k_n)\big(1 + +O\big(|k_n|^{s'-1}\big)\big).$$

Taking the limit $n \to \infty$, this leads to

$$-\zeta(\infty) H G^{+}_{\infty} \big(I + K^{+}\big)^{-1} |\psi_0\rangle\langle\phi^{+}| = 0. \tag{3.21}$$

Here $\zeta(\infty) := \lim_{n\to\infty} \zeta(k_n)$ can be computed by substituting $k^{2\nu}$ given by (3.20) in the expression for $\zeta(k)$ (this is the limit and one sees that it is nonzero), and similarly for $G^{+}_{\infty} := \lim_{n\to\infty} G_{k_n}$. We learn that

$$Hu^{+} = 0; \quad u^{+} := G^{+}_{\infty} f^{+}, \ f^{+} := \big(I + K^{+}\big)^{-1} \psi_0. \tag{3.22}$$

Now, the argument of integration by parts used in (3.8) applied to u^{+} leads to

$$0 = \sigma\left(\left|1 - \frac{\zeta(\infty)}{2\nu}\right|^2 - \left|\frac{\zeta(\infty)}{2\nu}\right|^2\right) \big|\langle\chi_2\phi_0, f^{+}\rangle\big|^2. \tag{3.23}$$

We claim that

$$\langle\chi_2\phi_0, f^{+}\rangle = 0. \tag{3.24}$$

In fact for any $k \in \Gamma^{+}_{\theta,\epsilon}$ obeying (3.20),

$$\left|1 - \frac{\zeta(\infty)}{2\nu}\right|^2 \Big/ \left|\frac{\zeta(\infty)}{2\nu}\right|^2 = \big|\mathrm{e}^{\sigma\pi} k^{-2\nu}\big|^2 = \mathrm{e}^{2\sigma(\pi - 2\arg k)} > 1, \tag{3.25}$$

whence indeed (3.24) follows from (3.23).

Using (3.24) we can mimic the rest of the proof of Proposition 3.3 and eventually conclude that $f^{+} = 0$. This is a contradiction since $\psi_0 \neq 0$. □

Combining (3.16)–(3.19) we obtain (possibly by taking $\epsilon > 0$ smaller)

$$\forall k \in \Gamma^{\pm}_{\theta,\epsilon}: \quad (H-k^2)G_k(I+K^{\pm})^{-1}\left(I - \frac{\zeta(k)}{\eta^{\pm}(k)}|\psi_0\rangle\langle\phi^{\pm}|\right)\left(I + O(|k|^{s'-1})\right) = I. \tag{3.26}$$

In particular we have derived a formula for the resolvent.

3.2. Asymptotics of resolvent

Let u be any nonzero *regular* solution to the equation

$$-u''(r) + \left(V_\infty(r) + V(r)\right)u(r) = 0. \tag{3.27}$$

By regular solution, we mean that the function $r \to \chi(r<1)u(r)$ belongs to $\mathcal{D}(H)$. It will be shown in Appendix A that the regular solution u is fixed up to a constant and can be chosen real-valued. See (3.33c) for a formula and for further elaboration. Let

$$R(k) := \left(H - k^2\right)^{-1} \quad \text{for all } k \in \Gamma^{\pm}_{\theta,\epsilon} \cap \{\operatorname{Im} k > 0\}. \tag{3.28}$$

Theorem 3.5. *There exist* (*finite*) *rational functions $f^{\pm}$ in the variable $k^{2\nu}$ for $k \in \Gamma^{\pm}_{\theta,\epsilon}$ for which*

$$\lim_{\Gamma^{\pm}_{\theta,\epsilon} \ni k \to 0} \operatorname{Im} f^{\pm}\left(k^{2\nu}\right) \textit{ do not exist,} \tag{3.29}$$

there exist Green's functions for H at zero energy, denoted by $R^{\pm}_0$, and there exists a real nonzero regular solution to (3.27), *denoted by u, such that the following asymptotics hold. For all $s > s' > 1$, $s' \leqslant 3$, there exists $C > 0$:*

$$\begin{aligned} &\forall k \in \Gamma^{\pm}_{\theta,\epsilon} \cap \{\operatorname{Im} k > 0\}: \\ &\quad \left\|(H - \mathrm{i})\left(R(k) - R^{\pm}_0 - f^{\pm}\left(k^{2\nu}\right)|u\rangle\langle u|\right)\right\|_{\mathcal{B}(\mathcal{H}_s,\mathcal{H}_{-s})} \leqslant C|k|^{s'-1}. \end{aligned} \tag{3.30}$$

Here

$$\forall s > 1: \quad (H-\mathrm{i})R^{\pm}_0 = I - \mathrm{i}R^{\pm}_0 \in \mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s}) \quad \textit{and} \quad (H-\mathrm{i})u = -\mathrm{i}u \in \mathcal{H}_{-s}. \tag{3.31}$$

Proof. By (3.26)

$$R(k) = G_k\left(I+K^{\pm}\right)^{-1}\left(I - \frac{\zeta(k)}{\eta^{\pm}(k)}|\psi_0\rangle\langle\phi^{\pm}|\right)\left(I + O\left(|k|^{s'-1}\right)\right) \tag{3.32}$$

for $k \in \Gamma^{\pm}_{\theta,\epsilon}$. We expand the product yielding up to errors of order $O(|k|^{s'-1})$

$$R(k) \approx R_0^\pm + \frac{\zeta(k)}{\eta^\pm(k)} |u_1^\pm\rangle\langle u_2^\pm|; \quad \text{where} \tag{3.33a}$$

$$R_0^\pm = G_0^\pm (I + K^\pm)^{-1}, \tag{3.33b}$$

$$u_1^\pm = -R_0^\pm \psi_0 + \chi_2 \phi_0, \tag{3.33c}$$

$$u_2^\pm = \phi^\pm = \big((I + K^\pm)^{-1}\big)^* \chi_2 \phi_0. \tag{3.33d}$$

Clearly, $u_2^\pm \neq 0$. According to (3.5), $Hu_1^\pm = -\psi_0 + H(\chi_2\phi_0) = 0$. In addition, $u_1^\pm \neq 0$. In fact for $r > 14$ (ensuring that $\chi_2(r) = 1$) one has

$$u_1^\pm = -R_0^D f^\pm + \phi_0, \quad \text{with } f^\pm = \chi_2 (1 + K^\pm)^{-1} \psi_0 \in \mathcal{H}_s,\ s > 1.$$

Using then (2.24b) and (2.6) we compute

$$r^{-1/2-\nu}\left(r\frac{\mathrm{d}}{\mathrm{d}r} - (1/2 - \nu)\right) u_1^\pm(r) = 1 - \int_r^\infty \tau^{\frac{1}{2}-\nu} f^\pm(\tau)\,\mathrm{d}\tau,$$

showing that $u_1^\pm(r) \neq 0$ for all r large enough. By the uniqueness of regular solutions, there exist constants $b_\pm \neq 0$ such that $u_1^\pm = b^\pm u$, where u is a real-valued nonzero regular solution to (3.27). Combining the duality relation $R(k)^* = R(\bar{k})$ and (3.33a), we obtain that

$$u_2^\pm = c^\pm u_1^\mp = c^\pm b^\mp u \quad \text{for some constants } c^\pm \neq 0. \tag{3.34}$$

Whence indeed (3.30) holds with

$$R_0^\pm = G_0^\pm (I + K^\pm)^{-1} \quad \text{and} \quad f^\pm(k^{2\nu}) = C^\pm \frac{\zeta(k)}{\eta^\pm(k)}, \tag{3.35}$$

where the constants $C^\pm = c^\pm b^\mp \overline{b^\pm}$ are nonzero. Whence indeed (3.29) holds. The properties (3.31) follow from the expressions (3.33b) and (3.33c). □

Corollary 3.6. *There is a limiting absorption principle at energies in* $]0, \epsilon^2]$:

$$\forall k' \in [-\epsilon, \epsilon] \setminus \{0\}\ \forall s > 1: \quad R(k') := \lim_{\Gamma_{\theta,\epsilon} \cap \{\operatorname{Im} k > 0\} \ni k \to k'} R(k) \text{ exists in } \mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s}). \tag{3.36}$$

In particular

$$]0, \epsilon^2] \cap \sigma_{\mathrm{pp}}(H) = \emptyset. \tag{3.37}$$

Moreover the bounds (3.30) *extend to* $\Gamma_{\theta,\epsilon}^\pm$.

Introducing the spectral density as an operator in $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$, $s > 1$,

$$\delta(H - k^2) := \frac{R(k) - R(-k)}{2\pi \mathrm{i}} \quad \text{for } 0 < k \leqslant \epsilon,$$

we have

$$\lim_{k \searrow 0} \delta\big(H - k^2\big) \text{ does not exist.} \tag{3.38}$$

Proof. Only (3.38) needs a comment: We represent $R(-k) = R(k)^*$ and use (3.30) yielding

$$\delta\big(H - k^2\big) \approx (2\pi \mathrm{i})^{-1}\big(R_0^+ - \big(R_0^+\big)^*\big) + \frac{\operatorname{Im} f^+(k^{2\nu})}{\pi} |u\rangle\langle u|.$$

The right-hand side does not converge, cf. (3.29). □

3.3. d-dimensional Schrödinger operator

As another application of Theorem 3.5, we consider a d-dimensional Schrödinger operator with spherically symmetric potential of the form

$$H = -\Delta + W\big(|x|\big)$$

in $L^2(\mathbb{R}^d)$, $d \geqslant 2$, where W is continuous and $W(|x|) = -\frac{\gamma}{|x|^2}$ for x outside some compact set and $\gamma > (\frac{d}{2} - 1)^2$. Assume that

$$\gamma \neq \left(l + \frac{d}{2} - 1\right)^2, \quad l \in \mathbb{N}. \tag{3.39}$$

Denote $\mathbb{N}_\gamma = \{l \in \mathbb{N} \cup \{0\} \mid (l + \frac{d}{2} - 1)^2 < \gamma\}$. Let π_l denote the spectral projection associated to the eigenvalue $l(l + d - 2)$, $l \in \mathbb{N} \cup \{0\}$, of the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ (and also its natural extension as operator on $\mathbb{H} = L^2(\mathbb{R}^d)$). Then H can be decomposed into a direct sum

$$H = \bigoplus_{l=0}^{\infty} \widetilde{H}_l \pi_l,$$

where

$$\widetilde{H}_l = -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + \frac{l(l+d-2)}{r^2} + W(r)$$

on $\widetilde{\mathcal{H}} := L^2(\mathbb{R}_+; r^{d-1}dr)$. When $l \in \mathbb{N}_\gamma$, we can apply Theorem 3.5 with $\nu = \nu_l$, $\nu_l^2 = (l + \frac{d}{2} - 1)^2 - \gamma < 0$, to expand the resolvent $(\widetilde{H}_l - k^2)^{-1}$ up to $O(|k|^\epsilon)$ (see Section 6 for a relevant reduction of $\widetilde{H}_l$ used here). For $l \notin \mathbb{N}_\gamma$, the resolvent $(\widetilde{H}_l - k^2)^{-1}$ may have singularities at zero, according to whether zero is an eigenvalue and/or a resonance of $\widetilde{H}_l$ (defined below).

Denote $\widetilde{\mathcal{H}}_s = \langle r\rangle^{-s}\widetilde{\mathcal{H}}$ and $\mathbb{H}_s = \langle x\rangle^{-s}\mathbb{H}$, $s \in \mathbb{R}$. Under the condition (3.39), we say that 0 is a *resonance* of H if there exists $u \in \mathbb{H}_{-1} \setminus \mathbb{H}$ such that $Hu = 0$. We call such function u a *resonance function*. (If the condition (3.39) is not satisfied, the definition of zero resonance has to be modified.) The number 0 is called a *regular point* of H if it is neither an eigenvalue nor a resonance of H. The same definitions apply for $\widetilde{H}_l$ on $\widetilde{\mathcal{H}}$. Clearly Lemma 4.2 stated below shows that for any resonance function u necessarily $\pi_l u = 0$ for all $l \in \mathbb{N}_\gamma$. In fact Lemma 4.2 shows that 0 is a regular point of $\widetilde{H}_l$, $l \in \mathbb{N}_\gamma$.

If $Hu = 0$ and $u \in \mathbb{H}_{-1}$, then by expanding u in spherical harmonics, one can show that (cf. Theorem 4.1 of [13])

$$u(r\theta) = \frac{\psi(\theta)}{r^{\frac{d-2}{2}+\mu}} + v, \tag{3.40a}$$

where $v \in L^2(|x| > 1)$,

$$\mu = \sqrt{\left(m + \frac{d}{2} - 1\right)^2 - \gamma}, \quad m = \min \mathbb{N} \setminus \mathbb{N}_\gamma, \tag{3.40b}$$

$$\psi(\theta) = \sum_{j=1}^{n_\mu} -\frac{1}{2\mu}\left\langle\left(W + \frac{\gamma}{|y|^2}\right)u, |y|^{-\frac{d-2}{2}+\mu}\varphi_\mu^{(j)}\right\rangle\varphi_\mu^{(j)}(\theta). \tag{3.40c}$$

Here $\{\varphi_\mu^{(j)},\ 1 \leqslant j \leqslant n_\mu\}$ is an orthonormal basis of the eigenspace of $-\Delta_{\mathbb{S}^{d-1}}$ with eigenvalue $m(m+d-2)$ and n_μ its multiplicity (cf. [12]):

$$n_\mu = \frac{(m+d-3)!}{(d-2)!(m-1)!} + \frac{(m+d-2)!}{(d-2)!m!}. \tag{3.40d}$$

The expansion (3.40a) implies that a solution u to $Hu = 0$ with $u \in \mathbb{H}_{-1}$ is a resonance function of H if and only if $\mu \in]0,1]$ and $\psi \neq 0$ and that if zero is a resonance, its multiplicity (cf. [6,13] for the definition) is at most n_μ. Conversely, if the equation $Hu = 0$ has a solution $u \in \mathbb{H}_{-1} \setminus \mathbb{H}$, then the equation

$$\widetilde{H}_m g = 0$$

has a nonzero regular solution $g \in \widetilde{\mathcal{H}}_{-1}$ decaying like $1/(r^{\frac{d-2}{2}+\mu})$ at infinity. It follows that $u_j = g \otimes \varphi_\mu^{(j)}$, $1 \leqslant j \leqslant n_\mu$, are all resonance functions of H. This proves that if 0 is a resonance of H its multiplicity is equal to n_μ.

Now let us come back to the asymptotics of the resolvent $R(k) = (H - k^2)^{-1}$ near 0. If 0 is a regular point of H (this is a generic condition and concerns by the discussion above only sectors $\widetilde{H}_l$ with $l \notin \mathbb{N}_\gamma$), then it is a regular point for all $\widetilde{H}_l$ with $l \notin \mathbb{N}_\gamma$. One deduces easily that there exists $R_0^{(l)} \in \mathcal{B}(\widetilde{\mathcal{H}}_s, \widetilde{\mathcal{H}}_{-s})$ for all $s > 1$, such that for any such s there exists $\epsilon > 0$:

$$\left(\widetilde{H}_l - k^2\right)^{-1} = R_0^{(l)} + O_l\left(|k|^\epsilon\right) \quad \text{in } \mathcal{B}(\widetilde{\mathcal{H}}_s, \widetilde{\mathcal{H}}_{-s}) \text{ for } |k| \text{ small and } k^2 \notin [0,\infty[. \tag{3.41}$$

The error term can be uniformly estimated in l as in [13], yielding an expansion for $R(k)$. If 0 is a resonance but not an eigenvalue of H, then 0 is a regular point for all $\widetilde{H}_l$ with $l \notin \mathbb{N}_\gamma \cup \{m\}$ and the expansion (3.41) remains valid for such l. When $l = m$, $(\widetilde{H}_m - k^2)^{-1}$ contains a singularity at 0 which can be calculated as in [14]. Let

$$k_\mu = \begin{cases} k^{2\mu}, & \text{if } \mu \in]0,1[, \\ k^2 \ln(k^2), & \text{if } \mu = 1. \end{cases} \tag{3.42}$$

Then there exist $g \in \widetilde{\mathcal{H}}_{-1} \setminus \widetilde{\mathcal{H}}$ verifying $\widetilde{H}_m g = 0$, a rank-one operator-valued entire function $\zeta \to F_m(\zeta) \in \mathcal{B}(\widetilde{\mathcal{H}}_s, \widetilde{\mathcal{H}}_{-s})$, $s > 1$, verifying $F_m(0) = 0$ and $R_0^{(m)} \in \mathcal{B}(\widetilde{\mathcal{H}}_s, \widetilde{\mathcal{H}}_{-s})$, $s > 3$, such that for any $s > 3$

$$\big(\widetilde{H}_m - k^2\big)^{-1} = \frac{e^{i\mu'\pi}}{k_\mu} |g\rangle\langle g| + \frac{1}{k_\mu} F_m\left(\frac{k^2}{k_\mu}\right) + R_0^{(m)} + O\left(\frac{|k|^2}{|k_\mu|}\right) \quad \text{in } \mathcal{B}(\widetilde{\mathcal{H}}_s, \widetilde{\mathcal{H}}_{-s}), \tag{3.43}$$

where μ' is the fractional part of μ: $\mu' = \mu$ if $\mu \in]0, 1[$ and $\mu' = 0$ if $\mu = 1$. Note that the sign "−" is missing in the constant c_1 corresponding to $\mu = 1$ given in (4.19) of [14]. In particular if $\mu \in]0, \frac{1}{2}]$ one has

$$\big(\widetilde{H}_m - k^2\big)^{-1} = \frac{e^{i\mu\pi}}{k_\mu} |g\rangle\langle g| + R_0^{(m)} + O\big(|k|\big) \quad \text{in } \mathcal{B}(\widetilde{\mathcal{H}}_s, \widetilde{\mathcal{H}}_{-s}),\ s > 3,$$

while in the "worse case", $\mu = 1$, the error term in (3.43) is of order $O(|\ln k|^{-1})$.

Summing up we have proved the following

Theorem 3.7. *Assume that $W(|x|)$ is continuous and $W(|x|) = -\frac{\gamma}{|x|^2}$ outside some compact set with $\gamma > (\frac{d}{2} - 1)^2$ satisfying* (3.39).

i) *Suppose that zero is a regular point of H. Then there exist $R_0^{\pm} \in \mathcal{B}(\mathbb{H}_s, \mathbb{H}_{-s})$ and $v_l \in \widetilde{\mathcal{H}}_{-s} \setminus \{0\}$ for all $s > 1$ and $l \in \mathbb{N}_\gamma$, such that for any $s > 1$ there exists $\epsilon > 0$:*

$$R(k) = \sum_{l \in \mathbb{N}_\gamma} f_l^{\pm}\big(k^{2\nu_l}\big)\big(|v_l\rangle\langle v_l|\big) \otimes \pi_l + R_0^{\pm} + O\big(|k|^{\epsilon}\big) \quad \text{in } \mathcal{B}(\mathbb{H}_s, \mathbb{H}_{-s}) \text{ for } k \in \Gamma_\theta^{\pm}. \tag{3.44}$$

Here $f_l^{\pm}(k^{2\nu_l})$ are the oscillatory functions given in Theorem 3.5 *with $\nu = \nu_l = -i\sqrt{\gamma - (l + \frac{d}{2} - 1)^2}$, $l \in \mathbb{N}_\gamma$.*

ii) *Suppose that zero is a resonance of H. Let m and μ be defined by* (3.40b). *Then $\mu \in]0, 1]$ and the multiplicity of the zero resonance of H is equal to*

$$\frac{(m + d - 3)!}{(d - 2)!(m - 1)!} + \frac{(m + d - 2)!}{(d - 2)!m!}.$$

Suppose in addition that zero is not an eigenvalue of H. Then there exist $g \in \widetilde{\mathcal{H}}_{-1} \setminus \widetilde{\mathcal{H}}$ with $\widetilde{H}_m g = 0$, a rank-one operator-valued analytic function $\zeta \to F_m(\zeta) \in \mathcal{B}(\widetilde{\mathcal{H}}_s, \widetilde{\mathcal{H}}_{-s})$, $s > 1$, defined for ζ near 0 verifying $F_m(0) = 0$, and $R_1^{\pm} \in \mathcal{B}(\mathbb{H}_s, \mathbb{H}_{-s})$, $s > 3$, such that for any $s > 3$

$$R(k) = \left(\frac{e^{i\mu'\pi}}{k_\mu} |g\rangle\langle g| + \frac{1}{k_\mu} F_m\left(\frac{k^2}{k_\mu}\right)\right) \otimes \pi_m + \sum_{l \in \mathbb{N}_\gamma} f_l^{\pm}\big(k^{2\nu_l}\big)\big(|v_l\rangle\langle v_l|\big) \otimes \pi_l + R_1^{\pm} + O\big(|\ln k|^{-1}\big) \quad \text{in } \mathcal{B}(\mathbb{H}_s, \mathbb{H}_{-s}) \text{ for } k \in \Gamma_\theta^{\pm}. \tag{3.45}$$

Here $f_l^{\pm}$ and v_l are the same as in i).

The case that 0 is an eigenvalue of H can be studied in a similar way. The zero eigenfunctions of H may have several angular momenta $l > m$ and the asymptotics of $R(k)$ up to $o(1)$ as $k \to 0$ contains many terms and we do not give details here. Note that if (3.39) is not satisfied and $\gamma = (l + \frac{d}{2} - 1)^2$ for some $l \in \mathbb{N} \cup \{0\}$, $(\widetilde{H}_l - k^2)^{-1}$ may contain a term of the order $\ln k$ as $k \to 0$.

4. Asymptotics for full Hamiltonian, more general perturbation

We shall "solve" the equation

$$-u''(r) + \big(V_\infty(r) + V(r)\big)u(r) = 0 \tag{4.1}$$

on the interval $I =]0, \infty[$ for a class of potentials V with faster decay than V_∞ at infinity (recall $V_\infty(r) = \frac{\nu^2 - 1/4}{r^2}\chi(r > 1)$). In particular we shall show absence of zero eigenvalue for a more general class of perturbations than prescribed by Condition 3.1. Explicitly we keep Conditions 3.1 1) and 3) but modify Condition 3.1 2) as

2)′ $V(r) = O(r^{-2-\epsilon})$, $\epsilon > 0$.

This means that we now impose

Condition 4.1.

1) $V \in C(]0, \infty[, \mathbb{R})$,
2) $V(r) = O(r^{-2-\epsilon})$, $\epsilon > 0$,
3) $\exists C_1, C_2 > 0$ $\exists \kappa > 0$: $C_1(r^{-2} + 1) \geqslant V(r) \geqslant (\kappa^2 - 1/4)r^{-2} - C_2$.

Lemma 4.2. *Under Condition* 4.1 *suppose u is a distributional solution to* (4.1) *obeying one of the following two conditions*:

1) $u \in L^2_{-1}$ *(at infinity)*.
2) $u(r)/\sqrt{r} \to 0$ *and* $u'(r)\sqrt{r} \to 0$ *for* $r \to \infty$.

Then

$$u = 0. \tag{4.2}$$

Proof. Let $\phi^\pm(r) = r^{1/2\pm\nu}$. Then $\phi^\pm$ are linear independent solutions to the equation

$$-u''(r) + V_\infty(r)u(r) = 0; \quad r > 2. \tag{4.3}$$

First we shall show that

$$u = O\big(r^{1/2-\epsilon}\big) \quad \text{and} \quad u' = O\big(r^{-1/2-\epsilon}\big). \tag{4.4}$$

Note that under the condition 1) in fact $u' \in L^2$ (at infinity) due to a standard ellipticity argument.

We shall apply the method of variation of parameters. Specifically, introduce "coefficients" a_2^+ and a_2^- of the ansatz

$$u = a^+\phi^+ + a^-\phi^-. \tag{4.5}$$

Using the differential equations for a^+ and a^- we shall derive estimates of these quantities.

The equations read

$$\begin{pmatrix} \phi^+ & \phi^- \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^+ & \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^- \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = V\begin{pmatrix} 0 & 0 \\ \phi^+ & \phi^- \end{pmatrix}\begin{pmatrix} a^+ \\ a^- \end{pmatrix}. \tag{4.6}$$

Note that the Wronskian $W(\phi^-,\phi^+) = \phi^- \frac{\mathrm{d}}{\mathrm{d}r}\phi^+ - \phi^+\frac{\mathrm{d}}{\mathrm{d}r}\phi^- = 2\nu$. (4.6) can be transformed into

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = N\begin{pmatrix} a^+ \\ a^- \end{pmatrix},$$

where

$$N = \frac{V}{2\nu}\begin{pmatrix} \phi^-\phi^+ & (\phi^-)^2 \\ -(\phi^+)^2 & -\phi^-\phi^+ \end{pmatrix}.$$

Clearly for V obeying Condition 4.1 the quantity $N = O(r^{-1-\epsilon})$ and whence it can be integrated to infinity. Whence there exist

$$a^\pm(\infty) = \lim_{r\to\infty} a^\pm(r);$$

in fact

$$a^\pm(\infty) - a^\pm(r) = O\big(r^{-\epsilon}\big). \tag{4.7}$$

We need to show that

$$a^\pm(\infty) = 0. \tag{4.8}$$

Note that

$$\begin{pmatrix} \phi^+ & \phi^- \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^+ & \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^- \end{pmatrix}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = \begin{pmatrix} u \\ u' \end{pmatrix}. \tag{4.9}$$

We solve for (a^+, a^-) and multiply the result by $r^{-1/2}$. Under the condition 1) each component of the right-hand side of the resulting equation is in L^2. Whence also $a^\pm(\infty)/r^{1/2} \in L^2$ and (4.8) and therefore (4.4) follow. We argue similarly under the condition 2).

To show (4.2) note that the considerations preceding (4.8) hold for all solutions distributional u (not only a solution u obeying 1) or 2)) yielding without 1) nor 2) the bounds (4.4) with $\epsilon = 0$. In particular for a solution $\tilde{u}$ with $W(u,\tilde{u}) = 1$ (assuming conversely that $u \neq 0$) we have

$$\int_1^r W(u,\tilde{u})(x)x^{-1}\,\mathrm{d}x = \ln r.$$

The right-hand side diverges while the left-hand side converges due to (4.4), and (4.2) follows. □

Using Lemma 4.2 we can mimic Section 3 and obtain similar results for $H = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_\infty + V$ with V satisfying (the more general) Condition 4.1. In particular Theorem 3.5 and Corollary 3.6 hold under Condition 4.1 provided that we in Theorem 3.5 impose the additional condition

$$s \leqslant 1 + \epsilon/2. \tag{4.10}$$

This is here needed to guarantee that the operators $K^\pm$ of (3.4) are compact on $\mathcal{H}_s$. Also Theorem 3.7 has a similar extension. We leave out further elaboration.

5. Regular positive energy solutions and asymptotics of phase shift

Under Condition 3.1, or in fact more generally under Condition 4.1, we can define the notion of regular positive energy solutions as follows: Let $k \in \mathbb{R}_+$. A solution u to the equation

$$-u''(r) + \big(V_\infty(r) + V(r)\big)u(r) = k^2 u(r) \tag{5.1}$$

is called *regular* if the function $r \to \chi(r < 1)u(r)$ belongs to $\mathcal{D}(H)$. Notice that this definition naturally extends the one applied in Section 3 in the case $k = 0$. Again we claim that the regular solution u is fixed up to a constant (and hence in particular can be taken real-valued): For the uniqueness we may proceed exactly as in Appendix A (uniqueness at zero energy). For the existence part we use the zero energy Green's function R_0^+ and the regular zero energy solution u appearing in Theorem 3.5. Consider the equation

$$u_{k^2} = u + k^2 R_0^+ \chi(\cdot < 1) u_{k^2}. \tag{5.2}$$

Notice that a solution to (5.2) indeed is a solution to (5.1) for $r < 1$ and hence it can be extended to a global solution $\tilde{u}_{k^2}$. Clearly $\chi(\cdot < 1)\tilde{u}_{k^2} \in \mathcal{D}(H)$ so $\tilde{u}_{k^2}$ is a regular solution. It remains to solve (5.2) for some nonzero u_{k^2}. For that we let $K = R_0^+ \chi(\cdot < 1)$ and note that K is compact on $\mathcal{H}_{-s}$ for any $s > 1$. Whence we have

$$u_{k^2} = \big(I - k^2 K\big)^{-1} u, \tag{5.3}$$

provided that

$$\mathrm{Ker}\big(I - k^2 K\big) = \{0\}. \tag{5.4}$$

We are left with showing (5.4). So suppose $u_0 = k^2 K u_0$ for some $u_0 \in \bigcap_{s>1} \mathcal{H}_{-s}$, then we need to show that $u_0 = 0$. Notice that $(H - k^2\chi(\cdot < 1))u_0 = 0$ and that here the second term can be absorbed into the potential V. The computation (3.8) shows that also in the present context

$$0 = \lim_{r\to\infty} \mathrm{Im}\big(\bar{u}_0 u_0'\big)(r) = \lim_{r\to\infty} \mathrm{Im}\big((1/2-\nu)|u_0|^2(r)/r\big). \tag{5.5}$$

From (5.5) we deduce the condition of Lemma 4.2 1) with $u \to u_0$ and whence from the conclusion of Lemma 4.2 that indeed $u_0 = 0$.

Now let u_{k^2} denote any nonzero real regular solution. By using the variation of parameters formula, more specifically by replacing the functions $\phi^{\pm}$ in the proof of Lemma 4.2 by $\cos(k\cdot)$ and $\sin(k\cdot)$ and repeating the proof (see Step I of the proof of Theorem 5.3 stated below for the details), we find the asymptotics

$$\lim_{r\to\infty}\big(u_{k^2}(r) - C\sin\big(kr + \sigma^{\mathrm{sr}}\big)\big) = 0. \tag{5.6}$$

Here $C = C(k) \neq 0$. Assuming (without loss of generality) that $C > 0$ the (real) constant $\sigma^{\mathrm{sr}} = \sigma^{\mathrm{sr}}(k)$ is determined modulo 2π.

Definition 5.1. The quantity $\sigma^{\mathrm{sr}} = \sigma^{\mathrm{sr}}(k)$ introduced above is called the *phase shift* at energy k^2.

Definition 5.2. The notation $\sigma^{\mathrm{per}} = \sigma^{\mathrm{per}}(t)$ signifies the continuous real-valued 2π-periodic function determined by

$$\begin{cases} \sigma^{\mathrm{per}}(0) = 0, \\ \mathrm{e}^{\pi\sigma}\mathrm{e}^{-\mathrm{i}t} - \mathrm{e}^{\mathrm{i}t} = r(t)\mathrm{e}^{\mathrm{i}(\sigma^{\mathrm{per}}(t)-t)}; \quad t \in \mathbb{R},\ r(t) > 0. \end{cases} \tag{5.7}$$

Theorem 5.3. *Suppose Condition* 4.1. *The phase shift* $\sigma^{\mathrm{sr}}(k)$ *can be chosen continuous in* $k \in \mathbb{R}_+$. *Any such choice obeys the following asymptotics as* $k \downarrow 0$: *There exist* $C_1, C_2 \in \mathbb{R}$ *such that*

$$\sigma^{\mathrm{sr}}(k) + \sigma \ln k - \sigma^{\mathrm{per}}(\sigma \ln k + C_1) \to C_2 \quad \textit{for } k \downarrow 0. \tag{5.8}$$

Proof. **Step I.** We shall show the continuity. From (5.2) and (5.3) we see that for any $r > 0$ the functions $]0,\infty[\ \ni k \to u_{k^2}(r)$ and $]0,\infty[\ \ni k \to u'_{k^2}(r)$ are continuous. Similar statements hold upon replacing $u_{k^2} \to \mathrm{Re}\, u_{k^2}$ and $u_{k^2} \to \mathrm{Im}\, u_{k^2}$ which are both real-valued regular solutions (solving (5.1) for $r < 1$). Since $u_{k^2} \neq 0$ one of these functions must be nonzero. Without loss of generality we can assume that u_{k^2} is a real-valued nonzero regular solution obeying that for $r = 1/2$ the functions $]0,\infty[\ \ni k \to u_{k^2}(r)$ and $]0,\infty[\ \ni k \to u'_{k^2}(r)$ are continuous. By a standard regularity result for linear ODE's with continuous coefficients these results then hold for any $r > 0$ too. Moreover (to used in Step II) we have (again for $r > 0$ fixed)

$$u_{k^2}(r) - u(r) = O\big(k^2\big) \quad \text{for } k \downarrow 0, \tag{5.9a}$$

$$u'_{k^2}(r) - u'(r) = O\big(k^2\big) \quad \text{for } k \downarrow 0. \tag{5.9b}$$

We introduce

$$\phi^+(r) = \cos kr \quad \text{and} \quad \phi^-(r) = \sin kr. \tag{5.10}$$

Mimicking the proof of Lemma 4.2 we write

$$u_{k^2} = a^+\phi^+ + a^-\phi^-. \tag{5.11}$$

Noting that the Wronskian $W(\phi^-, \phi^+) = -k$ we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = N\begin{pmatrix} a^+ \\ a^- \end{pmatrix}, \tag{5.12}$$

where

$$N = -k^{-1}(V_\infty + V)\begin{pmatrix} \phi^-\phi^+ & (\phi^-)^2 \\ -(\phi^+)^2 & -\phi^-\phi^+ \end{pmatrix}. \tag{5.13}$$

Since $N = O(r^{-2})$ there exist

$$a^\pm(\infty) = \lim_{r\to\infty} a^\pm(r). \tag{5.14}$$

By the same argument as before either $a^+(\infty) \neq 0$ or $a^-(\infty) \neq 0$. We write

$$\big(a^+(\infty), a^-(\infty)\big)\Big/\sqrt{a^+(\infty)^2 + a^-(\infty)^2} = \big(\sin\sigma^{\mathrm{sr}}, \cos\sigma^{\mathrm{sr}}\big) \tag{5.15}$$

and conclude the asymptotics (5.6) with some $C \neq 0$. It remains to see that $a^\pm(\infty)$ are continuous in k (then by (5.15) σ^{sr} can be chosen continuous too). For that we use the "connection formula"

$$\begin{pmatrix} u_{k^2} \\ u'_{k^2} \end{pmatrix} = \begin{pmatrix} \phi^+ & \phi^- \\ \phi^{+\prime} & \phi^{-\prime} \end{pmatrix}\begin{pmatrix} a^+ \\ a^- \end{pmatrix}$$

which is "solved" by

$$\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = -k^{-1}\begin{pmatrix} \phi^{-\prime} & -\phi^- \\ -\phi^{+\prime} & \phi^+ \end{pmatrix}\begin{pmatrix} u_{k^2} \\ u'_{k^2} \end{pmatrix}. \tag{5.16}$$

We use (5.16) at $r = 1/2$. By the comments at the beginning of the proof the right-hand side is continuous in k and therefore so is the left-hand side. Solving (5.12) by integrating from $r = 1/2$ and noting that (5.13) is continuous in k we then conclude that $a^\pm(r)$ are continuous in k for any $r > 1/2$. Since the limits (5.14) are taken locally uniformly in $k > 0$ we consequently deduce that indeed $a^\pm(\infty)$ are continuous in k.

Step II. We shall show (5.8) under Condition 3.1. We shall mimic Step I with (5.10) replaced by

$$\phi^+(r) = r^{1/2}H_\nu^{(1)}(kr) \quad \text{and} \quad \phi^-(r) = r^{1/2}\overline{H_{-\nu}^{(1)}(kr)}. \tag{5.17}$$

For completeness of presentation note that in terms of another Hankel function, cf. [11, (3.6.31)], $\phi^-(r) = r^{1/2}H_\nu^{(2)}(kr)$. We compute the Wronskian $W(\phi^-, \phi^+) = 4\mathrm{i}/\pi$, cf. (2.1c) and [11, (3.6.27)]. Since $V(r) = 0$ for $r \geqslant R$

$$a^\pm(r) = a^\pm(\infty) \quad \text{for } r \geqslant R. \tag{5.18}$$

Moreover (5.16) reads

$$\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = \frac{\pi}{4\mathrm{i}} \begin{pmatrix} \phi^{-\prime} & -\phi^- \\ -\phi^{+\prime} & \phi^+ \end{pmatrix} \begin{pmatrix} u_{k^2} \\ u'_{k^2} \end{pmatrix}. \tag{5.19}$$

We will use (5.19) at $r = R$. Clearly the right-hand side is continuous in $k > 0$ and therefore so is the left-hand side. From the asymptotics

$$\phi^+(r) - C_\nu \left(\frac{2}{\pi k}\right)^{1/2} \mathrm{e}^{\mathrm{i}kr} \to 0 \quad \text{for } r \to \infty, \tag{5.20}$$

$$\phi^-(r) - \overline{C_{-\nu}} \left(\frac{2}{\pi k}\right)^{1/2} \mathrm{e}^{-\mathrm{i}kr} \to 0 \quad \text{for } r \to \infty;$$

$$C_\nu := \mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}, \tag{5.21}$$

we may readily rederive the continuity statement shown more generally in Step I. The point is that now we can "control" the limit $k \to 0$. To see this we need to compute the asymptotics of the matrix in (5.19) as $k \to 0$ (with $r = R$). Using (2.1c) we compute

$$\phi^+(R) = \frac{1}{\mathrm{i}\sin(\nu\pi)} \left(\frac{2^\nu R^{\frac{1}{2}-\nu}}{\Gamma(1-\nu)} k^{-\nu} - \mathrm{e}^{-\sigma\pi} \frac{2^{-\nu} R^{\frac{1}{2}+\nu}}{\Gamma(1+\nu)} k^\nu + O\big(k^2\big) \right), \tag{5.22a}$$

$$\phi^-(R) = \frac{1}{-\mathrm{i}\sin(\nu\pi)} \left(\frac{2^\nu R^{\frac{1}{2}-\nu}}{\Gamma(1-\nu)} k^{-\nu} - \mathrm{e}^{\sigma\pi} \frac{2^{-\nu} R^{\frac{1}{2}+\nu}}{\Gamma(1+\nu)} k^\nu + O\big(k^2\big) \right), \tag{5.22b}$$

$$\phi^{+\prime}(R) = \frac{1}{\mathrm{i}\sin(\nu\pi)} \left(\big(2^{-1} - \nu\big) \frac{2^\nu R^{-\frac{1}{2}-\nu}}{\Gamma(1-\nu)} k^{-\nu} - \mathrm{e}^{-\sigma\pi} \big(2^{-1} + \nu\big) \frac{2^{-\nu} R^{-\frac{1}{2}+\nu}}{\Gamma(1+\nu)} k^\nu + O\big(k^2\big) \right), \tag{5.22c}$$

$$\phi^{-\prime}(R) = \frac{1}{-\mathrm{i}\sin(\nu\pi)} \left(\big(2^{-1} - \nu\big) \frac{2^\nu R^{-\frac{1}{2}-\nu}}{\Gamma(1-\nu)} k^{-\nu} - \mathrm{e}^{\sigma\pi} \big(2^{-1} + \nu\big) \frac{2^{-\nu} R^{-\frac{1}{2}+\nu}}{\Gamma(1+\nu)} k^\nu + O\big(k^2\big) \right). \tag{5.22d}$$

We combine (5.9a) and (5.9b) for $r = R$ with (5.18)–(5.22d) and obtain

$$u_{k^2}(r) = \left(\frac{2}{\pi k}\right)^{1/2} \left(\frac{\pi}{4\mathrm{i}} \frac{C_\nu}{\mathrm{i}\sin(\nu\pi)} \big(\mathrm{e}^{\sigma\pi} \overline{D} k^\nu - D k^{-\nu}\big) + O\big(k^2\big) \right) \mathrm{e}^{\mathrm{i}kr} + \text{h.c.} + o\big(r^0\big);$$

$$D := \frac{2^\nu R^{\frac{1}{2}-\nu}}{\Gamma(1-\nu)} \left(\frac{2^{-1} - \nu}{R} u(R) - u'(R) \right). \tag{5.23}$$

Here the term $O(k^2)$ depends on R but not on r and the term $o(r^0)$ depends on k. The second term, denoted by h.c., is given as the hermitian (or complex) conjugate of the first term. Note that $D \neq 0$.

We write $D = |D|\mathrm{e}^{\mathrm{i}\theta_0}$ yielding

$$\mathrm{e}^{\sigma\pi} \overline{D} k^\nu - D k^{-\nu} = \big|\mathrm{e}^{\sigma\pi} \overline{D} k^\nu - D k^{-\nu}\big| \mathrm{e}^{\mathrm{i}(\sigma^{\mathrm{per}}(\sigma \ln k + \theta_0) - (\sigma \ln k + \theta_0))}. \tag{5.24}$$

Next we substitute (5.24) into (5.23), use that $C_\nu = |C_\nu|\mathrm{e}^{-\mathrm{i}\pi/4}$ and conclude (5.8) with

$$C_1 = \theta_0 \quad \text{and} \quad C_2 = \pi/4 - \theta_0 + 2\pi p \quad \text{for some } p \in \mathbb{Z}. \tag{5.25}$$

Step III. We shall show (5.8) under Condition 4.1. This is done by modifying Step II using the proof of Step I too. Explicitly using again the functions $\phi^\pm$ of (5.17) "the coefficients" $a^\pm$ need to be constructed. Since V is not assumed to be compactly supported these coefficients will now depend on r. We first construct them at any large R, this is by the formula (5.19) (at $r = R$). Then the modification of (5.12)

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = N \begin{pmatrix} a^+ \\ a^- \end{pmatrix}, \tag{5.26}$$

with

$$N = \frac{\pi}{4\mathrm{i}} V \begin{pmatrix} \phi^-\phi^+ & (\phi^-)^2 \\ -(\phi^+)^2 & -\phi^-\phi^+ \end{pmatrix}, \tag{5.27}$$

is invoked. We integrate to infinity using that $N = O(r^{-1-\epsilon})$ uniformly in $k > 0$. This leads to

$$a^\pm(r) = a^\pm(\infty) + O\bigl(r^{-\epsilon}\bigr), \tag{5.28a}$$

$$a^\pm(R) = a^\pm(\infty) + O\bigl(R^{-\epsilon}\bigr), \tag{5.28b}$$

with the error estimates being uniform in $k > 0$. In particular for $r \geqslant R$

$$a^\pm(r) = a^\pm(R) + O\bigl(R^{-\epsilon}\bigr) + O\bigl(r^{-\epsilon}\bigr) \tag{5.29}$$

uniformly in $k > 0$.

From (5.29) we obtain the following modification of (5.23)

$$u_{k^2}(r) = \left(\frac{2}{\pi k}\right)^{1/2} \left(\frac{\pi}{4\mathrm{i}} \frac{C_\nu}{\mathrm{i}\sin(\nu\pi)}\bigl(\mathrm{e}^{\sigma\pi}\overline{D}k^\nu - Dk^{-\nu}\bigr) + O\bigl(k^2\bigr) + O\bigl(R^{-\epsilon}\bigr)\right)\mathrm{e}^{\mathrm{i}kr}$$
$$+ \text{h.c.} + o\bigl(r^0\bigr);$$
$$D = D(R) := \frac{2^\nu R^{\frac{1}{2}-\nu}}{\Gamma(1-\nu)}\left(\frac{2^{-1}-\nu}{R}u(R) - u'(R)\right).$$

The term $O(k^2)$ depends on R, and the term $O(R^{-\epsilon})$ depends on k but it is estimated uniformly in $k > 0$. By Lemma 4.2 there exist $\delta > 0$ and a sequence $R_n \to \infty$ such that

$$\bigl|D(R_n)\bigr| \geqslant \delta \quad \text{for all } n. \tag{5.30}$$

Using these values of D in (5.24) we can write

$$\mathrm{e}^{\sigma\pi}\overline{D}k^\nu - Dk^{-\nu} = \bigl|\mathrm{e}^{\sigma\pi}\overline{D}k^\nu - Dk^{-\nu}\bigr|\mathrm{e}^{\mathrm{i}(\sigma^{\mathrm{per}}(\sigma\ln k+\theta)-(\sigma\ln k+\theta))};$$
$$D = D(R_n), \qquad \theta = \theta_n \in [0, 2\pi[. \tag{5.31}$$

We can assume that for some $\theta_0 \in [0, 2\pi]$

$$\theta_n \to \theta_0 \quad \text{for } n \to \infty. \tag{5.32}$$

Using this number θ_0 we obtain again (5.8) with C_1 and C_2 given as in (5.25). □

6. Asymptotics of physical phase shift for a potential like $-\gamma\chi(r > 1)r^{-2}$

We shall reduce a d-dimensional Schrödinger equation to angular momentum sectors and discuss the asymptotics of the "physical" phase shift for small angular momenta in the low energy regime.

We consider for $d \geqslant 2$ the stationary d-dimensional Schrödinger equation

$$Hv = (-\Delta + W)v = \lambda v; \quad \lambda > 0,$$

for a radial potential $W = W(|x|)$ obeying

Condition 6.1.

1) $W(r) = W_1(r) + W_2(r)$; $W_1(r) = -\frac{\gamma}{r^2}\chi(r > 1)$ for some $\gamma > 0$,
2) $W_2 \in C(]0, \infty[, \mathbb{R})$,
3) $\exists \epsilon_1, C_1 > 0$: $|W_2(r)| \leqslant C_1 r^{-2-\epsilon_1}$ for $r > 1$,
4) $\exists \epsilon_2, C_2 > 0$: $|W_2(r)| \leqslant C_2 r^{\epsilon_2 - 2}$ for $r \leqslant 1$.

Under Condition 6.1 $H = -\Delta + W$ is self-adjoint as defined in terms of the Dirichlet form on $H^1(\mathbb{R}^d)$, cf. [4]. Let H_l, $l = 0, 1, \ldots$, be the corresponding reduced Hamiltonian corresponding to an eigenvalue $l(l + d - 2)$ of the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$

$$H_l u = -u'' + (V_\infty + V)u. \tag{6.1}$$

Here

$$V_\infty(r) = \frac{\nu^2 - 1/4}{r^2}\chi(r > 1); \quad \nu^2 = \left(l + \frac{d}{2} - 1\right)^2 - \gamma, \tag{6.2a}$$

$$V(r) = W_2(r) + \frac{(l + \frac{d}{2} - 1)^2 - 1/4}{r^2}\big(1 - \chi(r > 1)\big), \tag{6.2b}$$

and the stationary equation reads

$$-u'' + (V_\infty + V)u = \lambda u. \tag{6.3}$$

Notice that for

$$\gamma > \left(l + \frac{d}{2} - 1\right)^2, \tag{6.4}$$

and

$$(d,l) \neq (2,0), \tag{6.5}$$

indeed Condition 4.1 is fulfilled and H_l coincides with the Hamiltonian given by the construction of Section 4. The case $(d,l) = (2,0)$ needs a separate consideration which is given in Appendix B.

Under the conditions (6.4) and (6.5) let u_l be a regular solution to the reduced Schrödinger equation (6.3). Write

$$\lim_{r\to\infty}\bigl(u_l(r) - C\sin(\sqrt{\lambda}r + D_l)\bigr) = 0. \tag{6.6}$$

The standard definition of the phase shift (coinciding with the time-depending definition) is

$$\sigma_l^{\text{phy}}(\lambda) = D_l + \frac{d-3+2l}{4}\pi. \tag{6.7}$$

It is known from [16,4] that for a potential $W(r)$ behaving at infinity like $-\gamma r^{-\mu}$ with $\gamma > 0$ and $\mu \in \,]1,2[$

$$\exists \sigma_0 \in \mathbb{R}\colon \quad \sigma_l^{\text{phy}}(\lambda) - \int_{R_0}^{\infty} \bigl(\sqrt{\lambda} - \sqrt{\lambda - W(r)}\bigr)\,\mathrm{d}r \to \sigma_0 \quad \text{for } \lambda \downarrow 0. \tag{6.8}$$

Here R_0 is any sufficiently big positive number, and the integral does not have a (finite) limit as $\lambda \downarrow 0$. In the present case, $\mu = 2$, (6.8) indicates a logarithmic divergence. This is indeed occurring although (6.8) is incorrect for $\mu = 2$. The correct behaviour of the phase shift under the conditions (6.4) and (6.5) follows directly from Section 5:

Theorem 6.2. *Suppose Condition* 6.1 *and* (6.4) *for some* $l \in \mathbb{N} \cup \{0\}$. *Let*

$$\sigma = \sqrt{\gamma - \left(l + \frac{d}{2} - 1\right)^2}. \tag{6.9}$$

The phase shift $\sigma_l^{\text{phy}}(\lambda)$ *can be chosen continuous in* $\lambda \in \mathbb{R}_+$. *Any such choice obeys the following asymptotics as* $\lambda \downarrow 0$: *There exist* $C_1, C_2 \in \mathbb{R}$ *such that*

$$\sigma_l^{\text{phy}}(\lambda) + \sigma \ln\sqrt{\lambda} - \sigma^{\text{per}}(\sigma \ln\sqrt{\lambda} + C_1) \to C_2 \quad \textit{for } \lambda \downarrow 0. \tag{6.10}$$

Note that we have included the case $(d,l) = (2,0)$ in this result. The necessary modifications of Section 5 for this case are outlined in Appendix B.

Appendix A. Regular zero energy solutions

We shall elaborate on the notion of regular solutions as used in Sections 3 and 4. Recall from the discussion around (3.27) that we call a solution u to (3.27) for regular if $r \to \chi(r < 1)u(r)$ belongs to $\mathcal{D}(H)$ where H is defined in terms of a potential V satisfying Condition 3.1 (or Condition 4.1). The existence of a (nonzero) regular solution is shown explicitly by the formula (3.33c).

We shall show that the regular solution is unique up to a constant. Notice that as a consequence of this uniqueness result a regular solution is real-valued up to constant.

Suppose conversely that all solutions are regular. Due to [10, Theorem X.6(a)] there exists a nonzero solution v to

$$-v''(r) + \big(V_\infty(r) + V(r)\big)v(r) = \mathrm{i}v(r) \tag{A.1}$$

which is in L^2 at infinity. By the variation of parameter formula now based on the basis of regular solutions to (3.27), cf. the proof of [10, Theorem X.6(b)], we conclude that $v \in \mathcal{D}(H)$ and that $(H - \mathrm{i})v = 0$. This violates that H is self-adjoint.

Appendix B. Case $(d, l) = (2, 0)$

For $(d, l) = (2, 0)$ Condition 4.1 fails for the operator H_l of Section 6 (this example would require $\kappa = 0$ in Condition 4.1 3)). The form domain is not $H_0^1(\mathbb{R}_+)$ in this case. The form is given as follows:

$$\mathcal{D}(Q) = \left\{ f \in L^2(\mathbb{R}_+) \;\middle|\; g \in L^2(\mathbb{R}_+) \text{ where } g(r) = f'(r) - \frac{1}{2r} f(r) \right\}, \tag{B.1a}$$

$$Q(f) = \int_0^\infty \left(\left| f'(r) - \frac{1}{2r} f(r) \right|^2 + W(r) \left| f(r) \right|^2 \right) \mathrm{d}r; \quad f \in \mathcal{D}(Q). \tag{B.1b}$$

This is a closed semi-bounded quadratic form and the domain $\mathcal{D}(H)$ of the corresponding operator H (cf. [3,10]) is characterised as the subset of f's in $\mathcal{D}(Q)$ for which

$$h \in L^2(\mathbb{R}_+) \quad \text{where } h(r) := \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{4r^2} + W(r) \right) f(r) \text{ as a distribution on } \mathbb{R}_+, \tag{B.2}$$

and for $f \in \mathcal{D}(H)$ we have

$$(Hf)(r) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{4r^2} + W(r) \right) f(r). \tag{B.3}$$

To see the connection to the two-dimensional Hamiltonian of Section 6 defined with form domain $H^1(\mathbb{R}^2)$ let us note the alternative description of Q:

$$\mathcal{D}(Q) = \big\{ f \in L^2(\mathbb{R}_+) \;\big|\; \tilde{g}\big(|\cdot|\big) \in H^1\big(\mathbb{R}^2\big) \text{ where } \tilde{g}(r) = r^{-1/2} f(r) \big\}, \tag{B.4a}$$

$$Q(f) = (2\pi)^{-1} \int_{\mathbb{R}^2} \Big(\big|\nabla\big(|x|^{-1/2} f\big(|x|\big)\big)\big|^2 + W\big(|x|\big)\big||x|^{-1/2} f\big(|x|\big)\big|^2 \Big)\,\mathrm{d}x \quad \text{for } f \in \mathcal{D}(Q). \tag{B.4b}$$

Clearly the integral to the right in (B.4b) is the form of the two-dimensional Hamiltonian (applied to radially symmetric functions).

We also note that $H_0^1(\mathbb{R}_+) \subseteq \mathcal{D}(Q)$ and that

$$C_c^\infty(\mathbb{R}_+) + \operatorname{span}(f_0); \quad f_0(r) := r^{1/2}\chi(r < 1)$$

is a core for Q. In fact, although $f_0 \notin H_0^1(\mathbb{R}_+)$, the set $C_c^\infty(\mathbb{R}_+)$ is actually a core for Q. Whence H is the Friedrichs extension of the action (B.3) on $C_c^\infty(\mathbb{R}_+)$.

Due to (B.1b) and the description in (B.2) of the domain $\mathcal{D}(H)$ we can show the uniqueness of regular solutions exactly as in Appendix A. The existence of (nonzero) regular solutions follows from the previous scheme too. Indeed the basic operators $K^\pm$ of (3.4) are again compact on $\mathcal{B}(\mathcal{H}_s)$. To see this we need to see that various terms are compact. Let us here consider the contribution from the first term of (3.3)

$$-\left(\chi_1'' + 2\chi_1' \frac{\mathrm{d}}{\mathrm{d}r}\right)(H \mp \mathrm{i})^{-1}\chi_1 + \chi_1(\pm \mathrm{i})(H \mp \mathrm{i})^{-1}\chi_1 =: K_1^\pm + K_2^\pm.$$

(The contribution from the second term of (3.3) is treated in the same way as before.) We decompose using any $C > 0$ such that $H^0 \geqslant C + 1$

$$\begin{aligned} K_1^\pm &= B^\pm K; \\ B^\pm &= -\left(\chi_1''(r) + 2\chi_1'(r)\frac{1}{2r} + 2\chi_1'(r)\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{2r}\right)\right)(H \mp \mathrm{i})^{-1}(H - C)^{1/2}, \\ K &= (H - C)^{-1/2}\chi_1. \end{aligned}$$

The operator $B^\pm$ is bounded and the operator K is compact (the latter may be seen easily by going back to the space $L^2(\mathbb{R}^2)$ and there invoking standard Sobolev embedding); whence $K_1^\pm$ is compact. Clearly also $K_2^\pm$ is compact.

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