DECAY OF EIGENFUNCTIONS OF ELLIPTIC PDE'S, II

I. HERBST AND E. SKIBSTED

ABSTRACT. We study exponential decay rates of eigenfunctions of self-adjoint higher order elliptic operators on $\mathbb{R}^d$. We are interested in decay rates as a function of direction. We show that the possible decay rates are to a large extent determined algebraically.

CONTENTS

1. INTRODUCTION AND PREVIOUS RESULTS

Consider a real elliptic polynomial Q of degree q on $\mathbb{R}^d$. (Q elliptic means that for large $\xi \in \mathbb{R}^d$, $C|Q(\xi)| > |\xi|^q$ for some C.) We consider the operator $H = Q(p)+V(x)$, $p = -\mathrm{i}\nabla$, on $L^2(\mathbb{R}^d)$ with V bounded and measurable. For most of our results we assume $\lim_{|x|\to\infty} V(x) = 0$ and additional decay properties of the potential. By the assumptions on Q the operator $Q(p)$ is self-adjoint with domain the standard Sobolev space of order q which consequently is also the domain of H. The goal of the paper is to study exponential decay of L^2-eigenfunctions of H with eigenvalue $\lambda \in \mathbb{R}$ as a function of direction. It is the second in a series of two papers on exponential decay. The first one is [HS].

In [Ag1], Agmon investigated the asymptotic behavior of the Green's function (the integral kernel of the inverse of $Q(p)-\lambda$ for spectral parameter λ in the resolvent set of $Q(p)$). In certain cases he obtained rather precise asymptotics of this function. Since we are investigating the asymptotic behavior of eigenfunctions of $Q(p)+V(x)$ with $V(x)$ small at infinity, one might suspect that the asymptotic behavior of the Green's function would determine the exponential rate of fall-off of the eigenfunction. This is false in a rather spectacular way: First, the eigenvalue λ may actually be in the spectrum of $Q(p)$ where the Green's function decays (at most) like an inverse power of $|x|$ while the eigenfunction decays exponentially. And second, whether or not the eigenvalue is in the spectrum of $Q(p)$, there may be several (global or local) decay rates which occur for different potentials V of compact support. Of course at least one of these decay rates will not reflect the asymptotic behavior of the Green's function. Already in [HS] we gave examples of these phenomena. For

Key words and phrases. eigenfunctions, exponential decay, microlocal analysis, combinatorics.

another example see Section 4. These phenomena do not occur if $Q(\xi) = |\xi|^2$, at least if for example $V = o(|x|^{-1/2})$ at infinity (see Theorems 1.3 and 3.6).

We first summarize some of the results of [HS] which will be our starting point. References to previous work are given there. We define the *global decay rate* of $\phi \in L^2(\mathbb{R}^d)$ as

$$\sigma_g = \sup\{\sigma \geq 0 | \mathrm{e}^{\sigma|x|}\phi \in L^2\}. \tag{1.1}$$

It is intuitively clear that σ_g is determined by the directions of weakest exponential decay of ϕ.

In the rest of this section we assume that $(H-\lambda)\phi = 0$ with $\lambda \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R}^d)$. We will mostly assume there is a splitting of V, $V = V_1 + V_2$, into bounded functions, with V_1 smooth and real-valued and V_2 measurable, with additional assumptions depending on the result.

Theorem 1.1. *Under either of the following two conditions we can conclude that* $\sigma_g > 0$*:*

1) $\lambda \notin \mathrm{Ran}Q := \{Q(\xi) | \xi \in \mathbb{R}^d\}$ *and* $V(x) = o(1)$ *at infinity.*

2) $\lambda \in \mathrm{Ran}Q$ *but* λ *is not a critical value of* $Q(\xi)$*,* ξ *real, and in addition*

$$\forall \alpha : \partial^\alpha V_1(x) = o(|x|^{-|\alpha|}),$$
$$V_2(x) = o(|x|^{-1}).$$

Earlier work for the Laplacian can be found in [Oc, CT, FH, MP1]. Carleman type estimates which can be useful in proving part 2) of Theorem 1.1 for even more general operators were proved in [MP2].

The following theorem eliminates the possibility of super-exponential decay at the expense of rather strong decay assumptions on the potential:

Theorem 1.2. *Suppose* $V_2(x) = O(|x|^{-q/2-\delta})$ *and* $\partial^\alpha V_1(x) = O(|x|^{-(\delta+q+|\alpha|)/2})$, $1 \leq |\alpha| \leq q$*, where* $\delta > 0$*. Then* $\sigma_g < \infty$ *unless* $\phi = 0$*.*

For $Q(\xi) = |\xi|^2$ or $|\xi|^4$ (and perhaps for any real elliptic Q) one can do with weaker decay assumptions on V, see [HS]. In fact for $Q(\xi) = |\xi|^2$ or $|\xi|^4$, in the conditions on V, q can be replaced by $q/2$. (Of course the results given in [HS] for the Laplacian were known, see [BM, FHH2O1, FHH2O3, FHH2O2, FH].)

With the above two theorems we have conditions on V which guarantee that $0 < \sigma_g < \infty$. We will assume the latter in the rest of this paper.

The next theorem shows that σ_g must satisfy certain equations which in favorable situations determine its possible values.

Theorem 1.3. *Suppose* $0 < \sigma_g < \infty$*. If*

$$\forall \alpha : \partial^\alpha V_1(x) = o(|x|^{-|\alpha|}),$$
$$V_2(x) = o(|x|^{-1/2}),$$

then there exists $(\omega, \xi, \beta) \in S^{d-1} \times \mathbb{R}^d \times \mathbb{C}$ *such that*

$$Q(\xi + \mathrm{i}\sigma_g\omega) = \lambda, \tag{1.2a}$$
$$\nabla_\xi Q(\xi + \mathrm{i}\sigma_g\omega) = \beta\omega. \tag{1.2b}$$

Note that the number of real unknowns indicated by $\sigma_g, \omega, \xi, \beta$ equals the number of real equations in (1.2a) and (1.2b) and thus the set of σ_g occurring as solutions of these equations has a chance of being discrete. In fact except for a finite set of exceptional λ's this is true if Q is rotationally invariant (one might say in spite of the rotation invariance). In [HS] it is shown that except possibly for this finite set of λ's every solution $\sigma_g > 0$ of these equations actually occurs for a real, smooth V of compact support.

In this paper we will study quantities somewhat similar to (1.1). One of those is a rough measure of the asymptotics at infinity. It is the *local decay rate* of any $\phi \in L^2$ defined for $\omega \in S^{d-1}$ by

$$\sigma_{loc}(\omega) = \sup\{\sigma | \mathrm{e}^{\sigma|x|}\phi \in L^2(C) \text{ for some open cone C containing } \omega\}. \tag{1.3}$$

In the next section we introduce in addition two other measures of exponential rate of decay which also depend on direction. Those notions appear more amenable to analysis than the local decay rate, but as we will see our study of these other notions of decay yields information on $\sigma_{loc}(\omega)$. Our main result will be presented in Section 3, see Theorem 3.4. It allows us to some extent to calculate rates of decay of eigenfunctions in L^2, most notably for rotationally invariant Q's, see Theorem 3.6 (announced earlier in [HS]). This is in the spirit of Theorem 1.3, that is by solving a certain system of algebraic equations. We give another demonstration of our results for an example in Section 4 (a non-rotationally invariant case). In Section 5 we elaborate on a connection to previous works [Ag1, Ag2]. We show how the above mentioned system of algebraic equations relates to [Ag1, Ag2] and in fact, more generally, can be derived by a variational principle. Finally we have collected various considerations on possible smoothness of rates of decay of eigenfunctions in Section 6.

2. Directional decay rates, arbitrary ϕ

In this section ϕ is an arbitrary function in $L^2(\mathbb{R}^d)$ with $0 < \sigma_g < \infty$ where σ_g is defined in (1.1). Note that we do not assume that ϕ is an eigenfunction. The basic object which incorporates information on the directional decay rates of ϕ and which we find most amenable to analysis is the set

$$\mathcal{E} = \{\eta \in \mathbb{R}^d | \mathrm{e}^{\eta \cdot x}\phi \in L^2\}. \tag{2.1}$$

We introduce three exponential decay rates depending on a direction $\omega \in S^{d-1}$.

$$\sigma_c(\omega) = \sup\{\sigma | \mathrm{e}^{\sigma\omega\cdot x}\phi \in L^2\}$$
$$\sigma_s(\omega) = \sup\{\eta \cdot \omega | \eta \in \mathcal{E}\}$$
$$\sigma_{loc}(\omega) = \sup\{\sigma | \mathrm{e}^{\sigma|x|}\phi \in L^2(C) \text{ for some open cone C containing } \omega\}$$

It is easy to see that

$$\sigma_g \le \sigma_c(\omega) \le \sigma_s(\omega) \le \sigma_{loc}(\omega).$$

Note that σ_s, as the supremum of a family of continuous functions, is lower semi-continuous. In addition if we define $\sigma_s(t\omega) = t\sigma_s(\omega)$ for $t \ge 0$, then $\sigma_s(x)$ is the support function of the set $\mathcal{E}$ (by definition $0 \cdot \infty = 0$).

Here are some basic facts which are true for an arbitrary $\phi \in L^2$ if $\sigma_g \in (0, \infty)$. We allow $\sigma_c(\omega) = \infty$ in which case we define $1/\sigma_c(\omega) = 0$. Since $\sigma_g < \infty$ a simple compactness argument shows that $\sigma_c(\omega) < \infty$ for at least one ω, in fact $\sigma_g = \inf_\omega \sigma_{loc}(\omega)$. By $B_r(x)$ we mean the open ball in $\mathbb{R}^d$ of radius r centered at x.

Theorem 2.1. *1) $\mathcal{E}$ is convex and contains $B_{\sigma_g}(0)$.*

2) $1/\sigma_c(\omega)$ is Lipschitz. In fact $|1/\sigma_c(\omega_1) - 1/\sigma_c(\omega_2)| \le |\omega_1 - \omega_2|/\sigma_g$. In particular the set $\{\omega \in S^{d-1} | \sigma_c(\omega) < \infty\}$ is a relatively open subset of S^{d-1}.

3) $\partial\mathcal{E} = \{\sigma_c(\omega)\omega | \omega \in S^{d-1}, \sigma_c(\omega) < \infty\}$.

4) Suppose $f : \mathbb{R}^d \to [0, \infty)$ is convex and $f(tx) = tf(x)$ for all $t \ge 0$. Suppose in addition

$$\mathrm{e}^{tf}\phi \in L^2 \text{ for all } t < 1.$$

Then $f(x) \le \sigma_s(x)$.

5) The function σ_{loc} is lower semi-continuous. Suppose $\rho : \mathbb{R}^d \to [0, \infty)$ is continuous, $\rho(tx) = t\rho(x)$ for all $t \ge 0$ and $\rho(\omega) \le \sigma_{loc}(\omega)$ for all $\omega \in S^{d-1}$. Then

$$\mathrm{e}^{t\rho}\phi \in L^2 \text{ for all } t < 1.$$

Proof. **1)** Take $\eta_j \in \mathcal{E}, j = 1, 2$. By the Young inequality for any $s \in (0, 1)$

$$\mathrm{e}^{(s\eta_1 + (1-s)\eta_2)\cdot x} \le s\mathrm{e}^{\eta_1 \cdot x} + (1-s)\mathrm{e}^{\eta_2 \cdot x}.$$

This implies that $s\eta_1 + (1 - s)\eta_2 \in \mathcal{E}$. Similarly by the Cauchy-Schwarz inequality $B_{\sigma_g}(0) \subset \mathcal{E}$.

2) Given ω_1, ω_2, $\omega_1 \ne \omega_2$, and $\mu \in (0, \sigma_c(\omega_1))$ we will choose σ so that $\sigma\omega_2 \in \mathcal{E}$ as follows. We write

$$\sigma\omega_2 = \sigma\omega_1 + \sigma(\omega_2 - \omega_1),$$

and define $\eta_1 = \sigma\omega_1/(1 - t)$ and $\eta_2 = \sigma(\omega_2 - \omega_1)/t$ so that $\sigma\omega_2 = (1 - t)\eta_1 + t\eta_2$. In order to have $t \in (0, 1)$, $\eta_1 \in \mathcal{E}$ and $\eta_2 \in \mathcal{E}$ (so that 1) applies), we demand $0 < \sigma/(1-t) \le \mu$ and $\sigma|\omega_2-\omega_1|/t < \sigma_g$. We choose t so that $t\sigma_g/|\omega_2-\omega_1| = (1-t)\mu$. We find that if $1/\sigma > 1/\mu + |\omega_2 - \omega_1|/\sigma_g$ then indeed $\sigma\omega_2 \in \mathcal{E}$. It follows that $1/\sigma_c(\omega_2) \le 1/\sigma_c(\omega_1) + |\omega_2 - \omega_1|/\sigma_g$. Interchanging ω_2 and ω_1 gives the result.

3) We first show that if $\eta \in \bar{\mathcal{E}}$ then $s\eta \in \mathrm{int}(\mathcal{E})$ for $0 < s < 1$. Pick a sequence $\eta_j \in \mathcal{E}$ with $\eta_j \to \eta$. By 1) $s\eta_j + (1 - s)\zeta \in \mathcal{E}$ if $|\zeta| < \sigma_g$. This means $s\eta_j + B_{(1-s)\sigma_g}(0) \subset \mathcal{E}$. Since $s\eta_j \to s\eta$, $s\eta \in B_{(1-s)\sigma_g}(s\eta_j) \subset \mathcal{E}$ for large enough j. Whence indeed $s\eta \in \mathrm{int}(\mathcal{E})$. We next show that if $\eta \in \partial\mathcal{E}$ then with $\eta = |\eta|\omega, \omega \in S^{d-1}$, we have $\sigma_c(\omega) = |\eta|$. Since $\eta \in \bar{\mathcal{E}}, s\eta \in \mathrm{int}(\mathcal{E})$ for $0 < s < 1$. Thus $\sigma_c(\omega) \ge |\eta|$. Suppose $\sigma_c(\omega) > |\eta|$. Fix $\sigma \in (|\eta|, \sigma_c(\omega))$. Then $\sigma\omega \in \mathrm{int}(\mathcal{E})$. Fix $r > 0$ so that $\sigma\omega + B_r(0) \subset \mathcal{E}$. For $0 < s < 1$ we have $(1 - t)s\eta + t(\sigma\omega + B_r(0) \subset \mathcal{E}$ for any $t \in [0, 1]$. Let $t = \frac{|\eta| - s|\eta|}{\sigma - s|\eta|}$. Then $(1 - t)s\eta + t\sigma\omega = \eta$ so that $\eta + B_{tr}(0) \subset \mathcal{E}$ and thus η is not a boundary point. We have shown $\sigma_c(\omega) = |\eta|$ or in other words $\sigma_c(\omega) < \infty$ and $\eta = \sigma_c(\omega)\omega$. If on the other hand $\sigma_c(\omega) < \infty$ and $\eta = \sigma_c(\omega)\omega$, then $(1 - n^{-1})\sigma_c(\omega)\omega \in \mathcal{E}$ for all $n \in \mathbb{N}$ by the definition of $\sigma_c(\omega)$. Hence $\sigma_c(\omega)\omega \in \bar{\mathcal{E}}$. $\sigma_c(\omega)\omega$ cannot be in $\mathrm{int}(\mathcal{E})$ by the definition of $\sigma_c(\omega)$ so that $\sigma_c(\omega)\omega \in \partial\mathcal{E}$.

4) Since f is convex, for each $x_0 \in \mathbb{R}^d$ there exists a set of linear functions, $l_{\eta,x_0}(x) = f(x_0) + \eta \cdot (x - x_0), \eta \in G(x_0)$ (here $G(x_0)$ is our notation for the set of subgradients at x_0, see for example [Ro, p. 214]), so that

$$f(x) = \sup\{l_{\eta,x_0}(x) | x_0 \in \mathbb{R}^d, \eta \in G(x_0)\}.$$

Using $t^{-1}f(tx) = f(x)$ we have $f(x) \ge \eta \cdot x + t^{-1}l_{\eta,x_0}(0)$ for $x_0 \in \mathbb{R}^d, \eta \in G(x_0)$ Taking t to infinity we obtain $f(x) \ge \sup\{\eta \cdot x | \eta \in G)\}$ where $G = \cup_{x_0 \in \mathbb{R}^d} G(x_0)$. But since $0 = f(0) \geq l_{\eta,x_0}(0)$ we have $f(x) = \sup\{\eta \cdot x + l_{\eta,x_0}(0) | x_0 \in \mathbb{R}^d, \eta \in G(x_0)\} \leq \sup\{\eta \cdot x | x_0 \in \mathbb{R}^d, \eta \in G(x_0)\}$. Thus $f(x) = \sup\{\eta \cdot x | \eta \in G\}$. This means that if $\eta \in G$ then $e^{t\eta \cdot x}\phi \in L^2$ for all $t \in [0, 1)$ so that $G \subset \bar{\mathcal{E}}$. Thus

$$f(x) \leq \sup\{\eta \cdot x | \eta \in \bar{\mathcal{E}}\} = \sigma_s(x).$$

5) Given $\alpha \in \mathbb{R}$ the set $\{\omega \in S^{d-1} | \sigma_{loc}(\omega) > \alpha\}$ is open by definition of σ_{loc} (allowing $\sigma_{loc}(\omega) = \infty$). Thus again by definition, σ_{loc} is lower semi-continuous. Suppose $\sigma_{loc}(\omega_0) = \infty$. By the continuity of ρ we can find an open cone C_{ω_0} containing ω_0 and $\sigma_0 \in (0, \infty)$ so that if $\omega \in S^{d-1} \cap C_{\omega_0}$ then $\rho(\omega) < \sigma_0$. By shrinking C_{ω_0} if necessary, we can assume $e^{\sigma_0|x|}\phi \in L^2(C_{\omega_0})$. Thus $e^{t\rho}\phi \in L^2(C_{\omega_0})$ for all $t \in [0, 1]$. If $\sigma_{loc}(\omega_0) < \infty$, given $t \in [0, 1)$ we can find σ_0 such that $t\sigma_{loc}(\omega_0) < \sigma_0$ with $e^{\sigma_0|x|}\phi \in L^2(C_{\omega_0})$ where C_{ω_0} is an open cone containing ω_0. By assumption $t\rho(\omega_0) < \sigma_0$. Thus by continuity there is a smaller open cone $\tilde{C}_{\omega_0} \ni \omega_0$ so that $t\rho(\omega) < \sigma_0$ for $\omega \in \tilde{C}_{\omega_0}$ which implies $e^{t\rho}\phi \in L^2(\tilde{C}_{\omega_0})$. The result then follows by the compactness of S^{d-1} and a covering argument.

□

As we will see in the next section, if ϕ is an eigenfunction of $H = Q(p) + V(x)$ with eigenvalue λ, under favorable conditions we will be able to calculate the possible values of $\sigma_s(\omega)$ from our knowledge of $Q(\xi)$ and the eigenvalue λ. We do not have a direct method of calculating $\sigma_{loc}(\omega)$. Thus it is important to know when $\sigma_{loc}(\omega) = \sigma_s(\omega)$.

We call the (affine) hyperplane, $0 = (\eta - \eta_0) \cdot \omega_0$, with parameters $(\omega_0, \eta_0) \in S^{d-1} \times \partial\mathcal{E}$, a supporting hyperplane if $(\eta - \eta_0) \cdot \omega_0 \leq 0$ for all $\eta \in \bar{\mathcal{E}}$. Every point $\eta_0 \in \partial\mathcal{E}$ has at least one supporting hyperplane ([Ro], p.100). Note that by definition of σ_s, if the hyperplane with parameters (ω_0, η_0) is a supporting hyperplane, $\sigma_s(\omega_0) = \eta_0 \cdot \omega_0$. If there is a unique supporting hyperplane passing through $\eta_0 \in \partial\mathcal{E}$ we call η_0 a *regular point* of $\partial\mathcal{E}$. Otherwise we refer to $\eta_0 \in \partial\mathcal{E}$ as a *singular point.* If η_0 is a regular point then $\partial\mathcal{E}$, parametrized by $\eta = \sigma_c(\omega)\omega$ with $\omega \in S^{d-1}$, is differentiable at η_0. (Using the coordinates of some plane through the origin of dimension $d - 1$, $\partial\mathcal{E}$ can be written as the graph of a convex function f. The function f is differentiable at a point x_0 if and only if f has a unique subgradient at x_0 ([Ro], p. 242). This is the same as saying that $\bar{\mathcal{E}}$ has a unique supporting hyperplane at $(x_0, f(x_0))$. Note that from Theorem 2.1, $\partial\mathcal{E}$ is Lipschitz, so that by Rademacher's theorem it is given locally by a function differentiable almost everywhere.) Note also that if all points in $\partial\mathcal{E}$ are regular, then $\partial\mathcal{E}$ is C^1 ([Ro], p. 246).

Theorem 2.2. *Suppose $\omega_0 \in S^{d-1}$ is given so that for* some *regular point $\eta_0 \in \partial\mathcal{E}$ the hyperplane with parameters (ω_0, η_0) is a supporting hyperplane. Then*

$$\sigma_{loc}(\omega_0) = \sigma_s(\omega_0) = \eta_0 \cdot \omega_0. \tag{2.2}$$

Proof. We have $\sigma_s(\omega_0) = \eta_0 \cdot \omega_0$. Suppose $\sigma_{loc}(\omega_0) > \sigma_s(\omega_0)$, then we will obtain a contradiction. First note that $e^{t_0\eta_0 \cdot \omega_0|x|}\phi \in L^2(C_{\omega_0})$ for some open cone C_{ω_0} containing ω_0 and some $t_0 > 1$. The continuity of $\omega \mapsto \eta_0 \cdot \omega$ implies that by choosing $t_0 > 1$ smaller if necessary we can assume $e^{t_0\eta_0 \cdot x}\phi \in L^2(C_{\omega_0})$.

Now let $\theta \in S^{d-1}$ with $\theta \neq \omega_0$. By definition of $\sigma_s(\theta)$ we have $\eta_0 \cdot \theta \leq \sigma_s(\theta)$. If we have equality, then both of the hyperplanes with parameters (ω_0, η_0) and (θ, η_0) are supporting at η_0 contradicting the assumption that η_0 is a regular point. Thus $\eta_0 \cdot \theta < \sigma_s(\theta)$ for all $\theta \neq \omega_0$.

Given a unit vector θ in the complement of C_{ω_0}, it follows that there is an $\eta \in \mathcal{E}$ so that $\eta_0 \cdot \theta < \eta \cdot \theta \le \sigma_s(\theta)$. Since $\mathrm{e}^{\eta \cdot x}\phi \in L^2$ there is an open cone C_θ containing θ and a $t_\theta > 1$ such that $\mathrm{e}^{t_\theta \eta_0 \cdot x}\phi \in L^2(C_\theta)$. Hence by a covering argument $\mathrm{e}^{u\eta_0 \cdot x}\phi \in L^2(\mathbb{R}^d)$ with $u > 1$ contradicting the fact that $\eta_0 \in \partial\mathcal{E}$.

□

Corollary 2.3. *Suppose $\mathcal{E}$ is bounded and $\partial\mathcal{E}$ is C^1. Then $\sigma_{loc}(\omega) = \sigma_s(\omega)$ for all $\omega \in S^{d-1}$.*

Remark 2.4.

1) Notice the emphasis on the word "some" in Theorem 2.2. The point $\eta_0 \in \partial\mathcal{E}$ in that theorem may not be unique and there may be singular points and regular points which all satisfy $\eta_0 \cdot \omega_0 = \sigma_s(\omega_0)$. It is easy to show that if $\bar{\mathcal{E}}$ is strictly convex then $\sigma_s(\omega_0) = \eta(\omega_0) \cdot \omega_0$ for a unique $\eta(\omega_0) \in \partial\mathcal{E}$. However we will have no need to assume strict convexity.
2) See Section 4 for an operator $H = Q(p) + V(x)$ and a corresponding eigenfunction with a real eigenvalue $\lambda \notin \mathrm{Ran}Q$ such that the assumption of Theorem 2.2 is fulfilled for some values of ω_0 while for other values of ω_0 the conclusion of the theorem is false, that is $\sigma_{loc}(\omega) > \sigma_s(\omega)$ for some ω.

3. Calculating the decay rate, $H\phi = \lambda\phi$

In this section we assume that ϕ is an eigenfunction of H with eigenvalue λ. We assume that the global decay rate, σ_g, of ϕ is positive. We cannot completely eliminate the possibility that for some $\omega, \sigma_c(\omega) = \infty$ (unless $d = 1$), but the next result limits the size of the set where this might occur. See Theorem 3.4 for a very different result which under unrelated assumptions shows $\sigma_c(\omega) < \infty$ for all $\omega \in S^{d-1}$.

Proposition 3.1. *If $d = 1$ and $\phi \neq 0, \sigma_c(\pm 1) < \infty$ as long as V is bounded. If $d \ge 2$, under the hypotheses of Theorem 1.2 the set $\{\omega \in S^{d-1} | \sigma_c(\omega) = \infty\}$ lies in a hyperplane containing 0 unless $\phi = 0$.*

Proof. We use the notation $\langle x \rangle = (|x|^2 + 1)^{1/2}$. If $d = 1$ suppose $\sigma_c(1) = \infty$. Let $\phi_\sigma(x) = \mathrm{e}^{\sigma x}\phi(x)$. Then $(Q(p + \mathrm{i}\sigma) - \lambda)\phi_\sigma = -V\phi_\sigma$. $Q(z)$ has finitely many zeros so that $\lim_{\sigma\to\infty} \|(Q(p+\mathrm{i}\sigma)-\lambda)^{-1}V\| = 0$. Since for large σ, $\phi_\sigma = -(Q(p+\mathrm{i}\sigma)-\lambda)^{-1}V\phi_\sigma$, we obtain $\phi_\sigma = 0$. Suppose $d \ge 2$ and that $\sigma_c(\omega_j) = \infty$ for a set of linearly independent vectors $\omega_1, \cdots, \omega_d$. Since $\mathcal{E}$ is convex, there is an open cone C so that $\sigma_c(\omega) = \infty$ for $\omega \in C \cap S^{d-1}$. Choose $\omega_0 \in C \cap S^{d-1}$. Then for small enough $\delta > 0$, if $f(x) = \delta r + \omega_0 \cdot x$, we have $\mathrm{e}^{\sigma f}\phi \in L^2$ for all $\sigma > 0$. Here we take $r = r_\epsilon$, $r_\epsilon = \langle x \rangle - \langle x \rangle^{1-\epsilon} + 1$ as in [HS], because of its good convexity properties. The parameter $\epsilon > 0$ will be taken very small at the end of the proof. As in [HS], we let $\phi_\sigma = \mathrm{e}^{\sigma f}\phi$ and $a = p - \mathrm{i}\sigma\nabla f(x)$ and note that $(Q(a^*) + V_1 - \lambda)\phi_\sigma = -V_2\phi_\sigma$. Taking norms of both sides of this equation gives

$$\langle \phi_\sigma, ([Q(a), Q(a^*)] + |Q(a) + V_1 - \lambda|^2)\phi_\sigma \rangle = \langle \phi_\sigma, (2\mathrm{Re}\,[V_1, Q(a)] + |V_2|^2)\phi_\sigma \rangle.$$

The only properties of a and a^* which were used to prove Theorem 1.4 in [HS] are the form of the commutator $[a_j, a_k^*]$ and the form of $[a_j, V_1]$ which are virtually the same in the present situation: $p_{jk} = [a_j, a_k^*] = 2\sigma\delta\partial_j\partial_k r$ (and thus after a calculation $(p_{jk}) \ge c\sigma r^{-1-\epsilon}$). Similarly $[a_j, V_1] = -\mathrm{i}\partial_j V_1$ is the same as in [HS]. Thus the proof of Theorem 1.4 in [HS] works exactly in the same way to give the desired result after ϵ is chosen small enough (see [HS]). □

Note that if $\sigma_c(\omega_0) = \infty$, then $\sigma_{loc}(\omega) = \infty$ in the open half sphere $\{\omega \in S^{d-1} | \omega \cdot \omega_0 > 0\}$. We have not found examples of this phenomenon in the case where ϕ is an eigenfunction of $H = Q(p) + V(x)$ with $V(x) = o(1)$ at infinity and $\sigma_g < \infty$.

We now embark on a program to calculate the possibilities for $\sigma_c(\cdot)$. We assume as above that ϕ is an eigenfunction of H with eigenvalue λ and $\sigma_g > 0$. Our first result can put some restrictions on the pairs $(\lambda, \sigma_c(\omega))$.

Proposition 3.2. *Suppose $\omega_0 \in S^{d-1}$ with $\sigma_c(\omega_0) < \infty$. Suppose $V = o(1)$ at infinity. Then for some $\xi \in \mathbb{R}^d$*

$$Q(\xi + \mathrm{i}\sigma_c(\omega_0)\omega_0) = \lambda. \tag{3.1}$$

Proof. Abbreviate $\sigma_c(\omega_0) = \sigma_0$ and use $r = \langle x \rangle$. For $\epsilon > 0$ we consider

$$f(x) = (\sigma_0 - \epsilon)\omega_0 \cdot x + 2\epsilon r \text{ and } f_n = (\sigma_0 - \epsilon)\omega_0 \cdot x + 2\epsilon r/(1 + r/n),\ n \in \mathbb{N},$$

and $\phi_n = \mathrm{e}^{f_n}\phi$. We will show that unless (3.1) is satisfied for some ξ, $\|\phi_n\| \leq C$ with a constant C independent of n provided $\epsilon > 0$ is chosen small enough. Taking $n \to \infty$ yields $\mathrm{e}^f\phi \in L^2$ which is contradiction since $f(x) \geq (\sigma_0 + \epsilon)\omega_0 \cdot x$.

We introduce the notation of [HS]

$$X = \mathrm{Re}(Q(\xi + \mathrm{i}\nabla f_n(x)) - \lambda) \text{ and } Y = \mathrm{Im}Q(\xi + \mathrm{i}\nabla f_n(x)).$$

Suppose (3.1) does not have a solution. Then by a continuity and compactness argument and the fact that $|\nabla f_n(x) - \sigma_0\omega_0| \leq 3\epsilon$, we obtain

$$X^2 + Y^2 = |Q(\xi + \mathrm{i}\nabla f_n(x)) - \lambda|^2 \geq 2\kappa \text{ for some small } \kappa > 0.$$

Obviously here we needed $\epsilon > 0$ small.

Next we use the localization symbols $\chi_- = \chi(X^2 + Y^2 \leq \kappa)$ and $\chi_+ = \chi(X^2 + Y^2 \geq \kappa)$ of [HS] as well as their quantizations $\tilde{\chi}_\mp$, respectively. By construction $\tilde{\chi}_- = 0$, and whence by [HS, (4.9)] we have $I \leq \tilde{\chi}_+^2 + C/r^2$. Using the estimate $\|\tilde{\chi}_+\phi_n\|^2 \leq C(\|V\phi_n\|^2 + \|r^{-1/2}\phi_n\|^2)$ from Lemma 4.3 of [HS] we obtain

$$\|\phi_n\|^2 \leq C(\|V\phi_n\|^2 + \|r^{-1/2}\phi_n\|^2),$$

which easily leads to $\|\phi_n\| \leq C$ as desired. Here we mention that although [HS, Lemma 4.3] is stated only for $f_n(x) = r(\sigma + \gamma/(1 + r/n))$, for certain values of σ and γ, the proof given there works with minor modifications for our f_n. □

Remark 3.3. According to Propositions 3.1 and 3.2 if $d = 1, \sigma_g > 0$, and $V(x) = o(1)$ at infinity, then the possible decay rates $\sigma = \sigma_c(\pm 1)$ can be calculated from the equation $Q(\xi + \mathrm{i}\sigma) = \lambda$. Note that the reality condition shows that the totality of decay rates calculated from $Q(\xi + \mathrm{i}\sigma) = \lambda$ at $+\infty$ is the same as that at $-\infty$. In fact it is easy to see that if σ_1 and σ_2 are two positive solutions to this equation then there is a (complex) smooth compact support V and a smooth nonzero ϕ with decay rate σ_1 at $+\infty$ and decay rate σ_2 at $-\infty$ such that $(Q(p) + V - \lambda)\phi = 0$.

In the following we assume $d \geq 2$.

Our main result is the following theorem:

Theorem 3.4. *Suppose $(H - \lambda)\phi = 0, 0 < \sigma_g < \infty$ and V satisfies*

$$\forall \alpha : \ \partial^\alpha V_1(x) = o(|x|^{-|\alpha|}), \tag{3.2}$$

$$V_2(x) = o(|x|^{-1/2}). \tag{3.3}$$

For $\omega_0 \in S^{d-1}$ with $\sigma_0 := \sigma_c(\omega_0) < \infty$ let $\eta_0 = \sigma_0\omega_0$ and $\hat{C}_0 = \{\hat{x} \in S^{d-1} | \sigma_s(\hat{x}) = \eta_0 \cdot \hat{x}\}$. For any such ω_0 there exists $(\xi, \theta, \beta) \in \mathbb{R}^d \times \hat{C}_0 \times \mathbb{C}$ solving the pair of equations

$$Q(\xi + \mathrm{i}\eta_0) = \lambda, \tag{3.4a}$$

$$\nabla Q(\xi + \mathrm{i}\eta_0) = \beta\theta. \tag{3.4b}$$

If the set of η_0's which occur in the set of all solutions $(\xi, \theta, \beta, \eta_0) \in \mathbb{R}^d \times S^{d-1} \times \mathbb{C} \times \mathbb{R}^d$ to the pair of equations (3.4a) and (3.4b) is bounded, then $\sigma_c(\omega) < \infty$ for all $\omega \in S^{d-1}$.

Remarks 3.5.

1) There may be spurious solutions to the system of equations (3.4a) and (3.4b) which do not describe the exponential decay of an eigenfunction. This may happen for the finite set of exceptional eigenvalues λ which arises in rotationally invariant Q (see Theorem 3.6 below) and it happens for the example in Section 4. Both of these problems can be (at least partially) traced to the fact that the spectral parameter λ is a critical value of Q. It is well known that the set of critical values of $Q : \mathbb{C}^d \to \mathbb{C}$ is finite. In fact the number of these critical values can be bounded by $(q-1)^d$ (see [BR]).
2) Assume $\lambda \in \mathbb{R}$ is not such a critical value. Let us choose $\theta \in S^{d-1}$ and assume that there is a solution to the system of equations (3.4a) and (3.4b). We are interested in the set of $\eta = \mathrm{Im} z$ such that $\eta \cdot \theta$ is stationary with respect to variations of $z = \xi + \mathrm{i}\eta \in M = \{z | Q(z) = \lambda\}$. The vectors $\nabla_{(\xi,\eta)}(\mathrm{Re}Q)(\xi, \eta)$ and $\nabla_{(\xi,\eta)}(\mathrm{Im}Q)(\xi, \eta)$ are linearly independent by the Cauchy-Riemann equations. Introducing the Lagrange multipliers γ_1 and γ_2 and setting the derivatives of $\eta \cdot \theta + \gamma_1 \mathrm{Re}Q(z) + \gamma_2 \mathrm{Im}Q(z)$ with respect to ξ and η equal to zero we find that in fact $\eta \cdot \theta$ is indeed stationary at a point $z = \xi + \mathrm{i}\eta$ which solves (3.4a) and (3.4b) . Given the existence of the set $\mathcal{E}$, the meaning of θ is that of a unit vector perpendicular to a supporting hyperplane to $\bar{\mathcal{E}}$ at the point $\eta \in \partial\mathcal{E}$. Thus for $\eta' \in \bar{\mathcal{E}}$, $\eta' \cdot \theta$ has a global maximum or minimum at the point $\eta' = \eta$.
3) Consider the set $\mathcal{E}$ corresponding to an eigenfunction ϕ satisfying the assumptions of Theorem 3.4. For each point η in the boundary of $\mathcal{E}$ there must be a corresponding solution to (3.4a) and (3.4b). We must be able to put together a function $\eta = \sigma(\omega)\omega$ ($\omega = \eta/|\eta|, \sigma(\omega) = |\eta|$) from the (multiplicity of) solutions to (3.4a) and (3.4b) which satisfies the requirements coming from the convexity of $\mathcal{E}$ and the (related) Lipschitz continuity of $1/\sigma(\omega)$. If there is no such function then there is no such eigenfunction (see Remark 1.6 (4)) in [HS]). And clearly if the only such functions $\sigma(\omega)$ are bounded then $\sigma_c(\omega)$ corresponding to ϕ must be bounded. This generalizes a statement in Theorem 3.4.
4) Clearly Theorems 1.3 and 3.4 have a similar nature. Their proofs are also similar (partly explaining why the conditions on V are the same) although there are additional ideas necessary in the present paper. As noted in [HS] the proof of Theorem 1.3 is rather robust and applies with modifications to certain elliptic variable coefficient differential operators and even certain pseudodifferential operators with elliptic symbol being uniformly real-analytic in the ξ-variable assuming σ_g for the given eigenfunction is smaller than the

uniform analyticity radius, say denoted σ_a. The same can be said for Theorem 3.4 under the stronger condition $\sigma_c(\omega_0) < \sigma_a$ on the eigenfunction. For example our proof works for the symbol $(|\xi|^2 + s^2)^{1/2} + V(x)$ assuming $0 < \sigma_g \le \sigma_c(\omega_0) < s$.

We defer the proof of Theorem 3.4. We have the following corollary for rotationally invariant Q.

Theorem 3.6. *Suppose $(H - \lambda)\phi = 0$, V is as in Theorem 3.4, and $0 < \sigma_g < \infty$. Suppose Q is rotation invariant. Define the polynomial G of degree $q/2$ so that $G(\xi^2) = Q(\xi)$. We assume all the zeros of $G - \lambda$ have multiplicity one. (There are at most $\frac{q}{2} - 1$ values of λ for which this is not the case.) Then there are at most $q/2$ positive numbers σ_0 (being independent of $\omega_0 = \eta_0/|\eta_0|$) for which there is a solution to the pair of equations (3.4a) and (3.4b) with $|\eta_0| = \sigma_0$, and σ_g is one of them. In addition, $\sigma_{loc}(\omega) = \sigma_g$ for all $\omega \in S^{d-1}$.*

Proof. From the rotation invariance, (3.4a) and (3.4b) reduce to $G(z \cdot z) = \lambda$ and $2G'(z \cdot z)z = \beta\theta, z = \xi + \mathrm{i}\sigma_0\omega_0$. Our assumptions imply $z = \beta'\theta$ for some $\beta' \in \mathbb{C}$ and thus we have $z = (\alpha + \mathrm{i}\sigma_0)\omega_0$ with $\alpha \in \mathbb{R}$. It follows that the set of σ_0's which may occur is bounded. In fact the set of such positive σ_0's consists of at most $q/2$ constants independent of ω_0 and according to Theorem 1.3 σ_g is one of them. From the continuity of $1/\sigma_c(\omega)$, see Theorem 2.1, and the fact that S^{d-1} is connected it follows that $\sigma_c(\omega) = \sigma$ for some $\sigma \in (0, \infty)$ independent of ω. Whence $B_\sigma(0) \subset \mathcal{E} \subset \bar{B}_\sigma(0)$, which in turn by Corollary 2.3 implies that $\sigma = \sigma_c(\omega) = \sigma_{loc}(\omega)$ and therefore that $\sigma_{loc}(\omega) = \sigma_g$.

□

The main work of this section is in the next proposition which needs modified constructions defined as follows in terms of a large parameter m:

For a given $\phi \in L^2$ with $0 < \sigma_g < \infty$ and a given integer $m > 1/\sigma_g$ we replace the quantities $\mathcal{E}, \sigma_c$ and σ_s of Section 2 by $\mathcal{E}^m, \sigma_c^m$ and σ_s^m, respectively, given by replacing L^2 by $L^2_m(\mathbb{R}^d) = \mathrm{e}^{-r/m}L^2(\mathbb{R}^d)$ in the definitions in Section 2. Here and henceforth $r = \langle x\rangle$. Alternatively, this amounts to the old quantities with ϕ replaced by $\mathrm{e}^{r/m}\phi$. Whence by Proposition 2.1 we obtain that $\mathcal{E}^m$ is convex containing some ball, $1/\sigma_c^m$ is Lipschitz and $\partial\mathcal{E}^m = \{\sigma_c^m(\omega)\omega | \omega \in S^{d-1}, \sigma_c^m(\omega) < \infty\}$. Moreover for any $\omega_0 \in S^{d-1}$ with $\sigma_0 := \sigma_c(\omega_0) < \infty$ we can bound

$$\tfrac{1}{m} \le \sigma_0 - \sigma_c^m(\omega_0) \le \tfrac{1}{m}\tfrac{\sigma_0}{\sigma_g}, \tag{3.5}$$

which by Rademacher's theorem allows us to find a sequence $\eta^m = \sigma_c^m(\omega^m)\omega^m$ of regular points in $\partial\mathcal{E}^m$ with $\eta^m \to \eta_0 := \sigma_0\omega_0$ for $m \to \infty$. To obtain the second inequality in (3.5) note that if $0 < \sigma < \sigma_0(1 - (m\sigma_g)^{-1})$ then if $\omega_0 \cdot x/|x| \ge \sigma_g/\sigma_0$ we have $\sigma\omega_0 \cdot x/|x| + 1/m < \sigma_0\omega_0 \cdot x/|x|$ while if $\omega_0 \cdot x/|x| < \sigma_g/\sigma_0$ then $\sigma\omega_0 \cdot x/|x| + 1/m < \sigma_g$.

Proposition 3.7. *Suppose $(H - \lambda)\phi = 0, 0 < \sigma_g < \infty$, $m > 1/\sigma_g$, $\omega_0^m \in S^{d-1}$ with $\sigma_0^m := \sigma_c^m(\omega_0^m) < \infty$ and that V is as in Theorem 3.4. Let $\eta_0^m = \sigma_0^m\omega_0^m$ and $\hat{C}_0^m = \{\hat{x} \in S^{d-1} | \sigma_s^m(\hat{x}) = \eta_0^m \cdot \hat{x}\}$. Suppose η_0^m is a regular point of $\partial\mathcal{E}^m$ so that $\hat{C}_0^m$ consists of only one point, say θ_0^m. Then there exists $(\xi, \beta) \in \mathbb{R}^d \times \mathbb{C}$ solving the pair* *of equations*

$$Q(\xi + \mathrm{i}(\eta_0^m + \theta_0^m/m)) = \lambda, \tag{3.6a}$$

$$\nabla Q(\xi + \mathrm{i}(\eta_0^m + \theta_0^m/m)) = \beta\theta_0^m. \tag{3.6b}$$

Proof. We drop the superscript m. So fix $\omega_0 \in S^{d-1}$ with $\sigma_0 = \sigma_c(\omega_0) < \infty$.

Let

$$\Delta_1 = \max\{|\nabla Q(\xi + \mathrm{i}(\eta_0 + \theta_0/m))|^2 |\, \xi \in \mathbb{R}^d, Q(\xi + \mathrm{i}(\eta_0 + \theta_0/m)) = \lambda\}.$$

Note that indeed (3.6a) has a solution, cf. Proposition 3.2.

Letting $P_\perp(\theta)u = u - (u \cdot \theta)\theta$ we introduce

$$\delta_1 = \min\{|P_\perp(\theta_0)\nabla Q(\xi + \mathrm{i}(\eta_0 + \theta_0/m))|^2 |\, \xi \in \mathbb{R}^d, Q(\xi + \mathrm{i}(\eta_0 + \theta_0/m)) = \lambda\}.$$

We will show that $\delta_1 = 0$ proceeding by the way of contradiction. The contradiction if $\delta_1 > 0$ will arise by showing that $\mathrm{e}^{s\eta_0 \cdot x}\phi \in L^2_m(\mathbb{R}^d)$ for some $s > 1$. So suppose $\delta_1 > 0$.

Step I (Construction of phases.) Consider for (small) $\epsilon > 0$

$$\begin{aligned} f(x) &= (\sigma_0 - \epsilon)\omega_0 \cdot x + r/m, \\ f_n(x) &= f(x) + 2\epsilon r/(1 + r/n),\ n \in \mathbb{N}, \\ F(x) &= f(x) + 2\epsilon r. \end{aligned}$$

Note that $\phi_n := \mathrm{e}^{f_n}\phi \in L^2(\mathbb{R}^d)$. We will show that $\|\phi_n\| \le K$ with a constant K independent of n provided $\epsilon > 0$ is chosen small enough. Taking $n \to \infty$ yields $\mathrm{e}^F\phi \in L^2$ which is a contradiction since $F(x) \ge (\sigma_0 + \epsilon)\omega_0 \cdot x + r/m$. A necessary smallness condition on ϵ is

$$\delta_1/m > 2\epsilon\Delta_1. \tag{3.7}$$

Step II (Role of (3.7), convexity.) Noting that $\partial_i r = x_i/r$ we can compute ∇f_n and then estimate

$$|(\xi + \mathrm{i}(\eta_0 + \tfrac{x/r}{m}) - (\xi + \mathrm{i}\nabla f_n)| \le 3\epsilon.$$

If $Q(\xi + \mathrm{i}\nabla f_n) \approx \lambda$ this will for small ϵ allow us to exploit the positivity of δ_1 in a phase-space argument. More precisely we claim that there is an open cone $\tilde{C}_0 \supset C_0 := \mathbb{R}_+\theta_0 = \{c\theta_0 |\, c > 0\}$ so that the symbol

$$b_n(x,\xi) = \sum_{i,j} r\overline{\partial_i Q(\xi + \mathrm{i}\nabla f_n)}(\partial_j\partial_i f_n)\partial_j Q(\xi + \mathrm{i}\nabla f_n) \tag{3.8}$$

has a positive lower bound for $x \in \tilde{C}_0$ with $|x| \ge R$ and for $|Q(\xi + \mathrm{i}\nabla f_n) - \lambda|^2 \le 2\kappa$ provided $R^{-1}, \kappa, \epsilon > 0$ are small enough. The bound is uniform in n, x, ξ. This follows from the computations

$$\begin{aligned} \partial_j\partial_i(r/(1 + r/n)) &= (1 + r/n)^{-2}\partial_j\partial_i r - 2(r/n)(1 + r/n)^{-3}\partial_j r\partial_i r/r, \\ \partial_j\partial_i r &= (\delta_{ij} - x_j x_i r^{-2})/r. \end{aligned}$$

Note that the non-convex part $-4\epsilon(1/n)(1+r/n)^{-3}|x/r\rangle\langle x/r|$ of the Hessian $(\partial_j\partial_i f_n)$ has the lower bound $-2\epsilon I/r$, while the convex part has the lower bound $m^{-1}(I - |x/r\rangle\langle x/r|)/r$. Whence for x in a small open cone $\tilde{C}_0 \supset C_0$ and $R^{-1}, \kappa, \epsilon > 0$ small indeed we obtain a lower bound of the above form $b_n(x,\xi) \ge c_1$ where the constant c_1 can be chosen as close to $c_2 := -2\epsilon\Delta_1 + \delta_1/m$ as desired. The positivity of c_2 is exactly (3.7). In our application we may for convenience choose $c_1 = \frac{\delta_1}{2m}$ and consider only, say $\epsilon \le \frac{\delta_1}{8m\Delta_1}$. This allows us to consider $\tilde{C}_0$ as being independent of the parameters $R^{-1}, \kappa, \epsilon > 0$ provided they are small. Fix such a $\tilde{C}_0$.

Step III (Bounding on the complement of C_0.)

Note that $\mu(\hat{x}) := \sigma_s(\hat{x}) - \eta_0 \cdot \hat{x}$ is lower semi-continuous on S^{d-1}. Whence on any closed cone $C \subset \mathbb{R}^d \setminus C_0$ (for example $C = \mathbb{R}^d \setminus \tilde{C}_0$) there exists

$$\mu_C := \min\{\mu(x/|x|) |\, 0 \ne x \in C\} > 0,$$

and for $3\epsilon < \mu_C$ another compactness argument shows that $\mathrm{e}^F \phi \in L^2(C)$. We put this result in a more convenient form: For any smooth function χ_C on $\mathbb{R}^d$ taken homogeneous of degree zero for $|x| \ge 1$, $\chi_C(x) = 0$ in a neighbourhood of C_0, $\chi_C(x) = 0$ for $|x| \le 1/2$, and with $\chi_C(x) = 1$ for $|x| \ge 1$ and x outside another such neighbourhood,

$$\sup_n \|\chi_C \phi_n\| < \infty \text{ for small } \epsilon. \tag{3.9}$$

Step IV (Implementation of a scheme from [HS].) Consider the symbol $b_n = r\{X, Y\}$ (the Poisson bracket) where $X = \mathrm{Re}(Q(\xi + \mathrm{i}\nabla f_n(x)) - \lambda)$ and $Y = \mathrm{Im} Q(\xi + \mathrm{i}\nabla f_n(x))$. This is given by (3.8). We will freely use other notation from [HS], in particular the localization symbols $\chi_\mp$ and their quantizations $\tilde{\chi}_\mp$ also used in the proof of Proposition 3.2 (now with a different f_n but again in terms of a small parameter $\kappa > 0$). Pick any smooth function χ_C as in Step III with the property that if $\chi_{C_0}^2(x) := 1 - \chi_C^2(x) \ne 0$ then either $|x| < 1$ or $x \in \tilde{C}_0$. Of course we are going to use (3.9) as well as the lower bound of Step II. At this point we can consider the parameters $R^{-1}, \kappa, \epsilon > 0$ of Steps II and III as fixed (small), and with $c := c_1/3$ we conclude that $(b_n - 3c)\chi_-^2 \chi_{C_0}^2 \ge 0$ for $|x| \ge R$.

Noting also the uniform estimate $(b_n - 3c)\chi_-^2 \chi_C^2 \ge -K_1 \chi_C^2$ we can then mimic [HS, Section 7] and obtain with $A_c := \mathrm{Op}^{\mathrm{w}}(rY)$ and $\tilde{X} + \mathrm{i}\tilde{Y} := Q(p + \mathrm{i}\nabla f_n) + V - \lambda$:

$$\begin{aligned}
&2\mathrm{Im}(A_c(\tilde{X} + \mathrm{i}\tilde{Y}) \ge \\
&2c + \tilde{Y} r \tilde{Y} - K_1 \chi_C^2 - K_2 \tilde{\chi}_+ \langle p \rangle^{2q} \tilde{\chi}_+ + (c - K_3 r^{-1/2} \langle p \rangle^{2q} r^{-1/2}) \\
&\ge 2c + \tilde{Y} r \tilde{Y} - K_1 \chi_C^2 - K_4 \tilde{\chi}_+ \langle p \rangle^{2q} \tilde{\chi}_+ - K_5 \chi(r \le N) \langle p \rangle^{2q} \chi(r \le N).
\end{aligned}$$

In the last step we used a slightly modified version of [HS, Lemma 4.4]. Taking the expectation in the state ϕ_n and using a slightly modified version of [HS, Lemma 4.3] we get

$$\begin{aligned}
2c\|\phi_n\|^2 \le & -2\mathrm{Im}\,\langle \phi_n, A_c V \phi_n \rangle - \|r^{1/2} \tilde{Y} \phi_n\|^2 + K_1' \|V \phi_n\|^2 + \\
& K_1' \|r^{-1/2} \phi_n\|^2 + K_2' \|\langle p \rangle^q \phi\|^2 + K,
\end{aligned} \tag{3.10}$$

where $K = \sup_n K_1 \|\chi_C \phi_n\|^2$.

Taking into account (3.2) and (3.3) and the fact $A_c r^{-1} A_c \le \tilde{Y} r \tilde{Y} + K' r^{-1/2} \langle p \rangle^{2q} r^{-1/2}$ (and by invoking again [HS, Lemmas 4.3 and 4.4]) we estimate

$$\begin{aligned}
&-\mathrm{i}\langle \phi_n, [V_1, A_c] \phi_n \rangle \\
&\le \delta \|\langle p \rangle^q \phi_n\|^2 + K_1 \|\langle p \rangle^q \chi(r \le N) \phi_n\|^2 + K_2 \|\langle p \rangle^q r^{-1/2} \phi_n\|^2 \\
&\le \delta' \|\phi_n\|^2 + K_3 \|\phi\|^2,
\end{aligned}$$

and

$$
\begin{aligned}
&- 2\mathrm{Im}\,\langle A_c\phi_n, V_2\phi_n\rangle \\
&\le \|r^{1/2}V_2\phi_n\|^2 + \|r^{-1/2}A_c\phi_n\|^2 \\
&\le \delta\|\phi_n\|^2 + \|r^{-1/2}A_c\phi_n\|^2 + K_4\|\phi\|^2 \\
&\le \delta'\|\phi_n\|^2 + \|r^{1/2}\tilde{Y}\phi_n\|^2 + K_5\|\phi\|^2.
\end{aligned}
$$

We insert these estimates with δ' chosen smaller than $c/2$ into (3.10) and obtain finally the uniform bound

$$c\|\phi_n\|^2 \le \text{ constant},$$

accomplishing the goal of Step I. □

Proof of Theorem 3.4. There exists a sequence $\eta^m = \sigma_c^m(\omega^m)\omega^m$ of regular points in $\partial\mathcal{E}^m$ with $\eta^m \to \eta_0 = \sigma_0\omega_0$ for $m \to \infty$, cf. the discussion before Proposition 3.7. For all elements of this sequence this proposition applies and the equations (3.6a) and (3.6b) are satisfied. Using the ellipticity of Q and by going to a subsequence if necessary we can assume $(\xi^m, \theta^m, \beta^m, \eta^m) \to (\xi, \theta, \beta, \eta_0)$ which by the continuity of Q and ∇Q provides a solution to the equations (3.4a) and (3.4b). Since $\sigma_s^m(\theta^m) = \eta^m \cdot \theta^m$ we can by taking the limit show that $\sigma_s(\theta) = \eta_0 \cdot \theta$: For given $\epsilon, R > 0$ we have for all large m and all $\eta \in \mathcal{E}^m$ with $|\eta| \le R$

$$\eta_0 \cdot \theta + \epsilon \ge \eta^m \cdot \theta^m \ge \eta \cdot \theta^m \ge \eta \cdot \theta - R\epsilon.$$

Taking $\epsilon \to 0$ yields

$$\eta_0 \cdot \theta \ge \eta \cdot \theta \text{ for all } \eta \in \mathcal{E}^m \subset \mathcal{E} \text{ with } |\eta| \le R.$$

Then taking $m, R \to \infty$ using (3.5) and the (related) fact that $\sigma_c^m(\omega) = \infty$ if $\sigma_c(\omega) = \infty$ we obtain that $\sigma_s(\theta) \le \eta_0 \cdot \theta$. Obviously $\eta_0 \cdot \theta \le \sigma_s(\theta)$, so $\sigma_s(\theta) = \eta_0 \cdot \theta$ is proven.

The second result follows from the continuity of $1/\sigma_c(\omega)$, see Theorem 2.1, and the fact that S^{d-1} is connected. □

4. An example, $\sigma_{loc} \neq \sigma_s$

In this section we consider for $\epsilon \in (0, 1/2)$ the polynomial

$$Q(\xi) = |\xi|^4 + 2\epsilon\xi_d + \epsilon^2\xi_d^2$$

in dimension $d \ge 2$. A crude estimate gives $Q(\xi) \ge -2\epsilon^{4/3}$. We take $\lambda = -1$ so that $\lambda < \inf \sigma(Q(p))$, and we note

$$Q(\xi) - \lambda = (|\xi|^2 + \mathrm{i}(1 + \epsilon\xi_d))(|\xi|^2 - \mathrm{i}(1 + \epsilon\xi_d)). \tag{4.1}$$

We first solve the system

$$
\begin{aligned}
Q(\xi + \mathrm{i}\sigma\omega) &= \lambda, \\
\nabla Q(\xi + \mathrm{i}\sigma\omega) &= \beta\theta
\end{aligned}
$$

for $\sigma > 0$ given $\omega \in S^{d-1}$.

The result is that for $\omega_d \neq 0$

$$\sigma = \pm\epsilon\omega_d/2 + \sqrt{\lambda_0^2 - \epsilon^2(1 - \omega_d^2)/4}, \tag{4.2}$$

where $2\lambda_0^2 = (\epsilon/2)^2 + \sqrt{1 + (\epsilon/2)^4}$, $\lambda_0 > 0$.

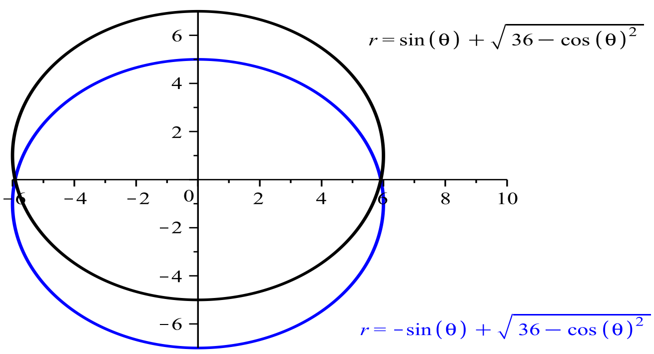
In addition there is another set of solutions which are only valid for $\omega_d = 0$. Namely for $d = 2$, $\sigma = 1/\epsilon$ and for $d \geq 3$, any $\sigma \geq 1/\epsilon$ independent of $\omega \in S^{d-1}$ such that $\omega_d = 0$.

Now suppose $(Q(p) + V + 1)\phi = 0$ for some $V \in C_c^\infty$ and for a nonzero $\phi \in L^2$. Combining the computation (4.2) with our general results we then conclude that $0 < \sigma_g < \infty$ and that $\sigma_c(\omega) < \infty$ for all $\omega \in S^{d-1}$ (note that near $\omega_d = 0$ the choices (4.2) stay well below $1/\epsilon$ so the choice $\sigma \geq 1/\epsilon$ is not relevant). Thus σ must be given by one of (4.2).

Thus there are the following possibilities for *continuous* $\sigma(\omega)$:

1) $\sigma(\omega) = \epsilon\omega_d/2 + \sqrt{\lambda_0^2 - \epsilon^2(1-\omega_d^2)/4}$
2) $\sigma(\omega) = -\epsilon\omega_d/2 + \sqrt{\lambda_0^2 - \epsilon^2(1-\omega_d^2)/4}$
3) $\sigma(\omega) = -\epsilon|\omega_d/2| + \sqrt{\lambda_0^2 - \epsilon^2(1-\omega_d^2)/4}$
4) $\sigma(\omega) = \epsilon|\omega_d/2| + \sqrt{\lambda_0^2 - \epsilon^2(1-\omega_d^2)/4}$

The case 4) cannot actually be $\sigma_c(\omega)$ for the eigenfunction ϕ because it does not describe the boundary of a convex set. For 1) and 2) the set $\partial\mathcal{E}$ is C^1 and Corollary 2.3 applies. For 3) we cannot apply Corollary 2.3 due to the wedge at $\omega_d = 0$ while indeed Theorem 2.2 applies near $\omega_d = \pm 1$ for example. The sets $\partial\mathcal{E}$ are depicted for the cases 1) and 2) for $d = 2$ by polar plots (this is for $2\lambda_0/\epsilon = 6$ and in terms of the unit $\epsilon/2$):



Note that in this picture $\partial\mathcal{E}$ for the case 3) is the union of the (closed) upper blue arch and the (closed) lower black arch, whence there is a wedge at $\omega_d = 0$ (in fact defined by $\sin\psi = \frac{\epsilon}{2\lambda_0} = \frac{1}{6}$ where $\pi - 2\psi$ is the apex angle). In general a computation for case 3) shows that Theorem 2.2 applies if and only if $|\omega_d| > \frac{\epsilon}{2\lambda_0}$ and in this case

$$\sigma_{loc}(\omega) = \sigma_s(\omega) = \lambda_0 - \tfrac{\epsilon}{2}|\omega_d|.$$

If on the other hand $|\omega_d| \leq \frac{\epsilon}{2\lambda_0}$ we compute for case 3)

$$\sigma_s(\omega) = (1-\omega_d^2)^{1/2}\sqrt{\lambda_0^2 - \epsilon^2/4}. \tag{4.3}$$

We present below an example of case 3) where $\sigma_{loc}(\omega) > \sigma_s(\omega)$ for $|\omega_d| < \frac{\epsilon}{2\lambda_0}$.

Let us first note that there are examples of 1) and 2): Indeed (motivated by (4.1)) we take

$$\phi_\pm = \chi + (1-\chi)(p^2 \pm \mathrm{i}(1+\epsilon p_d))^{-1}\delta,$$

where $\chi \in C_c^\infty$, $0 \le \chi \le 1$, χ is 1 in a small neighbourhood of 0 and has small support, and δ is the delta function at 0. If $g_\pm(x)$ denotes the Green's function $(p^2 \pm \mathrm{i} + \epsilon^2/4)^{-1}(x,0)$ then

$$\phi_\pm(x) = \chi(x) + \big(1-\chi(x)\big)\mathrm{e}^{\pm x_d\epsilon/2} g_\pm(x). \tag{4.4}$$

Using properties of $g_\pm(x)$ (see the discussion in [HS, Subsection 1.2] and note that $\sqrt{\mp\mathrm{i} - \epsilon^2/4} = \mathrm{i}\lambda_0 \mp (2\lambda_0)^{-1}$) we deduce that each choice $\phi = \phi_\pm$ fulfills $(Q(p)+V+1)\phi = 0$ for some $V \in C_c^\infty$. The choice ϕ_- is an example of the case 1) while the choice ϕ_+ is an example of the case 2). In general for these cases we have that for all $\omega \in S^{d-1}$

$$\sigma_{loc}(\omega) = \sigma_s(\omega) = \lambda_0 \mp \tfrac{\epsilon}{2}\omega_d,$$

respectively.

Now for an example of case 3), we consider

$$g(x) = (2\pi)^{-d} \int \mathrm{e}^{\mathrm{i}x\cdot\xi}(1+\epsilon\xi_d)(Q(\xi)+1)^{-1} d\xi;\ x \neq 0.$$

It is well-defined and smooth, and introducing as above

$$\phi(x) = \chi(x) + \big(1-\chi(x)\big)g(x), \tag{4.5}$$

for this ϕ we have $\sigma_g \in (0,\infty)$, cf. Paley-Wiener theory. We claim that $(Q(p)+V+1)\phi = 0$ for some $V \in C_c^\infty$ and that this is an example of case 3). Since $g(x) = \overline{g(-x)}$ we have $\sigma_c(\omega) = \sigma_c(-\omega)$ is valid for all ω. This excludes the cases 1) and 2) and we are left with case 3). Whence it remains to construct V. We use the function $g_-(x) = (p^2 - z)^{-1}(x,0)$, $z = \mathrm{i} - \epsilon^2/4$, from above and represent

$$2\mathrm{i}g(x) = \mathrm{e}^{-x_d\epsilon/2} g_-(x) - \mathrm{e}^{x_d\epsilon/2}\overline{g_-(x)} \text{ and } \operatorname{Re} g(x) = \cosh(x_d\epsilon/2)\operatorname{Im} g_-(x).$$

Next we use that $\operatorname{Im}\big((p^2-z)^{-1}(x,0)\big) > 0$ for all $|x| > 0$ small enough (this is valid for any $d \ge 2$ and for any $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$). For example in dimension $d = 3$ explicitly for $z = \mathrm{i} - \epsilon^2/4$ this property holds for $|x| < \pi(2\lambda_0)^{-1}$. Whence by possibly adjusting the support of χ we can safely define

$$V = -\{(Q(p)+1)\phi\}/\phi \in C_c^\infty. \tag{4.6}$$

Finally from the asymptotics of g_- we obtain $\sigma_{loc}(\omega) = \lambda_0 - \frac{\epsilon}{2}|\omega_d|$. Comparing with (4.3) we see that for the eigenfunction (4.5) indeed $\sigma_{loc}(\omega) > \sigma_s(\omega)$ for $|\omega_d| < \frac{\epsilon}{2\lambda_0}$.

Remarks. It is easy to check that the potential V of (4.6) satisfies $\overline{RV} = V$ where $Rf(x_\perp, x_d) = f(x_\perp, -x_d)$ using the fact that also $Q(p)$ has conjugate reflected symmetry. However there is no reason to believe that V is real-valued. If on the other hand we pick an arbitrary real nonzero $V \in C_c^\infty$, $V \ge 0$, the variational principle shows that for some $\kappa < 0$ the energy $\lambda = -1$ is an eigenvalue of $H = Q(p) + \kappa V$. If furthermore $RV = V$ then we can pick a corresponding eigenfunction ϕ obeying $\overline{R\phi} = \phi$. This ϕ is an example of case 3) with a real potential in C_c^∞. However it appears difficult to compute asymptotics for $|\omega_d| \le \frac{\epsilon}{2\lambda_0}$. We claim that for $d = 3$

at least $\sigma_{loc}(\omega) > \sigma_s(\omega)$ when $\omega_d = 0$. This can be done by first representing the Green's function without potential as

$$\big(Q(p)+1\big)^{-1}(x,0) = e^{-x_d\epsilon/2}\int g_-(x-y)e^{y_d\epsilon}g_+(y)dy,$$

where $g_\pm$ are given as above. For $d=3$ we may use the familiar expression $(p^2 - z)^{-1}(x,0) = (4\pi)^{-1}\mathrm{e}^{\mathrm{i}\sqrt{z}|x|}/|x|$, $\operatorname{Im}\sqrt{z} > 0$, and estimate this integral explicitly (after a suitable deformation of contour) and show that indeed $\sigma_{loc}(\omega) > \sqrt{\lambda_0^2 - \epsilon^2/4}$ when $\omega_d = 0$. We skip the details.

5. The Agmon metric and a variational principle

Here we discuss some connection to previous works [Ag1, Ag2] which applies for example to the case 3) of Section 4. As we will see we are not going to derive better bounds than we already have. Our analysis applies to an eigenvalue λ not in $\operatorname{Ran}Q$ and results in a set $\mathcal{E}_A$ whose boundary is described by the same equations as the boundary of the sets $\mathcal{E}$ which we have seen above. In other words the set $\partial\mathcal{E}_A$ is just a subset of the solutions of the equations (3.4a) and (3.4b). For the case 3) of Section 4 the boundary $\partial\mathcal{E} = \partial\mathcal{E}_A$, however this is not valid for the cases 1) and 2). As an additional bonus we will see that quite generally all the solutions to (3.4a) and (3.4b) can be obtained from the same variational principle which we use to derive the equations satisfied by the points of $\partial\mathcal{E}_A$. In the following we assume ϕ is an eigenfunction of H, $(H-\lambda)\phi = 0$. We assume $V = o(1)$ at infinity and $\lambda \notin \operatorname{Ran}Q$.

In analogy with what Agmon does [Ag2] for the Laplacian, we consider the set of all real-valued $f \in C^1(\mathbb{R}^d)$ such that

$$\|(Q(p+\mathrm{i}\nabla f(x)) + V(x) - \lambda)\psi\| \geq \delta\|\psi\| \tag{5.1}$$

with $\psi \in C_c^\infty(\mathbb{R}^d \setminus \bar{B}_R), B_R = B_R(0)$ for some R and positive δ. Since our V is $o(1)$ at infinity, it can be omitted from (5.1) and we get an equivalent estimate. We mention that the quadratic form estimate of Agmon in the case of a second order operator implies (5.1).

Let

$$\delta(f) = \lim_{R\to\infty} \inf\{\|(Q(p+\mathrm{i}\nabla f(x)) - \lambda)\psi\| \mid \psi \in C_c^\infty(\mathbb{R}^d \setminus \bar{B}_R), \|\psi\| = 1\}. \tag{5.2}$$

Note that $\delta(f)$ is invariant under translations: $\delta(f) = \delta(f_a)$, $f_a(x) = f(x-a)$. Thus $\delta(f)$ depends on the values of $\nabla f(x)$ for large x rather than for what x they are taken on. From the viewpoint of using psdo's to get an estimate such as (5.2) with positive $\delta(f)$ it is natural to look at f's which are symbols of order 1 for which $\nabla f(x)$ is in the set

$$\mathcal{E}_A := \{\eta \in \mathbb{R}^d | Q(\xi + \mathrm{i}t\eta) - \lambda \neq 0\ \forall\xi \in \mathbb{R}^d,\ \forall t \in [0,1]\}$$

for all large x. We do not show that this set of f satisfy an estimate such as (5.2) (although this can be done) but rather come at the question from a different point of view. We do mention an important reason for assuming that all smaller values of $\nabla f(x)$ in the same direction be in the set (the reason for t in the definition of $\mathcal{E}_A$). This is automatic with Agmon's quadratic form estimate but more importantly when trying to prove an estimate such as $\mathrm{e}^f\phi \in L^2$ one needs to approximate f with smaller functions f_ϵ for which one knows apriori that $\mathrm{e}^{f_\epsilon}\phi \in L^2$.

Let k be the Minkowski functional, $k(\eta) = \inf\{t > 0 | \eta/t \in \mathcal{E}_A\}$, of the bounded convex open set $\mathcal{E}_A$. (The convexity follows from [Hö].) It follows that $\mathcal{E}_A = \{\eta | k(\eta) < 1\}$. Following Agmon [Ag1] we introduce the polar $k_*(x) = \sup\{x \cdot \eta/k(\eta) | \eta \neq 0\} = \sup\{x \cdot \eta | \eta \in \mathcal{E}_A\}$. k_* is just the support function of the bounded convex set $\overline{\mathcal{E}_A}$. Finally the Agmon metric based on $\mathcal{E}_A$ is

$$\rho_A(x, y) = \inf\{\int_0^1 k_*(\dot\gamma(t))dt \ |\gamma(0) = y, \gamma(1) = x, \gamma(\cdot) \text{ absolutely continuous}\}.$$

Note that from Theorem 1.1 and Proposition 3.2 it follows that each $\eta \in \mathcal{E}_A$ satisfies $e^{\eta \cdot x}\phi \in L^2$ and thus $e^{tk_*(x)}\phi \in L^2$ for all $t \in [0, 1)$. (This can also be shown using the Combes-Thomas method [CT].) We claim that actually $\rho_A(x, 0) = k_*(x)$. First note that $\rho_A(x, 0) \leq \int_0^1 k_*(\dot\gamma(t))dt$ where $\gamma(t) = tx$. This gives $\rho_A(x, 0) \leq k_*(x)$. The opposite estimate, $\rho_A(x, 0) \geq k_*(x)$, follows readily from the fact that k_* is a norm. We give in the following a more informative proof, although it is more complicated. Let x be a point of differentiability of $k_*(x)$ (since k_* is convex it is differentiable a.e.). Pick a point $\eta \in \partial\mathcal{E}_A$ with $k_*(x) = \eta \cdot x$. Then $k_*(x + s\omega) \geq \eta \cdot (x + s\omega) = k_*(x) + s\eta \cdot \omega$ and thus taking $s > 0$ and then $s \to 0$ we obtain $\nabla k_*(x) \cdot \omega \geq \eta \cdot \omega$. Since $\omega \in \mathbb{R}^d$ is arbitrary we obtain $\nabla k_*(x) = \eta$. By definition of k this implies $k(\nabla k_*(x)) \leq 1$ which by [Ag2, Lemma 1.3] implies $k_*(x) \leq \rho_A(x, 0)$. Thus $k_*(x) = \rho_A(x, 0)$. Since we already knew that $e^{tk_*}\phi \in L^2$ for all $t \in [0, 1)$ the Agmon bound, $e^{t\rho_A(x,0)}\phi \in L^2$ for $t \in [0, 1)$, gives no new information.

The variational principle: We now turn to finding equations describing the set $\partial\mathcal{E}_A$. We fix $\omega_0 \in S^{d-1}$ and attempt to find the minimum value, say σ_0, of $\sigma > 0$ such that $Q(\xi + i\sigma\omega_0) = \lambda$ for some ξ. We are of course still in the situation where $\lambda \neq \text{Ran}Q$. The point $\eta_0 := \sigma_0\omega_0$ will then be in $\partial\mathcal{E}_A$. We want to use Lagrange multipliers. For this purpose define the two functions $f_1(\xi, t) = \text{Re}Q(\xi + it\omega_0)$ and $f_2(\xi, t) = \text{Im}Q(\xi + it\omega_0)$. If these functions are independent at a minimum point (ξ_0, σ_0) in the sense that the two gradients $\nabla_{(\xi,t)}f_j(\xi_0, \sigma_0)$ are linearly independent then defining the function $F(\xi, t) = t + \alpha_1 f_1(\xi, t) + \alpha_2 f_2(\xi, t)$ and setting the derivatives equal to zero gives

$$1 - \alpha_1\omega_0 \cdot \text{Im}\nabla Q(\xi_0 + i\eta_0) + \alpha_2\omega_0 \cdot \text{Re}\nabla Q(\xi_0 + i\eta_0) = 0, \tag{5.3a}$$

$$\alpha_1 \text{Re}\nabla Q(\xi_0 + i\eta_0) + \alpha_2 \text{Im}\nabla Q(\xi_0 + i\eta_0) = 0. \tag{5.3b}$$

Evidently $\alpha_1^2 + \alpha_2^2 > 0$ so that the real and imaginary parts of $\nabla Q(\xi_0 + i\eta_0)$ are linearly dependent which means that for some $\beta \in \mathbb{C}$ and $\theta \in S^{d-1}$ we have

$$Q(\xi_0 + i\eta_0) = \lambda, \tag{5.4a}$$

$$\nabla Q(\xi_0 + i\eta_0) = \beta\theta. \tag{5.4b}$$

On the other hand if the gradients of f_1 and f_2 are linearly dependent at the point (ξ_0, σ_0) then again the real and imaginary parts of $\nabla Q(\xi_0 + i\eta_0)$ are linearly dependent at this point and the equations (5.4a) and (5.4b) hold.

Conversely consider a point $\eta_0 = \sigma_c(\omega_0)\omega_0 \in \partial\mathcal{E}$ where we have in mind an eigenfunction ϕ of $H = Q(p) + V(x)$ with $\mathcal{E} = \{\eta \in \mathbb{R}^d | e^{\eta \cdot x}\phi \in L^2\}$. Suppose the eigenvalue λ is not a critical value of $Q(z)$ but we no longer assume that $\lambda \notin \text{Ran}Q$. There are $\xi_0 \in \mathbb{R}^d, \beta \in \mathbb{C} \setminus \{0\}$ and $\theta \in S^{d-1}$ such that $\sigma_s(\theta) = \eta_0 \cdot \theta$ and such that these quantities along with $\sigma_0 = \sigma_c(\omega_0)$ satisfy equations (5.4a) and (5.4b). We claim the the gradients of the functions f_j are linearly independent at this point. If for real α_1 and α_2 not both zero we have $\alpha_1\nabla_{(\xi,t)}f_1(\xi_0, \sigma_0) + \alpha_2\nabla_{(\xi,t)}f_2(\xi_0, \sigma_0) = 0$, we calculate $\alpha_1\nabla_\xi f_1 + \alpha_2\nabla_\xi f_2 = 0$ and $\omega_0 \cdot (-\alpha_1\nabla_\xi f_2 + \alpha_2\nabla_\xi f_1) = 0$. These two equations imply $\nabla_\xi f_1 \cdot \omega_0 = \nabla_\xi f_2 \cdot \omega_0 = 0$ or $\nabla Q(\xi_0 + i\eta_0) \cdot \omega_0 = 0$. Thus since λ is not a critical value of $Q(z)$, $\omega_0 \cdot \theta = 0$. This contradicts the geometry of $\partial\mathcal{E}$: Since $\bar{\mathcal{E}} \subset \{\eta|(\eta - \eta_0) \cdot \theta \le 0\}$ and since for example $\eta_0/2$ is an interior point of $\bar{\mathcal{E}}$ we can take $\eta = \eta_0/2 + u$ above where u is small and learn that $u \cdot \theta \le 0$ for all small u, a contradiction. Thus for some small $\epsilon > 0$, $\{(\xi, t)|Q(\xi+it\omega_0) = \lambda, |\xi-\xi_0|+|t-\sigma_0| < \epsilon\}$ is a co-dimension two smooth submanifold of $\mathbb{R}^{d+1}$ and (ξ_0, σ_0) is a critical point of the function $F(\xi, t) = t$ restricted to this submanifold. This is because given the equations (5.4a) and (5.4b) we can find α_1 and α_2 solving the equations (5.3a) and (5.3b). Thus the equations of Theorem 3.4 coming from an eigenfunction of H can be derived from this variational principle.

6. The set $\bar{\mathcal{E}}$

The set of η satisfying

$$Q(\xi + i\eta) = \lambda, \tag{6.1a}$$

$$\nabla Q(\xi + i\eta) = \beta\theta. \tag{6.1b}$$

for some ξ, β, θ is a semi-algebraic set (see [BCR]). By definition this means that it is a finite union of sets of the form $S_n = \{\eta \in \mathbb{R}^d | q_j(\eta) = 0, p_j(\eta) > 0, j = 1, \cdots, n\}$ where the p_j and q_j are real polynomials. This comes from the fundamental result that a projection of a semi-algebraic set is a semi-algebraic set. It would be interesting to know what restrictions this puts on the set of singular points of the boundary of the set $\mathcal{E}$ defined for an eigenfunction of H with $0 < \sigma_g < \infty$.

We give sufficient conditions for the local smoothness of solutions, $z = \xi + i\eta = h(\theta)$, of (6.1a) and (6.1b). We do not assume that solutions come from the exponential decay of an eigenfunction of $Q(p) + V(x)$. Let us assume λ is not a critical value of Q so that given a solution $(\xi_0, \eta_0, \beta_0, \theta_0)$, β_0 must be nonzero. Let us assume $Q''(z_0)$ is invertible ($z_0 = \xi_0 + i\eta_0$). Generically this is true when $Q(z_0) = \lambda$ except on a $d - 2$ dimensional manifold. Then we can define locally the Legendre transformation $P(w) = z \cdot w - Q(z), w = \nabla Q(z)$. ∇P is the inverse of ∇Q. We then have $\nabla P(\beta\theta) = z = \xi + i\eta$ so that $Q(\nabla P(\beta\theta)) = \lambda$. We can solve for β in terms of θ locally if $\frac{\partial}{\partial\beta}Q(\nabla P(\beta\theta)) \neq 0$ when $\beta = \beta_0, \theta = \theta_0$. A short calculation gives the requirement $\beta_0\theta_0 \cdot P''(\beta_0\theta_0)\theta_0 \neq 0$ so that we have:

Proposition 6.1. *Suppose $Q(z_0) = \lambda$, $\nabla Q(z_0) = \beta_0\theta_0$, $\beta_0 \neq 0$, $Q''(z_0)$ is invertible and*

$$\nabla Q(z_0) \cdot Q''(z_0)^{-1}\nabla Q(z_0) \neq 0.$$

Then there exists a neighborhood of (θ_0, β_0, z_0) in which the set of solutions to the system (6.1a) *and* (6.1b) *(with $z = \xi + i\eta$) is parametrized smoothly by θ.*

Given the assumptions of the proposition, we have $z = h(\theta)$ in a neighborhood of (z_0, θ_0). We can calculate the derivative $h'(\theta)$ by differentiating $\nabla Q(z) = \beta\theta$ and using the formula for $\beta'(\theta)$ from the above application of the implicit function theorem. We obtain as an identity on the tangent space $T_\theta(S^{d-1}) = \{x \in \mathbb{R}^d|\, x \cdot \theta = 0\}$

$$h'(\theta) = \beta Q''(z)^{-1}(I - R(\theta));\ R(\theta)x := \frac{\theta \cdot Q''(z)^{-1}x}{\theta \cdot Q''(z)^{-1}\theta}\theta.$$

To better understand the meaning of the relationship between η and θ let us take θ_1 near θ_0 and look for a critical point of the function $\eta \cdot \theta_1$ for $\eta = \mathrm{Im} h(\theta) =: g(\theta)$. Since $\theta \cdot h'(\theta) = 0$ by the above formula obviously $\theta = \theta_1$ is a critical point of the function $g \cdot \theta_1$. This is consistent with the geometric interpretation of $(\theta_1, g(\theta_1))$ being the parameters of a supporting hyperplane at the boundary point $\eta_1 = g(\theta_1)$ of the convex set $\mathcal{E}$ which comes from an L^2-function ϕ solving $(H - \lambda)\phi = 0$. In this case $\eta \cdot \theta_1$ would be maximized with $\eta = \eta_1$. If $\eta = g(\theta)$ describes the boundary of a convex set $\mathcal{E}$ which comes from an L^2-function ϕ solving $(H - \lambda)\phi = 0$ then the uniqueness of η corresponds to the strict convexity of $\bar{\mathcal{E}}$.

The conditions which allow us to conclude that η is a smooth function of its direction $\omega = \eta/|\eta|$ are more complicated. If $g(\theta) = \mathrm{Im} h(\theta)$ as above, we want to solve for (σ, θ) as a function of ω in the equation $\sigma\omega - g(\theta) = 0$ near $(\sigma\omega, \theta) = (\eta_0, \theta_0)$. The inverse function theorem gives the result that σ is locally a smooth function of ω if the only solution (x, μ) to the real linear equation $g'(\theta_0)x = \mu\eta_0$ is the trivial solution. Let us make the assumption that $g(\theta_0) \cdot \theta_0 \neq 0$ and the (generic) assumption $\ker g'(\theta_0) = 0$. Note that if the equation $\eta = g(\theta)$ represents the boundary of a set $\mathcal{E}$ which comes from a solution ϕ to $(H - \lambda)\phi = 0$ with $\sigma_g > 0$ and $\sigma_s(\theta) = g(\theta) \cdot \theta$, cf. Theorem 3.4, then obviously $g(\theta) \cdot \theta \neq 0$. Now differentiating $Q(z) = \lambda$ gives $\theta_0 \cdot h'(\theta_0) = 0$ and therefore also that $0 = \theta_0 \cdot g'(\theta_0)x = \mu\theta_0 \cdot \eta_0 = \mu g(\theta_0) \cdot \theta_0$ showing that $\mu = 0$ and then in turn $x = 0$. Whence the only solution to $g'(\theta_0)x = \mu\eta_0$ is the trivial one.

Acknowledgement: E. S. was supported by Grant 11-106598 FNU.

References

[Ag1] S. Agmon, *On the asymptotics of Green's functions of elliptic operators with constant coefficients*, Actes des Journées Mathématiques à la Mémoire de Jean Leray, 13–23, Sémin. Congr., **9**, Soc. Math. France, Paris, 2004.

[Ag2] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, Princeton, 1982.

[BCR] J. Bochnak, M. Coste, M.-F. Roy, *Real algebraic geometry*, Springer, Berlin, 1998.

[BM] C. Bardos, M. Merigot, *Asymptotic decay of the solution of a second-order elliptic equation in an unbounded domain. Applications to the spectral properties of a Hamiltonian*, Proc. R. Edinburgh, Sect A **76** (1977) 323–344.

[BR] R. Benedetti, J.-J. Rister, *Real algebraic and semi-algebraic sets, Actualit'es Mathematique*, Hermann, 1990.

[CT] J-M. Combes, L. Thomas, *Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators*, Commun. Math. Phys. **34** (1973), 251–270.

[FH] R. Froese, I. Herbst, *Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators*, Commun. Math. Phys. **87** no. 3 (1982/83), 429–447.

[FHH2O1] R. Froese, I. Herbst, M. Hoffmann-Ostenhof and T. Hoffmann-Ostenhof, *On the absence of positive eigenvalues for one-body Schrödinger operators*, J. d'Anal. Math. **41** (1982), 272–284.

[FHH2O2] R. Froese, I. Herbst, M. Hoffmann-Ostenhof and T. Hoffmann-Ostenhof, *L^2-exponential lower bounds to solutions of the Schrödinger equation*, Commun. Math. Phys. **87** (1982/83), 256–286.

[FHH2O3] R. Froese, I. Herbst, M. Hoffmann-Ostenhof and T. Hoffmann-Ostenhof, *L^2-lower bounds to solutions of one-body Schrödinger equations*, Proc. Royal Soc. Edingburgh **95A** (1983), 25–38.

[HS] I. Herbst, E. Skibsted, *Decay of eigenfunctions of elliptic PDE's I*, Adv. Math. **270** (2015), 138–180.

[Hö] L. Hörmander, *An introduction to complex analysis in several variables, 3rd ed.*, North Holland, Amsterdam, 1990.

[MP1] M. Mantoiu, R. Purice, *A-priori decay for eigenfunctions of perturbed periodic Schrödinger operators*, Ann. H. Poincaré **2** no. 3 (2001), 525–551.

[MP2] M. Mantoiu, R. Purice, *Hardy type inequalities with exponential weights for a class of convolution operators*, Ark. Mat. **45** (2007), 83–103.

[Oc] A. J. O'Connor, *Exponential decay of bound state wave functions*, Commun. Math. Phys. **32** (1973), 319–340

[Ro] R. T. Rockefellar, *Convex analysis* Princeton University Press, Princeton, 1970.

(I. Herbst) Department of Mathematics, University of Virginia, Charlottesville, VA 22904, U.S.A.

E-mail address: `iwh@virginia.edu`

(E. Skibsted) Institut for Matematiske Fag, Aarhus Universitet, Ny Munkegade 8000 Aarhus C, Denmark

E-mail address: `skibsted@imf.au.dk`