ABSENCE OF POSITIVE EIGENVALUES FOR HARD-CORE N-BODY SYSTEMS

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Abstract. We show absence of positive eigenvalues for generalized N-body hard-core Schrödinger operators under the condition of bounded obstacles with connected exterior. A particular example is atoms and molecules with the assumption of infinite mass and finite extent nuclei.

Contents

1. Introduction and results

Consider the N-body Schrödinger operator

$$H = \sum_{j=1}^{N} \Big(-\frac{1}{2m_j}\Delta_{x_j} + V_j(x_j) \Big) + \sum_{1\le i<j\le N} V_{ij}(x_i - x_j) \tag{1.1}$$

for a system of N d-dimensional particles in $\Omega_1 = \mathbb{R}^d \setminus \overline{\Theta}$ where Θ is a bounded and open subset of $\mathbb{R}^d$ given such that the exterior set Ω_1 is connected (for $N = 1$ the last term is omitted). Whence H is an operator on the Hilbert space $L^2(\Omega)$; $\Omega = (\Omega_1)^N$. It is defined more precisely by imposing the Dirichlet boundary condition. This operator models a system of N d-dimensional charged particles interacting with a fixed charged nucleus of finite extent, for example a ball. In particular we could have Coulomb potentials $V_j(y) = q_j q^{\mathrm{ncl}}|y|^{-1}$ and $V_{ij}(y) = q_i q_j |y|^{-1}$ in dimension $d \ge 2$

assuming $0 \in \Theta$ (the latter condition is reasonable due to Newton's shell theorem). We show in this particular case that H does not have positive eigenvalues. While this property is well-known for the one-body problem (see for example [RS, FHH2O, IS2]) it is a new result for $N \geq 2$. Moreover we extend the result to the case of molecules with fixed charged nuclei of finite extent.

We obtain absence of positive eigenvalues in a much more general setting, that is for so-called generalized N-body hard-core Schrödinger operators, see Theorem 1.7 for our main result. There are two a priori ingredients in our approach: 1) Suitable vector fields (applied before to usual N-body Schrödinger operators for different purposes). 2) A unique continuation property (in general a well-studied subject). Apart from those the paper is self-contained. In particular we give full proofs of a Mourre estimate and an exponential decay estimate of non-threshold eigenstates needed for the problem at hand. These results are part of a scheme of proof closely related to the one of [FH]. However technically there are significant differences. A main ingredient is the construction of a new vector field well-suited for our problem (and used somewhat differently), see (2.6). We addressed earlier the problem of proving absence of positive eigenvalues for generalized N-body hard-core Schrödinger operators in [IS3] using yet another approach, however obtaining there only partial results. In addition our use of the new vector field is considerably simpler, conceptionally as well as technically.

Since hard-core Schrödinger operators are conveniently defined by their quadratic form we naturally include relatively form-compact local singularities of the potentials. Those considered in [FH] are relatively operator-bounded ones. Whence our results in the case of no obstacles cover some stronger type of local singularities of the potentials than treated by [FH] (as for $d = 2$ with Coulomb interactions as considered above). However we do not recover all of the results of [FH] in this particular case for two reasons (the following conditions are not imposed in [FH]). In this paper: 1) The potential singularities are located in a bounded set. 2) We use the unique continuation principle which although being a general principle implicitly constitutes a condition on the local singularities, see a brief discussion after Proposition 1.6.

For previous works on N-body hard-core Schrödinger operators we refer to [BGS] and [Gri]. As the reader will see there is some overlap between this paper and [Gri], however the main problem of our paper, showing absence of positive eigenvalues, is not treated in [Gri]. To the contrary it is stated there explicitly as an open problem. A bi-product of our approach is absence of singularly continuous spectrum under virtually no regularity conditions on the obstacles (only boundedness is needed) generalizing [BGS, Theorem A], see Remark B.3. For further references to works on hard-core Hamiltonians and some further discussion we refer to [Gri, Section 1]. Finally, to put hard-core Hamiltonians into a broader perspective and not for claiming contribution, we call attention on an ongoing physics dispute: What is the size of a nucleus/proton? See [A] for a recent contribution.

1.1. **Usual generalized N-body systems.**

We will work in a generalized framework. We first review the analogue of this without obstacles, i.e. with "soft potentials". This is given by real finite dimensional vector space $\mathbf{X}$ with an inner product g, i.e. $(\mathbf{X}, g)$ is Euclidean space, and a finite family of subspaces $\{\mathbf{X}_a |\ a \in \mathcal{A}\}$ closed with respect to intersection. We refer to the elements of $\mathcal{A}$ as *cluster decompositions* (this terminology is not motivated here). The orthogonal complement of $\mathbf{X}_a$ in $\mathbf{X}$ is denoted $\mathbf{X}^a$, and correspondingly we decompose $x = x^a \oplus x_a \in \mathbf{X}^a \oplus \mathbf{X}_a$. We order $\mathcal{A}$ by writing $a_1 \subset a_2$ if $\mathbf{X}^{a_1} \subset \mathbf{X}^{a_2}$. It is assumed that there exist $a_{\min}, a_{\max} \in \mathcal{A}$ such that $\mathbf{X}^{a_{\min}} = \{0\}$ and $\mathbf{X}^{a_{\max}} = \mathbf{X}$. Let

$$\mathcal{B} = \mathcal{A} \setminus \{a_{\min}\}.$$

The length of a chain of cluster decompositions $a_1 \subsetneq \cdots \subsetneq a_k$ is the number k. Such a chain is said to connect $a = a_1$ and $b = a_k$. The maximal length of all chains connecting a given $a \in \mathcal{A} \setminus \{a_{\max}\}$ and $a_{\max}$ is denoted by $\#a$. We define $\#a_{\max} = 1$ and denoting $\#a_{\min} = N + 1$ we say the family $\{\mathbf{X}^a | a \in \mathcal{A}\}$ is of N-body type. The N-body Schrödinger operator H introduced above (now considered without an obstacle, i.e. with $\Omega_1 = \mathbb{R}^d$) is of the form $H = H_0 + V$, where $2H_0$ is (minus) the Laplace-Beltrami operator on the space $(\mathbf{X}, g)$

$$\mathbf{X} = (\mathbb{R}^d)^N, \quad g = \sum_{j=1}^{N} m_j |x_j|^2,$$

$V = V(x) = \sum_{b \in \mathcal{B}} V_b(x^b)$ and the relevant family $\{\mathbf{X}^a | a \in \mathcal{A}\}$ of subspaces is indeed of N-body type, see the proof of Corollary 1.8 for details. However this is just one example of a generalized N-body Schrödinger operator, see [DeGé, Section 5.1] for other examples. The general construction of such an operator H is similar, and under the following condition it is well-defined with form domain given by the Sobolev space $H^1(\mathbf{X})$, cf. [RS, Theorem X.17].

Condition 1.1. There exists $\varepsilon > 0$ such that for each (real-valued) potential V_b, $b \in \mathcal{B}$, there is a splitting $V_b = V_b^{(1)} + V_b^{(2)}$, where

(1) $V_b^{(1)}$ is smooth and

$$\partial_y^\alpha V_b^{(1)}(y) = O\big(|y|^{-\varepsilon - |\alpha|}\big). \tag{1.2}$$

(2) $V_b^{(2)}$ is compactly supported and

$$(-\Delta + 1)^{-1/2} V_b^{(2)} (-\Delta + 1)^{-1/2} \text{ is compact on } L^2(\mathbb{R}_y^{\dim \mathbf{X}^b}). \tag{1.3}$$

Let $-\Delta^a = (p^a)^2$ and $-\Delta_a = p_a^2$ denote (minus) the Laplacians on $L^2(\mathbf{X}^a)$ and $L^2(\mathbf{X}_a)$, respectively. Here $p^a = \pi^a p$ and $p_a = \pi_a p$ denote the "internal" (i.e. within clusters) and the "inter-cluster" components of the momentum operator $p = -\mathrm{i}\nabla$, respectively. For all $a \in \mathcal{B}$ we introduce

$$\begin{aligned}
V^a(x^a) &= \sum_{b \subset a} V_b(x^b), \\
H^a &= -\tfrac{1}{2}\Delta^a + V^a(x^a) \text{ on } L^2(\mathbf{X}^a), \\
H_a &= H^a \otimes I + I \otimes \big(-\tfrac{1}{2}\Delta_a\big) \text{ on } L^2(\mathbf{X}^a) \otimes L^2(\mathbf{X}_a), \\
I_a(x) &= \sum_{b \not\subset a} V_b(x^b).
\end{aligned}$$

We also define $H^{a_{\min}} = 0$ on $L^2(\mathbf{X}^{a_{\min}}) := \mathbb{C}$. The operator H^a is the sub-Hamiltonian associated with the cluster decomposition a, and I_a is the sum of all "inter-cluster" interactions. The detailed expression of H^a depends on the choice of coordinates on $\mathbf{X}^a$.

Given a family $\{\mathbf{X}^a | a \in \mathcal{A}\}$ of subspaces of N-body type and interactions obeying Condition 1.1 the generalized N-body Hamiltonian is $H = H^{a_{\max}}$. Let

$$\mathcal{T} = \cup_{a\in\mathcal{A},\#a\geq 2}\, \sigma_{\rm pp}(H^a) \tag{1.4}$$

denote the set of thresholds of H. By the HVZ theorem [RS, Theorem XIII.17] the essential spectrum of H is given by the formula

$$\sigma_{\rm ess}(H) = [\min \mathcal{T}, \infty). \tag{1.5}$$

It is also well-known that under rather general conditions H does not have positive eigenvalues and the negative non-threshold eigenvalues can at most accumulate at the thresholds and only from below, see [FH] and [Pe].

1.1.1. *Graf vector field and Mourre estimate.* We give a brief review of the construction of a family of conjugate operators for N-body Hamiltonians originating from [Sk1]. The phrase "conjugate" is here used (in agreement with conventions) to signify that there exists a so-called Mourre estimate. A slightly different proof of this Mourre estimate appears in [Sk2]. The construction is based on the vector field invented by Graf [Gra] which is a vector field satisfying the following properties, cf. [Gra, De, DeGé, Sk2]. We use throughout the paper the notation $\langle x\rangle = \sqrt{x^2+1}$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

Lemma 1.2. *There exist on* $\mathbf{X}$ *a smooth vector field* $\tilde\omega$ *with symmetric derivative* $\tilde\omega_*$ *and a partition of unity* $\{\tilde q_a\}$ *indexed by* $a\in\mathcal{A}$ *and consisting of smooth functions,* $0\leq \tilde q_a \leq 1$*, such that for some positive constants* r_1 *and* r_2

(1) $\tilde\omega_*(x) \geq \sum_a \pi_a \tilde q_a$.
(2) $\tilde\omega^a(x) = 0$ *if* $|x^a| < r_1$.
(3) $|x^b| > r_1$ *on* $\operatorname{supp}(\tilde q_a)$ *if* $b \not\subset a$.
(4) $|x^a| < r_2$ *on* $\operatorname{supp}(\tilde q_a)$.
(5) *For all* $\alpha \in \mathbb{N}_0^{\dim \mathbf{X}}$ *and* $k\in\mathbb{N}_0$ *there exist* $C\in\mathbb{R}$*:*

$$|\partial_x^\alpha \tilde q_a| + |\partial_x^\alpha (x\cdot\nabla)^k\big(\tilde\omega(x) - x\big)| \leq C. \tag{1.6}$$

Now, proceeding as in [Sk2], we introduce the rescaled vector field $\tilde\omega_R(x) := R\tilde\omega(\frac{x}{R})$ and the corresponding operator

$$A = A_R = \tilde\omega_R(x)\cdot p + p\cdot\tilde\omega_R(x);\ R>1. \tag{1.7}$$

We also introduce a function $d:\mathbb{R}\to\mathbb{R}$ by

$$d(E) = \begin{cases} \inf_{\tau\in\mathcal{T}(E)}(E-\tau),\ \mathcal{T}(E) := \mathcal{T}\cap\,]-\infty,E] \neq \emptyset,\\ 1,\ \mathcal{T}(E)=\emptyset. \end{cases} \tag{1.8}$$

These devices enter into the following Mourre estimate which is the relatively form compact version of the relatively operator compact one of [Sk2, Corollary 4.5]. We give a full proof in Appendix B covering inclusion of hard-core interactions. We remark that all inputs needed for the proof are contained in Lemma 1.2, whence the particular construction of the Graf vector field is irrelevant. For a different proof (also valid in the context of hard-core interactions) we refer to [Gri]. (For a different conjugate operator, see [BGS].)

Lemma 1.3. *For all* $E\in\mathbb{R}$ *and* $\kappa>0$ *there exists* $R_0>1$ *such that for all* $R\geq R_0$ *there is a neighbourhood* $\mathcal{V}$ *of* E *and a compact operator* K *on* $L^2(\mathbf{X})$ *such that*

$$f(H)^*\mathrm{i}[H,A_R]f(H) \geq f(H)^*\{4d(E)-\kappa-K\}f(H) \text{ for all } f\in C_{\rm c}^\infty(\mathcal{V}). \tag{1.9}$$

Here the commutator is *defined* by its formal expression, see (2.3). The possibly existing local singularities of the potential do not enter (for R large) due to Lemma 1.2 (2). This feature is a strong indication of the existence of a similar Mourre estimate for the hard-core model of Subsection 1.2, see Lemma 1.5 for such extension.

Two of the consequences of a Mourre estimate like the one stated above are that the set of thresholds $\mathcal{T}$ is closed and countable and that the eigenvalues of H can at most accumulate at $\mathcal{T}$, see Subsection B.2. We discuss a third consequence in Subsection 2.1 and Appendix C (for hard-core Hamiltonians), decay of non-threshold eigenstates.

1.2. **Generalized N-body hard-core systems.** The generalized hard-core model is a modification for the above model. For the generalized hard-core model we are given for each $a \in \mathcal{B}$ an open subset $\Omega_a \subset \mathbf{X}^a$ with $\mathbf{X}^a \setminus \Omega_a$ compact, possibly $\Omega_a = \mathbf{X}^a$. Let for $a_{\min} \neq b \subset a$

$$\Omega^a_b = \big(\Omega_b + \mathbf{X}_b\big) \cap \mathbf{X}^a = \Omega_b + \mathbf{X}_b \cap \mathbf{X}^a,$$

and for $a \neq a_{\min}$

$$\Omega^a = \cap_{a_{\min} \neq b \subset a} \Omega^a_b.$$

We define $\Omega^{a_{\min}} = \{0\}$ and $\Omega = \Omega^{a_{\max}}$.

Condition 1.4. There exists $\varepsilon > 0$ such that for all $b \in \mathcal{B}$ there is a splitting into (real-valued) terms $V_b = V_b^{(1)} + V_b^{(2)}$, where

(1) $V_b^{(1)}$ is smooth on the closure of Ω_b and

$$\partial_y^\alpha V_b^{(1)}(y) = O\big(|y|^{-\varepsilon-|\alpha|}\big). \tag{1.10}$$

(2) $V_b^{(2)}$ vanishes outside a bounded set in Ω_b and

$$V_b^{(2)} \in \mathcal{C}\big(H_0^1(\Omega_b), H_0^1(\Omega_b)^*\big). \tag{1.11}$$

Here and henceforth, given Banach spaces X_1 and X_2, the notation $\mathcal{C}\big(X_1, X_2\big)$ and $\mathcal{B}\big(X_1, X_2\big)$ refers to the set of compact and the set of bounded operators $T : X_1 \to X_2$, respectively.

We consider for $a \in \mathcal{B}$ the Hamiltonian $H^a = -\frac{1}{2}\Delta_{x^a} + V^a$ on the Hilbert space $L^2(\Omega^a)$ with the Dirichlet boundary condition on $\partial\Omega^a$, in particular

$$H = \tfrac{1}{2}p^2 + V = H_0 + V \text{ on } \mathcal{H} := L^2(\Omega)$$

with the Dirichlet boundary condition on $\partial\Omega$. More precisely the Hamiltonian H^a, henceforth called a hard-core Hamiltonian, is given by its form. The form domain is the standard Sobolev space $H_0^1(\Omega^a)$, and the corresponding action is the (naturally defined) Dirichlet form. Due to the continuous embedding $H_0^1(\Omega^a) \subset H_0^1(\Omega^a_b)$ for $a_{\min} \neq b \subset a$ we conclude that indeed H^a is self-adjoint, cf. [RS, Theorem X.17]. Again we define $H^{a_{\min}} = 0$ and the set of thresholds by (1.4). We claim that Lemma 1.3 holds for the hard-core Hamiltonian H upon replacing the Hilbert space $L^2(\mathbf{X})$ there by $\mathcal{H}$ (and with the same interpretation of the commutator). However we prefer to state the Mourre estimate slightly differently (see the comments after Lemma 1.5). There is the following estimate, cf. Appendix B and [Gri, Theorem 2.4]:

Lemma 1.5. *For all $\kappa \in (0,1]$ and compact $I \subset \mathbb{R}$ there exists $R_0 > 1$ such that for all $R \geq R_0$ and all $E \in I$ there is a neighbourhood $\mathcal{V}$ of E and a compact operator K on $\mathcal{H}$ such that*

$$f(H)^*\mathrm{i}[H, A_R]f(H) \geq 4f(H)^*\{d(E+\kappa) - 5\kappa - K\}f(H) \text{ for all } f \in C_c^\infty(\mathcal{V}). \quad (1.12)$$

Here the function d is defined by (1.8) now of course in terms of the set of thresholds for the hard-core Hamiltonian H. We also note that for $R > 1$ taken large enough the rescaled Graf vector field $\tilde{\omega}_R$ is complete on Ω. The latter is doable due to Lemma 1.2 (2) and (5). This allows for an interpretation of the "commutator" $\mathrm{i}[H, A_R]$ of (2.3) as a *commutator*, see Subsection B.2 and Appendix A for details. This feature is needed for showing exponential decay of non-threshold eigenstates. The local uniformity in energy of this version of the Mourre estimate is needed too for showing exponential decay. To the contrary Lemma 1.5 can be shown using only (2.4) (in particular only the formal expression of the commutator) and some of the properties of Lemma 1.2, see Subsection B.1. Using the fact that $\mathrm{i}[H, A_R]$ is a commutator one can obtain a version of Lemma 1.5 similar to Lemma 1.3 and various consequences of independent interest, see Subsection B.2.

1.3. **Results.**

Proposition 1.6. *Suppose $N \geq 1$ and Condition 1.4. Suppose the hard-core Hamiltonian H does not have positive thresholds. Suppose that any eigenstate of H vanishing outside a bounded set must be zero (the unique continuation property). Then H does not have positive eigenvalues.*

The unique continuation property (here used at infinity only) is a well-studied subject in particular for the one-body problem, see for example [Ge, JK, RS, Wo]. It is valid for some classes of potential singularities given the condition of connectivity of Ω although to our knowledge the state of art is presently not satisfactory for the N-body problem, see the proof of Corollary 1.8 below for a particular application. The following main result of this paper follows readily from Proposition 1.6 and induction in N. Recall $\mathcal{B} := \mathcal{A} \setminus \{a_{\min}\}$.

Theorem 1.7. *Suppose $N \geq 1$ and Condition 1.4. Suppose the unique continuation property for H and all sub-Hamiltonians H^b (more precisely that any eigenstate of H^b for $b \in \mathcal{B}$ vanishing outside a bounded set must be zero). Then H does not have positive eigenvalues.*

This result applies to (1.1) and the following generalization: Consider for given disjoint $R_1, \ldots, R_K \in \mathbb{R}^d$ the N-body Schrödinger operator

$$H = \sum_{j=1}^{N} \Big(-\frac{1}{2m_j}\Delta_{x_j} + \sum_{1 \leq k \leq K} V_j^k(x_j - R_k) \Big) + \sum_{1 \leq i < j \leq N} V_{ij}(x_i - x_j) \quad (1.13)$$

describing a system of N d-dimensional particles in $\Omega_1 = \mathbb{R}^d \setminus \overline{\Theta}$, where $\Theta = \cup_{1 \leq k \leq K}\Theta_k$ for given open and bounded subsets $\Theta_1, \ldots, \Theta_K$ of $\mathbb{R}^d$ such that $R_k \in \Theta_k$, $k = 1, \ldots, K$ (for $N = 1$ the last term to the right in (1.13) is omitted).

Corollary 1.8. *For N charged particles confined to $\Omega_1 \subset \mathbb{R}^d$ with the additional properties that $d \geq 2$ and this exterior set $\Omega_1 = \mathbb{R}^d \setminus \overline{\Theta}$ is connected the corresponding Hamiltonian H given by (1.13) with Coulomb interactions $V_j^k(y) = q_j q^k |y|^{-1}$ and $V_{ij}(y) = q_i q_j |y|^{-1}$ (and defined by the Dirichlet boundary condition) does not have positive eigenvalues.*

Proof. There is the following concrete description of the family $\{\mathbf{X}^a|a\in\mathcal{A}\}$: Consider $a=(C_1,\dots,C_p)$ where the C_q's are disjoint subsets of $\{1,\dots,N\}$. For $p\geq 2$ and $q<p$ we have $\#C_q\geq 2$ and we let $\mathbf{X}^{C_q}=\{x\in\mathbf{X}|x_j=0\text{ if }j\notin C_q\text{ and }\sum_{i\in C_q}m_ix_i=0\}$. Either similarly 1) $\mathbf{X}^{C_p}=\{x\in\mathbf{X}|x_j=0\text{ if }j\notin C_p\text{ and }\sum_{i\in C_p}m_ix_i=0\}$ (in that case we have $\#C_p\geq 2$), or 2) $\mathbf{X}^{C_p}=\{x\in\mathbf{X}|x_j=0\text{ if }j\notin C_p\}$. In both cases let correspondingly $\mathbf{X}^a=\mathbf{X}^{C_1}\oplus\cdots\oplus\mathbf{X}^{C_p}$. Moreover we supplement by writing $\mathbf{X}^{a_{\min}}=\{0\}$ where, for example, $a_{\min}:=\emptyset$. This is a concrete labeling of $\{\mathbf{X}^a|a\in\mathcal{A}\}$ for (1.13). The sub-Hamiltonians are given as

$$H^a=H^{C_1}\otimes I\otimes\cdots\otimes I+\cdots+I\otimes\cdots\otimes I\otimes H^{C_p},$$

where for $q<p$ the operator H^{C_q} is a usual Schrödinger operator defined on $L^2(\mathbf{X}^{C_q})$. The same is valid for H^{C_p} in case 1). To the contrary in case 2) the operator H^{C_p} has the same form as (1.13), but with only $N_p:=\#C_p$ particles involved, whence it is an operator on $L^2((\Omega_1)^{N_p})$. Note that either $\Omega_b=\mathbf{X}^b$ or $\Omega_b=\{x\in\mathbf{X}|x_j=0\text{ if }j\neq i\text{ and }x_i\in\Omega_1\}$ for some $i\leq N$.

Next we introduce the following subset of $\Omega=(\Omega_1)^N$; $\Omega_1=\mathbb{R}^d\setminus\overline{\Theta}$:

$$\widetilde{\Omega}=\begin{cases}\Omega & \text{for } N=1,\\ \Omega\setminus\{(x_1,\dots,x_N)\in(\mathbb{R}^d)^N|x_i=x_j\text{ for some }i\neq j\} & \text{for } N\geq 2.\end{cases}$$

We are going to use that $\widetilde{\Omega}$ is connected for $d\geq 2$ (and similarly upon replacing N by N_p). For this property we may argue as follows: Since Ω_1 is arcwise connected also $(\Omega_1)^N$ is arcwise connected. Any of the subspaces $\{(x_1,\dots,x_N)\in(\mathbb{R}^d)^N|x_i=x_j\}$, $i\neq j$, has co-dimension d. We conclude by using repeatedly Lemma 1.9 stated below and the fact that $d\geq 2$.

We need to check that the condition of Theorem 1.7 that the unique continuation property for H and all sub-Hamiltonians H^a holds. For that we shall use the version of the unique continuation property [RS, Theorem XIII.63]. Let us first consider the operator H: We argued above that $\widetilde{\Omega}$ is connected. Since moreover the subset $\{(x_1,\dots,x_N)\in\Omega|x_i=x_j\text{ for some }i\neq j\}$ of Ω has measure zero we can indeed apply [RS, Theorem XIII.63] for this case. For the unique continuation property for any sub-Hamiltonians H^a we use the above tensor decomposition. Any eigenstate has the form

$$\phi^a=\sum_l\psi_l\otimes\phi_l^{C_p},$$

where $\{\psi_l\}$ is an at most countable orthonormal set of vectors and each $\phi_l^{C_p}$ is an eigenstate of H^{C_p}. If ϕ^a vanishes outside a bounded set also each $\phi_l^{C_p}$ vanishes outside a bounded set (seen by multiplying by ψ_l and integrating). In case 2) we then argue as above replacing N by N_p and conclude that each $\phi_l^{C_p}=0$ and whence that $\phi^a=0$. In case 1) the same type of arguments works (we omit the details). We conclude by Theorem 1.7. □

Lemma 1.9. *Suppose U is an open connected subset of $\mathbb{R}^n$, $n\geq 2$, and L is a closed submanifold of U with co-dimension at least 2 then also $U\setminus L$ is connected.*

Proof. We can assume that $U\cap L\neq\emptyset$. Suppose $U\setminus L=U_1\cup U_2$ where U_1 and U_2 are open and disjoint. We need to show that either U_1 or U_2 is empty. Look for $j=1,2$

at the boundary ∂U_j of U_j considering U_j as a subset of $\mathbb{R}^n$, and let us denote the part of this boundary inside U, that is $U \cap \partial U_j$, by V_j. Clearly $U \cap L = V_1 \cup V_2$. Moreover since L has co-dimension at least 2 the set $V_1 \cap V_2 = \emptyset$ (easily seen by using a suitable "local" arc). Assuming that $V_1 \neq \emptyset$ (can be done since V_1 and V_2 are not both empty) we will show that $U_2 = \emptyset$: Pick $x_1 \in V_1$ and suppose that there exists $x_2 \in U_2$. Then we pick a arc $\gamma : [0,1] \to U$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$ and consider

$$t_1 = \sup\{t \in [0,1] | \gamma(s) \in U_1 \cup V_1 \text{ for } s \in [0,t]\}.$$

We have $\gamma(t_1) \in V_1 \cap V_2$, contradicting that $V_1 \cap V_2 = \emptyset$. □

We shall need some more notation. We fix a non-negative $\chi \in C^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$ and

$$\chi(t) = \begin{cases} 0 & \text{for } t \leq 5/4, \\ 1 & \text{for } t \geq 7/4. \end{cases}$$

We shall frequently use the rescaled functions given in terms of parameters $\nu' \geq 2\nu > 0$ as

$$\begin{aligned} \chi_\nu(t) &= \chi_\nu^+(t) = \chi(t/\nu), \\ \bar{\chi}_\nu &= \chi_\nu^- = 1 - \chi_\nu, \\ \chi_{\nu,\nu'} &= \chi_\nu \bar{\chi}_{\nu'}. \end{aligned} \tag{1.14}$$

For any self-adjoint operator T and state ϕ we abbreviate $\langle T \rangle_\phi = \langle \phi, T\phi \rangle$.

2. Preliminaries, ideas and proof

We collect some preliminaries, explain the ideas of the proof of Proposition 1.6 and then give the proof. The latter is done in Subsection 2.3 using results of appendices.

2.1. **Exponential decay.** According to [Gri, Theorem 2.5(1)] stated there without proof (a reference to Grisemer's thesis is given) any non-threshold eigenstate decays exponentially to an order determined by the thresholds above the corresponding eigenvalue, cf. the result of [FH] for usual N-body Hamiltonians. This is a consequence of the hard-core Mourre estimate by arguments rather similar to the ones of [FH]. Below we briefly explain some preliminary ingredients of our proof of Lemma 1.5 and the proof of this application, see Appendices B and C for complete proofs in the present context. More precisely we show in Appendix C that if H does not have positive thresholds then any $\phi \in \mathcal{D}(H)$ with $(H-E)\phi = 0$ for some $E > 0$ is exponentially decaying to any order, whence super-exponentially decaying, that is $e^{\sigma|x|}\phi \in \mathcal{H}$ for all $\sigma \geq 0$.

2.1.1. *Potential function.* Since the derivative $\tilde{\omega}_*$ of Lemma 1.2 is symmetric we can write

$$\tilde{\omega} = \nabla r^2/2.$$

The function $r = r(x)$ can be chosen positive, smooth and convex, see the proof of [De, Proposition 3.4] (we remark that [De] also uses the Graf construction although with a different regularization procedure). Let us introduce

$$\omega = \nabla r \text{ and } \partial^r f = \mathrm{i}p^r f := \omega \cdot \nabla f.$$

Note that $\tilde\omega = r\omega$. From the convexity of r we learn that

$$\partial^r |\mathrm{d}r|^2 \geq 0. \tag{2.1a}$$

We have a slight extension of a part of (1.6), cf. [De, Lemma 3.3 (f)] and [DeGé, (5.2.8)],

$$\forall \alpha \in \mathbb{N}_0^{\dim \mathbf{X}} \text{ and } k \in \mathbb{N}_0 : |\partial_x^\alpha (x\cdot\nabla)^k \big(r^2 - x^2\big)| \leq C_\alpha. \tag{2.1b}$$

In particular we obtain yet another property

$$\forall \alpha \in \mathbb{N}_0^{\dim \mathbf{X}} : |\partial_x^\alpha \big(|\mathrm{d}r|^2 - 1\big)| \leq C_\alpha \langle x\rangle^{-2}. \tag{2.1c}$$

In fact letting $f = r^2 - x^2$ the bounds (2.1c) follow from (2.1b) and the identity

$$|\mathrm{d}r|^2 - 1 = \frac{x\cdot\nabla f + 4^{-1}|\mathrm{d}f|^2 - f}{x^2 + f}.$$

The rescaled r reads

$$r_R(x) = Rr(x/R),$$

so that $\tilde\omega_R = \nabla r_R^2/2$. Clearly the bounds (2.1a)-(2.1c) are also valid for the rescaled r (possibly with R-dependent constants). We also rescale the partition of unity functions of Lemma 1.2 introducing $\tilde q_{a,R}(x) = \tilde q_a(x/R)$, and similarly for the "quadratic" partition of unity functions

$$q_b(x) := \tilde q_b(kx)\Big(\sum_c \tilde q_c(kx)^2\Big)^{-1/2};\ k = r_1/r_2.$$

Using that

$$\tilde q_c(x)\tilde q_b(kx) = 0 \text{ if } c \not\subseteq b,$$

and Lemma 1.2 (1) we conclude that

$$\tilde\omega_*(x) \geq \sum_b \pi_b q_b^2(x), \tag{2.2}$$

and similarly for the rescaled quantities.

2.1.2. *Commutator calculation.* We formally calculate (or more precisely *define*)

$$\mathrm{i}[H, A_R] = 2p\tilde\omega_*(x/R)p - (4R^2)^{-1}\big(\Delta^2 r^2\big)(x/R) - 2\tilde\omega_R\cdot\nabla V, \tag{2.3}$$

and using (2.2) we thus deduce

$$\begin{aligned}\mathrm{i}[H, A_R] &\geq 2\sum_b q_{b,R}\, p_b^2\, q_{b,R} + O\big(R^{-2}\big) - 2\tilde\omega_R\cdot\nabla V\\ &= 2\sum_b q_{b,R}\, p_b^2\, q_{b,R} + O\big(R^{-\min\{2,\varepsilon\}}\big).\end{aligned} \tag{2.4}$$

2.2. **Super-exponentially decaying states.** We are heading at proving absence of positive eigenvalues using the following three-step procedure: 1) Using the assumption of absence of positive thresholds and the hard-core Mourre estimate we deduce that any eigenstate with corresponding positive eigenvalue decays super-exponentially, cf. Subsection 2.1. 2) We show that any such state must vanish outside a bounded set. 3) We invoke the unique continuation property and conclude that any such state vanishes.

2.2.1. *Homogeneous vector field and potential function, and distortions.* We shall use a potential function introduced in [Ya]. Its construction and some properties are somewhat similar to those for the previously discussed potential function. As before the specific construction will not be relevant for us. We only need the following result, cf. [Ya].

Lemma 2.1. *There exists a real-valued $m \in C^\infty(\mathbf{X} \setminus \{0\})$ with following properties:*

(1) *m is homogeneous of degree one.*
(2) *$m(x) \geq 1$ for $|x| = 1$.*
(3) *m is convex.*
(4) *There exists $\delta \in (0,1)$ such that for all $a \in \mathcal{A}$*

$$m(x) = m(x_a) \text{ if } |x_a| > (1-\delta)|x|. \tag{2.5}$$

We construct a distorted version of the function m of Lemma 2.1 and the associated vector field as follows. This is in terms of a (small) parameter $\epsilon \in (0,1)$ and a (large) parameter $R > 1$ to be fixed below. Pick $g \in C^\infty([0,\infty))$ with $g(s) = s - s^{1-\epsilon}$ for s large, $g(s) = R$ for $s < R$ and $g'(s), g''(s) \geq 0$ for all $s > 0$. Define then

$$r(x) = g(m(x)) \text{ and } \omega(x) = \nabla r(x). \tag{2.6}$$

This is doable since indeed the function $\tilde{g}(s) := s - s^{1-\epsilon}$ obeys $\tilde{g}'(s), \tilde{g}''(s) \geq 0$ for $s \geq 1$. (Our construction is somewhat inspired by [RT] in which the convexity of the function $|x| - (1+|x|)^{1-\epsilon}$ on $\mathbb{R}^n$, $n \geq 3$, is used although in a different context.) Note also that we used the same notation as before although the new functions r and ω are different from the old functions r and ω, respectively. We are going to use the new quantities below and in Subsection 2.3. The analogue of (1.7) is

$$A = \tfrac{1}{2}(\nabla r^2 \cdot p + p \cdot \nabla r^2) = r\omega \cdot p + p \cdot r\omega. \tag{2.7}$$

We fix the parameter R so big that $\nabla r^2 = 2r\omega$ is complete in Ω, cf. the properties (4) (of Lemma 2.1) and $\mathbf{X}^a \setminus \Omega_a$ be compact (see Remark A.3 for more details), and so big that $\omega \cdot \nabla V_b^{(2)} = 0$, cf. the properties (4) and $V_b^{(2)}$ be supported in a bounded set. The parameter $\epsilon \in (0,1)$ is chosen such that $\epsilon < \varepsilon$ where ε is given in Condition 1.4.

It is not known whether there is a Mourre estimate for the operator (2.7) (as for the one defined by (1.7)). On the other hand, as the reader will see, the new A yields some useful bound for super-exponentially decaying eigenstates. To our knowledge such bound does not follow from using the previous operator A.

We compute

$$\nabla^2 r = g' \nabla^2 m + g'' \mathrm{d}m \otimes \mathrm{d}m, \tag{2.8a}$$

$$\tfrac{1}{2}\nabla^2 r^2 = r\nabla^2 r + \mathrm{d}r \otimes \mathrm{d}r, \tag{2.8b}$$

$$\omega \cdot \nabla |\mathrm{d}r|^2 = 2\nabla^2 r(\omega,\omega). \tag{2.8c}$$

In particular (seen by using (3) and (2.8a) twice) r and r^2 are convex, and

$$\omega \cdot \nabla |\mathrm{d}r|^2 \geq 2(g')^2 g'' |\mathrm{d}m|^4 \geq 0. \tag{2.9}$$

By (1), (2) and the Euler's homogeneous function theorem

$$\hat{x} \cdot \nabla m(x) = m(\hat{x}) \geq 1;\ \hat{x} := x/|x|.$$

Whence $|\mathrm{d}m| \geq 1$, and we get from (2.9) the lower bound

$$r\omega \cdot \nabla |\mathrm{d}r|^2 \geq c r^{-\epsilon} \text{ for } |x| \text{ large.} \tag{2.10}$$

2.2.2. *Idea of procedure.* Let us explain (formally) how we are going to use these properties: For $\sigma \geq 1$ we compute

$$\begin{aligned} \mathrm{i}[H_0, \mathrm{e}^{\sigma r} A \mathrm{e}^{\sigma r}] &= \mathrm{e}^{\sigma r}\big(\mathrm{i}[H_0^\sigma, A] + \sigma(p_r A + A p_r)\big)\mathrm{e}^{\sigma r}; \\ H_0^\sigma &= H_0 - \tfrac{\sigma^2}{2}|\mathrm{d}r|^2, \\ p_r &= \tfrac{1}{2}(\omega \cdot p + p \cdot \omega). \end{aligned} \tag{2.11}$$

Here (formally)

$$\mathrm{i}[H_0^\sigma, A] = p\nabla^2 r^2 p - \tfrac{1}{4}\Delta^2 r^2 + \sigma^2 r\omega \cdot \nabla |\mathrm{d}r|^2.$$

Noting the formulas

$$\begin{aligned} A &= 2rp_r - \mathrm{i}|\mathrm{d}r|^2, \\ A &= 2p_r r + \mathrm{i}|\mathrm{d}r|^2, \end{aligned}$$

this leads to the identity (cf. (2.3))

$$\mathrm{i}[H_0, \mathrm{e}^{\sigma r} A \mathrm{e}^{\sigma r}] = \mathrm{e}^{\sigma r}\big(p\nabla^2 r^2 p + 4\sigma p_r r p_r - \tfrac{1}{4}\Delta^2 r^2 + \sigma(\sigma r - 1)\omega \cdot \nabla |\mathrm{d}r|^2\big)\mathrm{e}^{\sigma r}.$$

Applied to states localized at infinity and using (2.10) we then obtain that

$$\mathrm{e}^{-\sigma r}\mathrm{i}[H_0, \mathrm{e}^{\sigma r} A \mathrm{e}^{\sigma r}]\mathrm{e}^{-\sigma r} \geq 4\sigma p_r r p_r - C r^{-2} + \sigma^2 c r^{-\epsilon}.$$

Computing and estimating

$$\mathrm{e}^{-\sigma r}\mathrm{i}[V, \mathrm{e}^{\sigma r} A \mathrm{e}^{\sigma r}]\mathrm{e}^{-\sigma r} = -2r\omega \cdot \nabla V \geq -C r^{-\varepsilon},$$

we conclude using that $\epsilon < \min(\varepsilon, 1)$ that at infinity

$$\mathrm{i}[H, \mathrm{e}^{\sigma r} A \mathrm{e}^{\sigma r}] \geq \sigma^2 \tilde{c} r^{-\epsilon} \mathrm{e}^{2\sigma r}.$$

Roughly our idea (to be implemented in Subsection 2.3) is to apply this bound to a localization of any super-exponentially decaying eigenstate (localized at infinity). By undoing the commutator (a virial type argument) we shall obtain a bound from which we can deduce that the eigenstate vanishes outside a bounded set (done by letting $\sigma \to \infty$).

2.3. **Implementation of idea.** Under Condition 1.4 we shall show that

$$\begin{gathered} (H - E)\phi = 0,\ E \in \mathbb{R},\ \text{and } \forall \sigma \geq 0:\ \mathrm{e}^{\sigma r}\phi \in \mathcal{H} = L^2(\Omega) \\ \Rightarrow \phi = 0 \text{ outside a bounded set.} \end{gathered} \tag{2.12}$$

Here and below r is given by (2.6) with the parameters R and ϵ as specified (although above r could be replaced by $|x|$ of course). For ϕ given as in (2.12) we let for $\sigma, \nu \geq 1$

$$\phi_\sigma = \phi_{\sigma,\nu} := \chi_\nu \mathrm{e}^{\sigma(r - 4\nu)}\phi;\ \chi_\nu = \chi_\nu(r); \tag{2.13}$$

here (1.14) is used. By assumption $\phi_\sigma \in \mathcal{H}$. Putting $H^\sigma = H - \frac{\sigma^2}{2}|\mathrm{d}r|^2$ we note that (formally)

$$(H^\sigma - E)\phi_\sigma = -\mathrm{i}\sigma p_r \phi_\sigma - \mathrm{i}\mathrm{e}^{\sigma(r-4\nu)}R(\nu)\phi, \tag{2.14}$$

where $R(\nu) = \mathrm{i}[H_0, \chi_\nu] = \mathrm{Re}\,\big(\chi_\nu' p^r\big)$. Here and henceforth

$$p^r f = -\mathrm{i}\partial^r f = -\mathrm{i}\omega \cdot \nabla f;$$

whence $p_r = \mathrm{Re}\, p^r$.

2.3.1. *Undoing the commutator.* Using Lemma A.10, Corollary A.11 and Lemma A.12 we can indeed "undo the commutator"

$$\mathrm{i}[H^\sigma, A] := p\nabla^2 r^2 p - \tfrac{1}{4}\Delta^2 r^2 + \sigma^2 r\partial^r|\mathrm{d}r|^2 - 2r\omega\cdot\nabla V, \tag{2.15}$$

and use (2.14). Whence

$$\langle \mathrm{i}[H^\sigma, A]\rangle_{\phi_\sigma} = -2\sigma\,\mathrm{Re}\,\langle p_r A\rangle_{\phi_\sigma} - 2\,\mathrm{Re}\,\langle R(\nu)\mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi. \tag{2.16}$$

The first term of (2.16) is computed

$$\begin{aligned} &-2\sigma\,\mathrm{Re}\,(p_r A)\\ &= -\sigma p_r(2rp_r - \mathrm{i}|\mathrm{d}r|^2) + \mathrm{h.c.}\\ &= -4\sigma p_r r p_r + \sigma(\partial^r|\mathrm{d}r|^2). \end{aligned} \tag{2.17}$$

As for the second term we estimate (recall the notation $\bar\chi_\nu = 1-\chi_\nu$)

$$\begin{aligned} &-2\,\mathrm{Re}\,\langle R(\nu)\mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi\\ &\le \|\mathrm{e}^{\sigma(r-4\nu)}R(\nu)\phi\|^2 + \|\bar\chi_{2\nu}A\chi_\nu\mathrm{e}^{\sigma(r-4\nu)}\phi\|^2\\ &\le \big\{\|\chi_\nu'\mathrm{e}^{\sigma(r-4\nu)}p^r\phi\| + \tfrac12\|(\chi_\nu''|\mathrm{d}r|^2 + \chi_\nu'(\Delta r))\mathrm{e}^{\sigma(r-4\nu)}\phi\|\big\}^2\\ &\quad + \big\{\|2r\bar\chi_{2\nu}\chi_\nu\mathrm{e}^{\sigma(r-4\nu)}p^r\phi\| + \|\bar\chi_{2\nu}(2r|\mathrm{d}r|^2\chi_\nu' + 2\sigma r\chi_\nu|\mathrm{d}r|^2 + \tfrac12(\Delta r^2)\chi_\nu)\mathrm{e}^{\sigma(r-4\nu)}\phi\|\big\}^2\\ &\le C\nu^2\|\chi_{\nu/2}|p\phi|\|^2 + C\nu^2\sigma^2\|\phi\|^2\\ &\le C\nu^2\langle p^2\rangle_\phi + C\nu^2\sigma^2\|\phi\|^2. \end{aligned}$$

Using infinitesimal smallness of the potential we have for some $C>0$

$$\langle p^2\rangle_\phi \le \langle 4H + C\rangle_\phi = (4E+C)\|\phi\|^2,$$

and we deduce that

$$-2\,\mathrm{Re}\,\langle R(\nu)\mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu\mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi \le C\nu^2\sigma^2\|\phi\|^2. \tag{2.18}$$

2.3.2. *Doing the commutator.* On the other hand using (2.10) we obtain

$$\langle \mathrm{i}[H^\sigma, A]\rangle_{\phi_\sigma} \ge \langle \sigma^2 r\partial^r|\mathrm{d}r|^2 - Cr^{-\min(2,\varepsilon)}\rangle_{\phi_\sigma} \ge \sigma^2\tilde c\langle r^{-\epsilon}\rangle_{\phi_\sigma} + \langle\sigma\partial^r|\mathrm{d}r|^2\rangle_{\phi_\sigma}; \tag{2.19}$$

this is provided $\nu\ge 1$ is sufficiently large. We fix any such ν.

2.3.3. *Final estimate and conclusion.* We combine (2.16)–(2.19) and obtain that

$$C\nu^2\sigma^2\|\phi\|^2 \ge 4\sigma\langle r\rangle_{p_r\phi_\sigma} + \sigma^2\tilde c\langle r^{-\epsilon}\rangle_{\phi_\sigma} \ge \sigma^2\tilde c\langle r^{-\epsilon}\rangle_{\phi_\sigma}.$$

Letting then $\sigma\to\infty$ we deduce that $\chi_{4\nu}(r)\phi\equiv 0$.

Finally Proposition 1.6 follows by invoking the unique continuation property (assumed to hold).

Appendix A. Justifying computations

In this appendix we show how to "undo the commutator" $\mathrm{i}[H,A]$. We do it simultaneously for the A given by (1.7) and the A given by (2.7) to justify (C.12) and (2.16), respectively. For a different regularization procedure, see [Gri, Lemma 3.15].

A.1. **Setting.** We shall work in a generalized setting on a Riemannian manifold (see [Cha] for geometric notions), and present all conditions needed for the argument independently of the previous sections. The case of a constant metric is sufficient for application to (C.12) and (2.16). The verification of the conditions below under Condition 1.4 is straightforward.

Let (Ω, g) be a Riemannian manifold of dimension $d \geq 1$, and consider the Schrödinger operator on $\mathcal{H} = L^2(\Omega) = L^2(\Omega, (\det g)^{1/2}\mathrm{d}x)$:

$$H = H_0 + V; \quad H_0 = -\tfrac{1}{2}\Delta = \tfrac{1}{2}p_i^* g^{ij} p_j, \quad p_i = -\mathrm{i}\partial_i.$$

We realize H_0 as a self-adjoint operator by imposing the Dirichlet boundary condition, i.e. H_0 is the unique self-adjoint operator associated with the closure of the quadratic form

$$\langle H_0 \rangle_\psi = \langle \psi, -\tfrac{1}{2}\Delta\psi \rangle, \quad \psi \in C_{\mathrm{c}}^\infty(\Omega).$$

We denote the form closure and the self-adjoint realization by the same symbol H_0. Moreover, we consider the weighted spaces

$$\mathcal{H}^s = (H_0 + 1)^{-s/2}\mathcal{H}, \quad s \in \mathbb{R},$$

and H_0 may also be understood as $\mathcal{H}^s \to \mathcal{H}^{s-2}$, $s \in \mathbb{R}$. For the realization of $H = H_0 + V$ we assume the following condition:

Condition A.1. The potential V is a locally integrable real-valued function, and there exist $\delta \in [0, 1)$ and $C > 0$ such that for any $\psi \in C_{\mathrm{c}}^\infty(\Omega)$

$$|\langle V \rangle_\psi| \leq \delta \langle H_0 \rangle_\psi + C\|\psi\|^2.$$

By this condition we can extend the form domain of V as $Q(V) = \mathcal{H}^1$, and the extended form defines a bounded operator $V\colon \mathcal{H}^1 \to \mathcal{H}^{-1}$. Henceforth we consider $H = H_0 + V$ as a closed quadratic form on $Q(H) = \mathcal{H}^1$ or, alternatively, as a bounded operator $\mathcal{H}^1 \to \mathcal{H}^{-1}$. In Subsection A.3 we shall also consider the self-adjoint realization of H on $\mathcal{H}$ (also denoted by H), which is the restriction of $H\colon \mathcal{H}^1 \to \mathcal{H}^{-1}$ to the domain:

$$\mathcal{D}(H) = \{\psi \in \mathcal{H}^1 \,|\, H\psi \in \mathcal{H}\} \subset \mathcal{H}.$$

We next assume a regularity condition for the (virtual) boundary of Ω:

Condition A.2. There exists a real-valued function $r \in C^\infty(\Omega)$ such that:

(1) The gradient vector field $\operatorname{grad} r^2$ on Ω is complete.
(2) The following bounds hold:

$$\sup|\mathrm{d}r| < \infty, \quad \sup|\nabla^2 r^2| < \infty, \quad \sup|\mathrm{d}\Delta r^2| < \infty. \tag{A.1}$$

Remark A.3. The function r of Condition A.2 is indeed a generalization of the two r's of Section 2. We refer to (2.1b) and Lemmas 1.2 and 2.1 for properties. Note that in both cases the vector field $\operatorname{grad} r^2$ is defined and complete on $\mathbf{X} \supset \Omega$, cf. Lemma 1.2 (5) and Lemma 2.1 (1). The completeness on Ω is then valid intuitively because the vector fields are tangent to the boundary $\partial\Omega$, cf. Lemma 1.2 (2) and Lemma 2.1 (4). Rigorously Condition A.2 (1) can be seen as a consequence of Lemma 1.2 (2), Lemma 2.1 (4) and the inclusion

$$\partial\Omega \subset \cup_{b \in \mathcal{B}}(\partial\Omega_b + \mathbf{X}_b),$$

indeed excluding that the flow can hit the boundary in finite time. Clearly Condition A.2 (2) follows from (2.1b) and Lemma 2.1 (1).

By Condition A.2 (1) the vector field $\operatorname{grad} r^2$ generates a one-parameter group of diffeomorphisms on Ω, which we denote by

$$\mathrm{e}^{2\cdot}\cdot\colon \mathbb{R}\times\Omega\to\Omega,\quad (t,x)\mapsto \mathrm{e}^{2t}x. \tag{A.2}$$

This satisfies by definition, in local coordinates,

$$\partial_t(\mathrm{e}^{2t}x)^i = g^{ij}(\mathrm{e}^{2t}x)(\partial_j r^2)(\mathrm{e}^{2t}x). \tag{A.3}$$

We define the group of *dilations* $\mathrm{e}^{\mathrm{i}tA}\colon \mathcal{H}\to\mathcal{H}$ with respect to r as the one-parameter group of unitary operators

$$\mathrm{e}^{\mathrm{i}tA}u(x) = J(\mathrm{e}^{2t};x)^{1/2}\left(\frac{\det g(\mathrm{e}^{2t}x)}{\det g(x)}\right)^{1/4} u(\mathrm{e}^{2t}x),$$

where J is the relevant Jacobian. Note that there is another expression:

$$\mathrm{e}^{\mathrm{i}tA}u(x) = \exp\left(\int_0^t \tfrac12(\Delta r^2)(\mathrm{e}^{2s}x)\,\mathrm{d}s\right) u(\mathrm{e}^{2t}x). \tag{A.4}$$

We let A be the generator of $\mathrm{e}^{\mathrm{i}tA}$. By the unitarity of $\mathrm{e}^{\mathrm{i}tA}$ the operator A is self-adjoint, and $C^\infty_{\mathrm{c}}(\Omega)\subseteq\mathcal{D}(A)$ is a core for it. In fact, the dense subspace $C^\infty_{\mathrm{c}}(\Omega)\subseteq\mathcal{H}$ is invariant under $\mathrm{e}^{\mathrm{i}tA}$, and for any $u\in C^\infty_{\mathrm{c}}(\Omega)$ the limit

$$\lim_{t\to 0} t^{-1}(\mathrm{e}^{\mathrm{i}tA}u-u)$$

exists in $\mathcal{H}$. Note that by (A.4) (when applied to vectors in $C^\infty_{\mathrm{c}}(\Omega)$) the operator A takes the form

$$A = \mathrm{i}[H_0,r^2] = \tfrac12\{(\partial_i r^2)g^{ij}p_j + p_i^* g^{ij}(\partial_j r^2)\} = rp^r + (p^r)^* r,$$

where $p^r = -\mathrm{i}\partial^r = -\mathrm{i}(\partial_i r)g^{ij}\partial_j$.

Let us first consider the commutator $\mathrm{i}[H,A]$ as a quadratic form defined for $\psi\in C^\infty_{\mathrm{c}}(\Omega)$ by

$$\langle \mathrm{i}[H,A]\rangle_\psi = \mathrm{i}\langle H\psi, A\psi\rangle - \mathrm{i}\langle A\psi, H\psi\rangle.$$

In order to discuss its extension we impose the following abstract form bound condition, which is not quite independent of Conditions A.1 and A.2 (see for example [IS1, Corollary 4.2]).

Condition A.4. There exists $C>0$ such that for any $\psi\in C^\infty_{\mathrm{c}}(\Omega)$

$$|\langle \mathrm{i}[H,A]\rangle_\psi| \le C\langle H_0+1\rangle_\psi.$$

Similarly to the above, we henceforth regard $\mathrm{i}[H,A]$ as a quadratic form on $Q(\mathrm{i}[H,A])=\mathcal{H}^1$ (which may not be closed) or as a bounded operator $\mathcal{H}^1\to\mathcal{H}^{-1}$.

A.2. **Regularity properties of flow.** We prove a regularity properties of the flow (A.2) and its quantum implementation (A.4).

Lemma A.5. *There exist $d,C>0$ such that for any $t\in\mathbb{R}$ and $x\in\Omega$*

$$d\mathrm{e}^{-C|t|} \le g^{ij}(x)g_{kl}(\mathrm{e}^{2t}x)[\partial_i(\mathrm{e}^{2t}x)^k][\partial_j(\mathrm{e}^{2t}x)^l] \le d\mathrm{e}^{C|t|}. \tag{A.5}$$

Proof. We note that the expression in the middle of (A.5) is independent of choice of coordinates. Fix $x \in \Omega$ and choose coordinates such that $g_{ij}(x) = \delta_{ij}$. Consider the vector fields along $\{\mathrm{e}^{2t}x\}_{t\in\mathbb{R}}$ given by $\partial_i \mathrm{e}^{2t}x$ and $\partial_j \mathrm{e}^{2t}x$. Since the Levi-Civita connection ∇ is compatible with the metric,

$$\begin{aligned}\tfrac{\partial}{\partial t} g_{kl}(\mathrm{e}^{2t}x)[\partial_i(\mathrm{e}^{2t}x)^k][\partial_j(\mathrm{e}^{2t}x)^l] &= \tfrac{\partial}{\partial t}\langle \partial_i \mathrm{e}^{2t}x, \partial_j \mathrm{e}^{2t}x\rangle \\ &= \langle \nabla_{\partial_t \mathrm{e}^{2t}x}\partial_i \mathrm{e}^{2t}x, \partial_j \mathrm{e}^{2t}x\rangle + \langle \partial_i \mathrm{e}^{2t}x, \nabla_{\partial_t \mathrm{e}^{2t}x}\partial_j \mathrm{e}^{2t}x\rangle.\end{aligned} \tag{A.6}$$

(The definition of $\nabla_{\partial_t \mathrm{e}^{2t}x}$ is given below.) From (A.3) it follows that

$$\begin{aligned}\nabla_{\partial_t \mathrm{e}^{2t}x}\partial_i(\mathrm{e}^{2t}x)^\bullet &= \partial_t\partial_i(\mathrm{e}^{2t}x)^\bullet + [\partial_t(\mathrm{e}^{2t}x)^k]\Gamma^\bullet_{kl}\partial_i(\mathrm{e}^{2t}x)^l \\ &= \partial_i\partial_t(\mathrm{e}^{2t}x)^\bullet + (g^{km}\partial_m r^2)\Gamma^\bullet_{kl}\partial_i(\mathrm{e}^{2t}x)^l \\ &= [\partial_i(\mathrm{e}^{2t}x)^k]\partial_k(g^{\bullet l}\partial_l r^2) + [\partial_i(\mathrm{e}^{2t}x)^l]\Gamma^\bullet_{kl}g^{km}\partial_m r^2 \\ &= \nabla_{\partial_i \mathrm{e}^{2t}x}(g^{\bullet l}\partial_l r^2) \\ &= g^{\bullet l}[\partial_i(\mathrm{e}^{2t}x)^k](\nabla^2 r^2)_{kl}.\end{aligned}$$

Thus, plugging this into (A.6) and taking a contraction with $g^{ij}(x) = \delta^{ij}$, we obtain

$$\Big|\tfrac{\partial}{\partial t} g^{ij}(x) g_{kl}(\mathrm{e}^{2t}x)[\partial_i(\mathrm{e}^{2t}x)^k][\partial_j(\mathrm{e}^{2t}x)^l]\Big| \le C g^{ij}(x) g_{kl}(\mathrm{e}^{2t}x)[\partial_i(\mathrm{e}^{2t}x)^k][\partial_j(\mathrm{e}^{2t}x)^l].$$

Noting $g^{ij}(x) g_{kl}(\mathrm{e}^{2t}x)[\partial_i(\mathrm{e}^{2t}x)^k][\partial_j(\mathrm{e}^{2t}x)^l]\big|_{t=0} = d$, we have (A.5). ☐

Lemma A.6. *For any $s \in [-1,1]$ the inclusion $\mathrm{e}^{\mathrm{i}tA}\mathcal{H}^s \subseteq \mathcal{H}^s$ holds, and*

$$\sup_{|t|<1} \|\mathrm{e}^{\mathrm{i}tA}\|_{\mathcal{B}(\mathcal{H}^s)} < \infty. \tag{A.7}$$

Moreover, $\mathrm{e}^{\mathrm{i}tA}\colon \mathcal{H}^s \to \mathcal{H}^s$ is strongly continuous in $t \in \mathbb{R}$.

Proof. Let us first set $s = 1$. For any $\psi \in C^\infty_{\mathrm{c}}(\Omega)$ we can compute by (A.4)

$$\begin{aligned} & p_i(\mathrm{e}^{\mathrm{i}tA}\psi)(x) \\ &= \left(\int_0^t \tfrac12 [p_i(\mathrm{e}^{2s}x)^j](\partial_j \Delta r^2)(\mathrm{e}^{2s}x)\,\mathrm{d}s\right)(\mathrm{e}^{\mathrm{i}tA}\psi)(x) + [\partial_i(\mathrm{e}^{2t}x)^j](\mathrm{e}^{\mathrm{i}tA}p_j\psi)(x).\end{aligned} \tag{A.8}$$

Here and below we slightly abuse notation writing $(\mathrm{e}^{\mathrm{i}tA}p_j\psi)(x)$ rather than the expression $\mathrm{e}^{\int\cdots}(p_j\psi)(e^{2t}x)$. Then by (A.1) and Lemma A.5 for any $|t| \le T$

$$\begin{aligned}\|\mathrm{e}^{\mathrm{i}tA}\psi\|^2_{\mathcal{H}^1} &= \|\psi\|^2_{\mathcal{H}} + \|p\mathrm{e}^{\mathrm{i}tA}\psi\|^2_{\mathcal{H}} \\ &\le \|\psi\|^2_{\mathcal{H}} + C_T\|\mathrm{e}^{\mathrm{i}tA}\psi\|^2_{\mathcal{H}} + C_T\|\mathrm{e}^{\mathrm{i}tA}p\psi\|^2_{\mathcal{H}} \\ &\le C_T\|\psi\|^2_{\mathcal{H}^1}.\end{aligned}$$

By a density argument this implies $\mathrm{e}^{\mathrm{i}tA}\mathcal{H}^1 \subseteq \mathcal{H}^1$, and moreover for any $\psi \in \mathcal{H}^1$ and $|t| \le T$

$$\|\mathrm{e}^{\mathrm{i}tA}\psi\|^2_{\mathcal{H}^1} \le C_T\|\psi\|^2_{\mathcal{H}^1}.$$

Thus (A.7) follows for $s = 1$. As for the strong continuity as $\mathcal{H}^1 \to \mathcal{H}^1$, we can show it first on $C^\infty_{\mathrm{c}}(\Omega)$ using (A.8) and standard regularity properties for flows, and then extend it by the boundedness.

We can show the same results for $s = -1$ by taking the adjoint, and then the assertions are proved for $s \in (-1,1)$ by interpolation. ☐

Lemma A.7. *There exists $C > 0$ such that for any $|t| < 1$*
$$\|H\mathrm{e}^{\mathrm{i}tA} - \mathrm{e}^{\mathrm{i}tA}H\|_{\mathcal{B}(\mathcal{H}^1,\mathcal{H}^{-1})} \le C|t|$$

Proof. As a quadratic form on $C_\mathrm{c}^\infty(\Omega)$, or as an operator $C_\mathrm{c}^\infty(\Omega) \to \mathcal{H}^{-1}$,
$$\begin{aligned} H\mathrm{e}^{\mathrm{i}tA} - \mathrm{e}^{\mathrm{i}tA}H &= \int_0^t \tfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{\mathrm{i}(t-s)A}H\mathrm{e}^{\mathrm{i}sA}\,\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{\mathrm{i}sA}\mathrm{i}[H,A]\mathrm{e}^{\mathrm{i}(t-s)A}\,\mathrm{d}s. \end{aligned}$$
Then by Lemma A.6 and the density of $C_\mathrm{c}^\infty(\Omega) \subseteq \mathcal{H}^1$ the assertion follows. □

Lemma A.8. *The following strong limit to the right exists in $\mathcal{B}(\mathcal{H}^1,\mathcal{H}^{-1})$, and the following equality holds*
$$\mathrm{i}[H,A] = \operatorname*{s-lim}_{t\to 0} t^{-1}[H\mathrm{e}^{\mathrm{i}tA} - \mathrm{e}^{\mathrm{i}tA}H]. \tag{A.9}$$

Proof. For any $\psi \in C_\mathrm{c}^\infty(\Omega)$
$$t^{-1}(H\mathrm{e}^{\mathrm{i}tA} - \mathrm{e}^{-\mathrm{i}tA}H)\psi - \mathrm{i}[H,A]\psi = t^{-1}\int_0^t \big\{\mathrm{e}^{\mathrm{i}sA}\mathrm{i}[H,A]\mathrm{e}^{\mathrm{i}(t-s)A} - \mathrm{i}[H,A]\big\}\psi\,\mathrm{d}s.$$
We use the strong continuity of $\mathrm{e}^{\mathrm{i}tA}$ of Lemma A.6 to obtain (A.9) on $C_\mathrm{c}^\infty(\Omega)$. Then by Lemma A.7 and the density argument, the strong limit of (A.9) exists in $\mathcal{B}(\mathcal{H}^1,\mathcal{H}^{-1})$. □

Remark A.9. Using terminology of [GGM] Lemmas A.6 and A.7 assert that $H \in C^1\big(A_{\mathcal{H}^1}, A_{\mathcal{H}^{-1}}\big)$. The statement (A.9) can be viewed as a consequence of this abstract property, see [GGM, Proposition 2.29], however we gave the proof in our concrete setting.

A.3. **Applications.** We shall henceforth use and consider the cutoff-functions of (1.14) as being functions of the r from Condition A.2 (i.e. as composite functions).

Lemma A.10. *Let $\phi \in \mathcal{D}(H)$. Suppose $\mathrm{e}^{\sigma r}\phi, \mathrm{e}^{\sigma r}H\phi \in \mathcal{H}$ for some $\sigma \ge 0$. Then $\mathrm{e}^{\sigma r}\phi \in \mathcal{D}(H)$ and there exists a sequence $\tilde\phi_m \in C_\mathrm{c}^\infty(\Omega)$ such that, as $m \to \infty$,*
$$\|\mathrm{e}^{\sigma r}(\phi - \tilde\phi_m)\| + \|p\mathrm{e}^{\sigma r}(\phi - \tilde\phi_m)\| \to 0, \tag{A.10}$$

Proof. Step I. We first prove that $\mathrm{e}^{\sigma r}\bar\chi_\nu\phi \in \mathcal{D}(H)$. Since $\phi \in \mathcal{H}^1$, we have
$$\mathrm{e}^{\sigma r}\bar\chi_\nu\phi, \mathrm{e}^{\sigma r}\bar\chi_\nu p_j\phi \in \mathcal{H},$$
and hence $p_j\mathrm{e}^{\sigma r}\bar\chi_\nu\phi \in \mathcal{H}$ by (A.1). Choose a sequence $\phi_n \in C_\mathrm{c}^\infty(\Omega)$ such that, as $n \to \infty$,
$$\|\phi - \phi_n\| + \|p(\phi - \phi_n)\| \to 0. \tag{A.11}$$
Then by using (A.1) again we obtain
$$\mathrm{e}^{\sigma r}\bar\chi_\nu\phi_n \to \mathrm{e}^{\sigma r}\bar\chi_\nu\phi, \quad p_j\mathrm{e}^{\sigma r}\bar\chi_\nu\phi_n \to p_j\mathrm{e}^{\sigma r}\bar\chi_\nu\phi \quad \text{in } \mathcal{H}.$$
This implies that $\mathrm{e}^{\sigma r}\bar\chi_\nu\phi \in \mathcal{H}^1$. Note the distributional identity
$$H\mathrm{e}^{\sigma r}\bar\chi_\nu\phi = \mathrm{e}^{\sigma r}\bar\chi_\nu H\phi - \mathrm{e}^{\sigma r}(\sigma\bar\chi_\nu + \bar\chi_\nu')\partial^r\phi - \tfrac12(\Delta\mathrm{e}^{\sigma r}\bar\chi_\nu)\phi. \tag{A.12}$$
Then since $\phi, p_j\phi, H\phi \in \mathcal{H}$, and by (A.1)
$$\chi_\nu|\Delta r| = \tfrac{1}{2r}\chi_\nu|(\Delta r^2) - 2|\mathrm{d}r|^2| \le C, \tag{A.13}$$

we have $He^{\sigma r}\bar\chi_\nu\phi\in\mathcal H$. Hence $e^{\sigma r}\bar\chi_\nu\phi\in\mathcal D(H)$.

Step II. We next show that $e^{\sigma r}p_j\phi\in\mathcal H$. Noting that $e^{\sigma r}\bar\chi_\nu\phi\in\mathcal H^1$ as in Step I, we commute and estimate by Conditions A.1 and A.2

$$\begin{aligned}\|e^{\sigma r}\bar\chi_\nu p\phi\|^2&=\|pe^{\sigma r}\bar\chi_\nu\phi\|^2-\langle|\nabla e^{\sigma r}\bar\chi_\nu|^2-\tfrac12(\Delta e^{2\sigma r}\bar\chi_\nu^2)\rangle_\phi\\&\le 4\langle H\rangle_{e^{\sigma r}\bar\chi_\nu\phi}+C_{1,\sigma}\|e^{\sigma r}\phi\|^2.\end{aligned}$$

Whence, by reversing a commutation used above,

$$\begin{aligned}\|e^{\sigma r}\bar\chi_\nu p\phi\|^2&\le 4\mathrm{Re}\,\langle e^{\sigma r}\bar\chi_\nu\phi,e^{\sigma r}\bar\chi_\nu H\phi\rangle+C_{2,\sigma}\|e^{\sigma r}\phi\|^2\\&\le\|e^{\sigma r}H\phi\|^2+C_{3,\sigma}\|e^{\sigma r}\phi\|^2.\end{aligned}$$

Now we let $\nu\to\infty$ invoking the Lebesgue dominated convergence theorem, and we conclude that $e^{\sigma r}p_j\phi\in\mathcal H$.

Step III. We show that $e^{\sigma r}\phi\in\mathcal H^1$, and then we complete the proof. Note that $p_je^{\sigma r}\phi\in\mathcal H$ by Step II. We choose a sequence $\phi_n\in C_c^\infty(\Omega)$ satisfying (A.11) as $n\to\infty$, and consider the quantity

$$\|e^{\sigma r}\phi-e^{\sigma r}\bar\chi_\nu\phi_n\|+\|p(e^{\sigma r}\phi-e^{\sigma r}\bar\chi_\nu\phi_n)\|.\tag{A.14}$$

The first term of (A.14) is bounded by

$$\|e^{\sigma r}\phi-e^{\sigma r}\bar\chi_\nu\phi_n\|\le\|e^{\sigma r}\chi_\nu\phi\|+\|e^{\sigma r}\bar\chi_\nu(\phi-\phi_n)\|,$$

and the second term bounded by

$$\begin{aligned}&\|p(e^{\sigma r}\phi-e^{\sigma r}\bar\chi_\nu\phi_n)\|\\&\le\|pe^{\sigma r}\chi_\nu\phi\|+\|pe^{\sigma r}\bar\chi_\nu(\phi-\phi_n)\|\\&\le\|e^{\sigma r}\chi_\nu p\phi\|+C_\sigma\|e^{\sigma r}\bar\chi_{2\nu}\phi\|+\|e^{\sigma r}\bar\chi_\nu p(\phi-\phi_n)\|+C_\sigma\|e^{\sigma r}\bar\chi_{2\nu}(\phi-\phi_n)\|.\end{aligned}$$

We can make (A.14) arbitrarily small by first fixing ν large and then taking n large. Whence we obtain a sequence of states $\phi_{n(\cdot)}$ verifying

$$\|e^{\sigma r}\phi-e^{\sigma r}\bar\chi_{\nu(m)}\phi_{n(m)}\|+\|p(e^{\sigma r}\phi-e^{\sigma r}\bar\chi_{\nu(m)}\phi_{n(m)})\|\to0\tag{A.15}$$

as $m\to\infty$, and hence $e^{\sigma r}\phi\in\mathcal H^1$.

Finally using the distributional identity

$$He^{\sigma r}\phi=e^{\sigma r}H\phi-\sigma e^{\sigma r}\partial^r\phi-\tfrac12(\Delta e^{\sigma r})\phi$$

we learn, cf. (A.12) and (A.13), that $He^{\sigma r}\phi\in\mathcal H$ and hence that $e^{\sigma r}\phi\in\mathcal D(H)$. Clearly (A.10) follows from (A.15) by taking $\tilde\phi_m=\bar\chi_{\nu(m)}\phi_{n(m)}$. □

Corollary A.11. *Suppose $\phi\in\mathcal D(H)$ satisfies $e^{\sigma r}\phi, e^{\sigma r}H\phi, Ae^{\sigma r}\phi\in\mathcal H$ for some $\sigma\ge0$. Then $e^{\sigma r}\phi\in\mathcal D(H)\cap\mathcal D(A)$, and for all $\nu\ge1$ also $\psi=\chi_\nu e^{\sigma r}\phi\in\mathcal D(H)\cap\mathcal D(A)$.*

The following lemma is used in Subsection 2.3 to a state ψ of this type, i.e. $\psi=\chi_\nu e^{\sigma r}\phi\in\mathcal D(H)\cap\mathcal D(A)$. Another application is given in Appendix C (to derive (C.12)).

Lemma A.12. *Suppose $\psi\in\mathcal D(H)\cap\mathcal D(A)$. Then*

$$\langle i[H,A]\rangle_\psi=i\langle H\psi,A\psi\rangle-i\langle A\psi,H\psi\rangle.$$

Proof. By Lemma A.8

$$\langle i[H,A]\rangle_\psi=\lim_{t\to0}\langle t^{-1}[He^{itA}-e^{itA}H]\rangle_\psi=i\langle H\psi,A\psi\rangle-i\langle A\psi,H\psi\rangle.$$

□

Appendix B. Mourre estimate

We shall prove Lemma 1.5 along the lines of the proof of [Sk2, Lemma 4.1, Corollary 4.2, Proposition 4.4] and give some spectral consequences. The local uniformity in energy property of the lemma is not included in the analogous version [Sk2, Proposition 4.4], cf. Lemma 1.3. It is needed for the application in Appendix C.

B.1. **Squeezing lemma.** We introduce the following functions:

$$\eta_\delta(t)=\bar\chi_\delta(|t|),\ F(t<C)=1_{(-\infty,C)}(t) \text{ and } F(t\geq C)=1-F(t<C).$$

For any $c\in\mathcal{A}$ we introduce

$$\mathcal{H}_c=L^2(\Omega^c)\otimes L^2(\mathbf{X}_c)=L^2(\Omega^c+\mathbf{X}_c),$$

and note that for all $b\subset c$ there is an embedding $\mathcal{H}_c\subset\mathcal{H}_b$ due to the relation $\Omega^c+\mathbf{X}_c\subset\Omega^b+\mathbf{X}_b$. In particular $\mathcal{H}=\mathcal{H}_{a_{\max}}\subset\mathcal{H}_c\subset\mathcal{H}_{a_{\min}}=L^2(\mathbf{X})$. Recall that $H_c=H^c\otimes I+I\otimes(\frac12 p_c^2)$ and $H^{a_{\min}}=0$ on $L^2(\mathbf{X}^{a_{\min}})=\mathbb{C}$. By an approximation argument the operator H_c is realized as a hard-core Hamiltonian, more precisely as the operator associated with the (naturally defined) Dirichlet form on $Q(H_c):=H_0^1(\Omega^c+\mathbf{X}_c)$, cf. [Gri, Theorem 3.1]. Moreover a partial Fourier transform takes it to the direct integral $\int\oplus(H^c+\frac12\xi_c^2)\mathrm{d}\xi_c$. We shall also use the local compactness result

$$\forall c\in\mathcal{B}:\langle x^c\rangle^{-1}\langle p^c\rangle^{-1}\in\mathcal{C}(L^2(\Omega^c)),\tag{B.1}$$

where $\langle p^c\rangle=(-\Delta_{x^c}+1)^{1/2}$ is defined using the Dirichlet boundary condition. The set of thresholds $\mathcal{T}$ is defined in terms of hard-core sub-Hamiltonians by (1.4) and the function d is defined in terms of this set by (1.8).

Lemma B.1. *Let $\epsilon\in(0,1]$, $c\in\mathcal{A}$ and $h^c\in L^\infty(\mathbf{X}^c)$ with compact support, $\kappa>0$ and $E\in\mathbb{R}$ be given. Consider for $\delta>0$ the operator*

$$B_c=h^c(x^c)F(\tfrac12 p_c^2<d(E+\epsilon)-2\epsilon)\eta_\delta(H-E)\in\mathcal{B}(\mathcal{H},\mathcal{H}_c).$$

For all small enough $\delta>0$ there exists $K\in\mathcal{C}(\mathcal{H},\mathcal{H}_c)$ such that

$$\|B_c-K\|\leq\kappa.\tag{B.2}$$

Proof. Step I. We shall reduce the proof to proving the "uniform squeezing" result (B.7). Note that indeed $B_c=(h^c(x^c)\otimes F_c(p_c^2))\eta_\delta(H-E)\in\mathcal{B}(\mathcal{H},\mathcal{H}_c)$ with an obvious meaning of F_c. We abbreviate $B_c=(h^c\otimes F_c)\eta_\delta(H-E)$. Obviously for $c=a_{\max}$ it follows from (B.1) that B_c is compact, so in that case we can put $K=B_c$. To treat the general case we pick a family $\{j_a\}$, $a\in\mathcal{A}\setminus\{a_{\max}\}$, of functions on $\mathbf{X}$ each one being smooth and homogeneous of degree 0 outside a compact set. We assume that $0\leq j_a\leq1$, $\sum_a j_a=1$ and that

$$\exists C>0\forall b\not\subset a:\ |x|j_a(x)\leq C|x^b|j_a(x);\ x\in\mathbf{X}.\tag{B.3}$$

We decompose

$$B_c=\sum_a(h^c\otimes F_c)j_a\eta_\delta(H-E),\tag{B.4}$$

and distinguish between two cases: 1) $c\not\subset a$, and 2) $c\subset a$. Terms in the case 1) are compact due to (B.1) and (B.3). Now consider case 2). We consider for $\delta'\geq2\delta$ and large $R>1$

$$\begin{aligned} j_a\eta_\delta(H-E) &= \eta_{\delta'}(H_a-E)\chi_R(|x|)j_a\eta_\delta(H-E)+K+K(\delta')\eta_\delta(H-E); \quad \text{(B.5)}\\ K &= \bar\chi_R(|x|)j_a\eta_\delta(H-E),\\ K(\delta') &= \chi_R(|x|)j_a\eta_{\delta'}(H-E)-\eta_{\delta'}(H_a-E)\chi_R(|x|)j_a. \end{aligned}$$

Here K is compact due to (B.1), and therefore $(h^c\otimes F_c)K$ is compact. We claim that also $K(\delta')$ is compact, however to see this the following mapping property is needed: Due to (B.3) for $R>1$ large enough

$$\chi_R(|x|)j_a\in\mathcal{B}(Q(H_a),Q(H))=\mathcal{B}(H^1_0(\Omega^a+\mathbf{X}_a),H^1_0(\Omega)).$$

Using a standard commutation formula, cf. [DeGé, Section C.2], we then obtain that

$$K(\delta')=\int_{\mathbb{C}}(H_a-z)^{-1}\big(-\mathrm{iRe}\,\big(p\cdot\nabla(\chi_Rj_a)\big)-\sum_{b\not\subset a}V_b\chi_Rj_a\big)(H-z)^{-1}d\mu_{\delta'}(z),$$

showing that indeed $K(\delta')\in\mathcal{C}(\mathcal{H},\mathcal{H}_a)\subset\mathcal{C}(\mathcal{H},\mathcal{H}_c)$, cf. (B.1).

We have shown that $(h^c\otimes F_c)K(\delta')\in\mathcal{C}(\mathcal{H},\mathcal{H}_c)$, and it remains to consider the contribution from the first term in (B.5). We claim that $\|(h^c\otimes F_c)\eta_{\delta'}(H_a-E)\|$ is arbitrarily small provided $\delta'\geq 2\delta$ is small enough finishing the proof. For that it suffices (more precisely) to show that

$$\lim_{\delta'\to 0}\ \sup_{\xi_a\in\mathbf{X}_a}\|(h^c\otimes F_c)\eta_{\delta'}(H^a+\tfrac12\xi_a^2-E)\|_{\mathcal{B}(L^2(\Omega^a),\mathcal{H}_c)}=0. \quad \text{(B.6)}$$

Reintroducing arguments, (B.6) in turn follows from

$$\lim_{\delta'\to 0}\ \sup_{E'\leq E}\|h^c(x^c)F(\tfrac12(p^a_c)^2<d(E'+\epsilon)-2\epsilon)\eta_{\delta'}(H^a-E')\|_{\mathcal{B}(L^2(\Omega^a),\mathcal{H}_c)}=0. \quad \text{(B.7)}$$

Note that since $c\subset a$ we can write $p_c^2=(p^a_c)^2+p_a^2$, and therefore due to the property

$$d(t)\leq d(s)+t-s \text{ for } t\geq s, \quad \text{(B.8)}$$

indeed

$$\begin{aligned} F(\tfrac12p_c^2<d(E+\epsilon)-2\epsilon) &= F(\tfrac12(p^a_c)^2<d(E+\epsilon)-2\epsilon-\tfrac12p_a^2)\\ &\leq F(\tfrac12(p^a_c)^2<d(E'+\epsilon)-2\epsilon);\ E'=E-\tfrac12p_a^2. \end{aligned}$$

Step II. We formulate a slightly stronger statement than (B.7). Introduce for $c'\subset a'$ the set $\mathcal{P}^{a'}_{c'}=\cup_{c'\subset b\subset a'}\sigma_{\mathrm{pp}}(H^b)$ and the following distance function given for $c\subset a$ and for fixed $E\in\mathbb{R}$,

$$d^a_c(t)=\begin{cases}\inf_{\tau\in\mathcal{P}^a_c(t)}(t-\tau) \text{ for } \mathcal{P}^a_c(t):=\mathcal{P}^a_c\cap\,]-\infty,t]\neq\emptyset,\\ \max(1,E+1-\inf\mathcal{P}^{a_{\max}}_{a_{\min}}) \text{ for } \mathcal{P}^a_c(t)=\emptyset.\end{cases} \quad \text{(B.9)}$$

Now we replace d in the argument of the factor F in (B.7) by d^a_c. More precisely we introduce

$$F^a_c=F^a_c((p^a_c)^2)=F(\tfrac12(p^a_c)^2<d^a_c(E'+\epsilon)-2\epsilon),$$

and we claim the following statement for all $\epsilon>0$:

$$\begin{aligned} &\lim_{\delta\to 0}\ \sup_{E'\leq E}\|B^a_c(\delta,E',\epsilon)\|_{\mathcal{B}(L^2(\Omega^a),\mathcal{H}_c)}=0; \quad \text{(B.10)}\\ &B^a_c(\delta,E',\epsilon):=(h^c(x^c)\otimes F^a_c((p^a_c)^2))\eta_\delta(H^a-E'). \end{aligned}$$

From the definitions (1.8) and (B.9) it follows that $d(t) \leq d_c^a(t)$ for all $t \leq E+1$ (for $\epsilon \in (0,1]$, $E \in \mathbb{R}$ and $c \subset a \neq a_{\max}$). Whence indeed (B.10) is stronger than (B.7).

Step III. We shall show (B.10) by induction in $a \in \mathcal{A}$ (more precisely in $\#a$) considering arbitrary $c \subset a$ allowing $c = a_{\min}$ and (for convenience) also $c = a_{\max}$. We shall use that the bound (B.8) is valid with d replaced by d_c^a leading to the general properties:

$$d_c^a(t+\epsilon) - 2\epsilon \leq d_c^a(s+\epsilon/2) - \epsilon \text{ for } |t-s| \leq \epsilon/2, \tag{B.11a}$$

$$d_{c''}^{a''}(t) \leq d_{c'}^{a'}(t) \text{ for } c'' \subset c' \subset a' \subset a'' \text{ and } t \leq E+1. \tag{B.11b}$$

First in the case $a = c$ (the start of induction) we look at fixed $E' \leq E$ only, amounting to showing

$$\lim_{\delta \to 0} \|B_c^a(\delta, E', \epsilon)\|_{\mathcal{B}(L^2(\Omega^a), \mathcal{H}_c)} = 0. \tag{B.12}$$

In this case $(p_c^a)^2 = 0$ so that we can assume that $d_c^a(E'+\epsilon) > 2\epsilon$ excluding that $E' \in \sigma_{\mathrm{pp}}(H^a)$. Whence by compactness, cf. (B.1), indeed (B.12) follows for $a = c$. Moreover we can show that the limit (B.12) is attained uniformly in $E' \leq E$ (still for $a = c$): Suppose not. Then there exist $\delta_n \to 0$ and $E'_n \leq E$ such that

$$\liminf_{n \to \infty} \|B_c^a(\delta_n, E'_n, \epsilon)\|_{\mathcal{B}(L^2(\Omega^a), \mathcal{H}_c)} > 0. \tag{B.13}$$

We can assume that $E'_n \to E'$. But due to (B.11a) we can decompose for any $\delta > 0$ and for all large n

$$B_c^a(\delta_n, E'_n, \epsilon) = B_n B_c^a(\delta, E', \epsilon/2) \eta_{\delta'_n}(H^a - E'_n),$$

where $B_n = F(\tfrac{1}{2}(p_c^a)^2 < d_c^a(E'_n + \epsilon) - 2\epsilon)$. The middle factor has arbitrarily small norm (when taking $\delta \to 0$), while the other factors have norm ≤ 1. This contradicts (B.13).

Now suppose we have proven (B.10) for all $a \supset c$ with $a \neq a_{\max}$ (the induction hypothesis). Then it remains to verify the statement with a replaced by $a_{\max}$. By the previous argument we can assume $c \subsetneq a_{\max}$. We proceed decomposing as in (B.4) and (B.5) with E and F_c there replaced by $E' \leq E$ and $F_c^{a_{\max}}$, respectively. In case 1) $c \not\subset a$ we argue as above taking $\delta \to 0$ (the terms vanish identically for $E' \in \sigma_{\mathrm{pp}}(H)$, and whence they vanish in the limit uniformly in $E' \leq E$). In case 2) $c \subset a$ there are again three terms to consider. The (compact) terms involving K and $K(\delta')$ (where $\delta' \geq 2\delta$) are treated as above (again the terms vanish identically for $E' \in \sigma_{\mathrm{pp}}(H)$, and whence they vanish uniformly in E'). So it remains to consider the contribution from the first term. Using (B.8) (for $d_c^{a_{\max}}$) and (B.11b) we estimate (using that $p_c^2 = (p_c^a)^2 + p_a^2$, $E' \leq E$ and $\epsilon \leq 1$)

$$d_c^{a_{\max}}(E'+\epsilon) - \tfrac{1}{2}p_a^2 \leq d_c^{a_{\max}}(E' - \tfrac{1}{2}p_a^2 + \epsilon) \leq d_c^a(E' - \tfrac{1}{2}p_a^2 + \epsilon),$$

yielding the following bounds:

$$\begin{aligned}
&\|(h^c \otimes F_c^{a_{\max}})\eta_{\delta'}(H_a - E')\chi_R(|x|) j_a \eta_\delta(H-E)\| \\
&\leq \|(h^c \otimes F_c^{a_{\max}})\eta_{\delta'}(H_a - E')\| \\
&\leq \sup_{E'' \leq E'} \|B_c^a(\delta', E'', \epsilon)\| \\
&\leq \sup_{E'' \leq E} \|B_c^a(\delta', E'', \epsilon)\|.
\end{aligned}$$

The last expression vanishes in the limit $\delta'\to 0$ (by the induction hypothesis). So by first choosing (and fixing) small $\delta'>0$ (to make this expression small) and then letting $(0,\delta'/2]\ni\delta\to 0$ in agreement with the previous considerations the proof is complete. □

Proof of Lemma 1.5. Let $\epsilon\in(0,1]$ and $I\subset\mathbb{R}$ be compact (this ϵ will play the role of the κ in the lemma). Clearly the second term to the right in (2.4) is small for all large R. As for the first term we estimate

$$2\sum_b q_{b,R}\,p_b^2\,q_{b,R}\geq 4k-4k\sum_b q_{b,R}\,\bar\chi_\epsilon\big(\tfrac12 p_b^2-d(E+\epsilon)+4\epsilon\big)\,q_{b,R};$$
$$k=\max(0,d(E+\epsilon)-4\epsilon).$$

By commutation we obtain that uniformly in $E\in I$ and $\epsilon\in(0,1]$

$$q_{b,R}\,\bar\chi_\epsilon\big(\tfrac12 p_b^2-d(E+\epsilon)+4\epsilon\big)\,q_{b,R}=\operatorname{Re}\big(q_{b,R}^2\bar\chi_\epsilon\big(\tfrac12 p_b^2-d(E+\epsilon)+4\epsilon\big)\big)+O(R^{-2}).$$

Due to Lemma 1.2 (4) we can freely insert a factor of $h^b(x^b)=\bar\chi_{r_2R}(|x^b|)$ at the factor of $q_{b,R}^2$ to the right. Using the Cauchy-Schwarz inequality and the bound $\sum_b q_{b,R}^4\leq 1$ we then obtain for any $\kappa>0$

$$2\sum_b q_{b,R}\,p_b^2\,q_{b,R}\geq 4k-\kappa-\tfrac{4k^2}{\kappa}\sum_b T_b^*T_b-CR^{-2};$$
$$T_b=h^b(x^b)\bar\chi_\epsilon\big(\tfrac12 p_b^2-d(E+\epsilon)+4\epsilon\big),$$

here C is a positive constant that can be chosen independently of E and ϵ as above. In particular we have for all $R\geq R(\kappa)$ (with $R(\kappa)$ large) that

$$\mathrm{i}[H,A_R]\geq 4k-2\kappa-\tfrac{4k^2}{\kappa}\sum_b T_b^*T_b.$$

Next we localize in energy and obtain in agreement with Lemma B.1 that for any given $\epsilon'>0$

$$\sum_b \eta_{2\delta}(H-E)T_b^*T_b\eta_{2\delta}(H-E)=K+T,$$

where K is compact and $\|T\|\leq\epsilon'$; this is provided $\delta>0$ is taken small enough. Whence we have for all $R\geq R(\kappa)$ and all small $\delta>0$

$$\eta_\delta(H-E)\mathrm{i}[H,A_R]\eta_\delta(H-E)\geq\eta_\delta(H-E)\big(4k-2\kappa-\tfrac{4k^2}{\kappa}\epsilon'-K\big)\eta_\delta(H-E)$$

with K compact. We fix $\kappa,\epsilon'>0$ such that $2\kappa+\frac{4k^2}{\kappa}\epsilon'\leq 4\epsilon$ for all $E\in I$, leading to

$$\begin{aligned}&\eta_\delta(H-E)\mathrm{i}[H,A_R]\eta_\delta(H-E)\\&\geq 4\eta_\delta(H-E)\big(\max(0,d(E+\epsilon)-4\epsilon)-\epsilon-K\big)\eta_\delta(H-E)\\&\geq 4\eta_\delta(H-E)\big(d(E+\epsilon)-5\epsilon-K\big)\eta_\delta(H-E)\end{aligned}$$

for all $R\geq R(\epsilon,I)>0$ and all sufficiently small $\delta>0$. We have proved Lemma 1.5 with $\kappa=\epsilon$. □

B.2. **Spectral consequences.** Although these properties are not needed for the main theme of this paper we briefly discuss some consequences of Lemma 1.5 of independent interest, cf. [Sk2, Corollary 4.5].

Corollary B.2. *The following properties hold for the Hamiltonian H:*

i) The set of thresholds $\mathcal{T}$ is closed and countable, and the set of eigenvalues $\sigma_{\mathrm{pp}}(H)\setminus\mathcal{T}$ is discrete in $\mathbb{R}\setminus\mathcal{T}$. These eigenvalues have finite multiplicity.
ii) The singularly continuous spectrum $\sigma_{\mathrm{sc}}(H)=\emptyset$.

Proof. We will use tacitly below that $\mathrm{i}[H,A]$ is a commutator in the sense of Lemma A.8. In particular (for i)), since $\sigma_{\mathrm{pp}}(H^{a_{\min}})=\{0\}$, it follows from Lemma 1.5 that $\sigma_{\mathrm{pp}}(H^a)\setminus\{0\}$ is discrete in $\mathbb{R}\setminus\{0\}$ for all 2-body sub-Hamiltonians H^a. By induction, repeating this argument, indeed i) follows.

As for ii) we note that the version of the Mourre estimate of Lemma 1.3 (with $L^2(\mathbf{X})$ there replaced by $\mathcal{H}$) follows from Lemma 1.5 and the closedness of $\mathcal{T}$. Combining this with the property $H\in C^2\big(A_{\mathcal{H}^1},A_{\mathcal{H}^{-1}}\big)$, cf. Remark A.9, there is a limiting absorption principle away from $\mathcal{T}$ which is immediately seen from the proof of [BMP, Theorem 2.1], see also [ABG, Theorem 7.5.4]. In particular $\sigma_{\mathrm{sc}}(H)=\emptyset$ in agreement with [BMP, Theorem 2.1]. □

Remark B.3. The first part on the structure of the sets of thresholds and eigenvalues is also proved in [BGS, Gri]. The second part on absence of singularly continuous spectrum is not discussed in [Gri]. It is proved in [BGS] under some regularity conditions, see [BGS, Theorem A]. These conditions are not needed in our approach.

Appendix C. Exponential decay

In this appendix we are using the potential function r from Subsection 2.1 and the operator $A=A_R$ given by (1.7). (The parameter $R>1$ needs to be adjusted according to an applicaton of Lemma 1.5.) We shall show

Proposition C.1. *Suppose H does not have positive thresholds. If $\phi\in\mathcal{D}(H)$ satisfies $H\phi=E\phi$ for some $E>0$, then $\mathrm{e}^{\sigma r}\phi\in\mathcal{H}$ for any $\sigma\geq 0$.*

We shall use the (familiar) notation

$$\omega=\nabla r,\quad p^r=-\mathrm{i}\partial^r=\omega\cdot p,\quad p_r=\operatorname{Re}p^r,$$
$$\mathcal{H}^1=\mathcal{D}\big(|H_0|^{1/2}\big),\quad \mathcal{H}^{-1}=(\mathcal{H}^1)^*.$$

We shall use the cutoff-functions $\chi_{\nu,\nu'}=\chi_\nu\bar\chi_{\nu'}$ of (1.14) considered as functions of the function r. We introduce the regularized weights

$$\Theta(r)=\Theta_m^{\sigma;\delta}(r)=\sigma r+\delta r(1+\tfrac{r}{m})^{-1}$$

for $\sigma,\delta\geq 0$ and $m\geq 1$, and denote the first and the second derivatives in r by

$$\Theta'=\sigma+\delta(1+\tfrac{r}{m})^{-2},\quad \Theta''=-\tfrac{2\delta}{m}(1+\tfrac{r}{m})^{-3}. \tag{C.1}$$

Set

$$H_\Theta=\mathrm{e}^{\Theta}H\mathrm{e}^{-\Theta}:=H-\tfrac12|\mathrm{d}\Theta|^2+\mathrm{i}p_\Theta; \tag{C.2}$$
$$p_\Theta=\operatorname{Re}p^\Theta,\quad p^\Theta=\nabla\Theta\cdot p=\Theta'p^r.$$

We are going to consider H_Θ as an operator acting on suitable functions with bounded support, see (C.11). For the following intermediate result the interpretation is partially different (to be reversed in (C.12)). More precisely for the left hand side of (C.3) stated below we have, cf. (2.11),

$$2\operatorname{Im}(A(H_\Theta - E)) = \mathrm{i}[H, A] + r\partial^r|\mathrm{d}\Theta|^2 + (p_\Theta A + Ap_\Theta),$$

and here the first term to the right is given by its formal expression (2.3), that is

$$\mathrm{i}[H, A] = p\nabla^2 r^2 p - \tfrac{1}{4}\Delta^2 r^2 - 2r\partial^r V.$$

Note that due to (2.1b) this expression as well as the second term are bounded quadratic forms on $\mathcal{H}^1$. Similarly the third term has a clean meaning as a form on the set of functions in $\mathcal{H}^1$ with bounded support. The term $\operatorname{Re}(B(H_\Theta - E))$ of (C.3) is also a bounded quadratic form on $\mathcal{H}^1$.

Lemma C.2. *Suppose H does not have positive thresholds. Let $E > 0$ and $\sigma_0 \geq 0$ be given. Then there exist $R_0 > 1$ such that for all $R \geq R_0$ there exist $\epsilon, \delta_0 > 0$: For all $\sigma \in [0, \sigma_0]$ there exists $B \in \mathcal{B}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}^{-1})$ such that for all large $\nu \geq 1$, and all $\nu' \geq 2\nu$, $\delta \in (0, \delta_0]$ and $m \geq 1$, as a quadratic form on $\mathcal{H}^1$,*

$$\chi_{\nu,\nu'} 2\operatorname{Im}(A_R(H_\Theta - E))\chi_{\nu,\nu'} \geq \chi_{\nu,\nu'}\big(\epsilon - \operatorname{Re}(B(H_\Theta - E))\big)\chi_{\nu,\nu'}. \tag{C.3}$$

Proof. Fix any $\epsilon < 4E$, $\epsilon > \max(0, 4E - 96)$. We will apply Lemma 1.5 to $\kappa = \kappa'/16 > 0$ given by $\epsilon = 4E - 6\kappa'$ and with $I = E + [0, \frac{1}{2}\sigma_0^2]$. This fixes $R_0 > 1$. Whence we consider $R \geq R_0$ and energies $\tilde{E} \in I$. Fix such R. There is a neighbourhood $\mathcal{V} = \mathcal{V}(\sigma)$ of $\tilde{E} := E + \frac{1}{2}\sigma^2$, $\sigma \in [0, \sigma_0]$, and a compact operator $K = K(\sigma)$ on $\mathcal{H}$ such that

$$f(H)^*\mathrm{i}[H, A_R]f(H) \geq f(H)^*\big(4\tilde{E} - \kappa' - K\big)f(H) \text{ for all } f \in C_\mathrm{c}^\infty(\mathcal{V}).$$

By using the Cauchy-Schwarz inequality and (2.3) we obtain from this bound that for some positive constants C and $\tilde{\delta}$

$$\mathrm{i}[H, A_R] \geq 4E - 2\kappa' - K - C\bar{\chi}_{\tilde{\delta}}(|H - \tilde{E}|)\langle H\rangle. \tag{C.4}$$

Next by a compactness argument we can find $C > 0$ and $\tilde{\delta}$ independent of $\sigma \in [0, \sigma_0]$ such that (C.4) is valid for all $\sigma \in [0, \sigma_0]$. Given these properties we fix

$$B := C\bar{\chi}_{\tilde{\delta}}(|H - \tilde{E}|)\langle H\rangle(H - \tilde{E})^{-1}.$$

Note that $B = B(\sigma) \in \mathcal{B}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}^{-1})$ with norms being uniformly bounded in $\sigma \in [0, \sigma_0]$, and that we have shown

$$\mathrm{i}[H, A_R] \geq 4E - 2\kappa' - K - B(H - \tilde{E}). \tag{C.5}$$

It remains to choose a $\delta_0 \in (0, 1]$ independently of $\sigma \in [0, \sigma_0]$, to pick $\nu_0 = \nu_0(\sigma) > 1$ and to verify (C.3) uniformly in the parameters $\nu' \geq 2\nu \geq 2\nu_0$, $\delta \in (0, \delta_0]$ and $m \geq 1$.

We obtain from (C.5), omitting for simplicity here and below factors of $\chi_{\nu,\nu'}$ to the left and to the right,

$$\begin{aligned} 2\operatorname{Im}(A(H_\Theta - E)) &\geq 4E - 2\kappa' - K - \operatorname{Re}(B(H_\Theta - E)) \\ &\quad + r(\partial^r|\mathrm{d}\Theta|^2) - \tfrac{1}{2}\operatorname{Re}(B(|\mathrm{d}\Theta|^2 - \sigma^2)) - \operatorname{Im}(Bp_\Theta) + 2\operatorname{Re}(Ap_\Theta). \end{aligned} \tag{C.6}$$

For $\nu \geq \nu_0$ (with ν_0 sufficiently large)

$$\chi_\nu K\chi_\nu \geq -\kappa'. \tag{C.7}$$

We shall demonstrate the lower bound $-3\kappa'$ for the sum of the last four terms on the right hand side of (C.6) completing the proof. Let us recall that (2.1a) is valid for the rescaled $r = r_R$ we are using here. Note also that according to (2.1c) we have for some $C_0 > 0$

$$||\mathrm{d}r|^2 - 1| \leq C_0 \langle x/R \rangle^{-2}. \tag{C.8}$$

By using these properties and (C.1) we obtain for any $\delta_0 > 0$ small enough

$$\begin{aligned} r\partial^r |\mathrm{d}\Theta|^2 &= (\Theta')^2 r\partial^r |\mathrm{d}r|^2 + 2r\Theta''\Theta' |\mathrm{d}r|^4 \\ &\geq -4\delta_0(\sigma_0 + \delta_0)|\mathrm{d}r|^4 \\ &\geq -4\delta_0(\sigma_0 + \delta_0)(1 + C_0)^2 \geq -\kappa'. \end{aligned}$$

Similarly using (C.1), (C.8), the Cauchy-Schwarz inequality and the uniform boundedness of B stated above we derive

$$-\tfrac{1}{2}\chi_\nu \operatorname{Re}(B(|\mathrm{d}\Theta|^2 - \sigma^2))\chi_\nu \geq -\kappa'. \tag{C.9}$$

This is for any small $\delta_0 > 0$ (smallness being independent of $\sigma \in [0, \sigma_0]$) and for all $\nu \geq 1$ sufficiently large.

Next, noting the expressions

$$\begin{aligned} A &= rp^r + (p^r)^* r = 2p_r r + \mathrm{i}|\mathrm{d}r|^2, \\ p_\Theta &= \Theta' p_r - \tfrac{\mathrm{i}}{2}|\mathrm{d}r|^2\Theta'', \end{aligned}$$

we compute with $S = \big(|\mathrm{d}r|^2 + \tfrac{1}{2}B\big)\Theta' + r|\mathrm{d}r|^2\Theta''$ and any $\epsilon' > 0$,

$$\begin{aligned} &-\operatorname{Im}(Bp_\Theta) + 2\operatorname{Re}(Ap_\Theta) \\ &= 4p_r r\Theta' p_r - 2\operatorname{Im}\big(Sp_r\big) + \operatorname{Re}\big((|\mathrm{d}r|^4 + \tfrac{1}{2}B|\mathrm{d}r|^2)\Theta''\big) \\ &\geq 4p_r\{r\Theta' - \tfrac{1}{4\epsilon'}|S|^2\}p_r - \epsilon' + \operatorname{Re}\big((|\mathrm{d}r|^4 + \tfrac{1}{2}B|\mathrm{d}r|^2)\Theta''\big). \end{aligned}$$

We shall use this estimate with factors of $\chi_{\nu,\nu'}$ to the left and to the right heading at proving (C.10) below. Fix $\epsilon' = \kappa'/2$. The contribution from the third term on the right hand side is small, say $\geq -\kappa'/2$, when δ_0 is taken small (obviously deduced from (C.1), (C.8) and the uniform boundedness of B). This fixes a small $\delta_0 \in (0, 1]$ (which indeed is independent of σ). The contribution from the first term is non-negative as shown as follows: It suffices to show that

$$\chi_{\nu/2,2\nu'}\{r\Theta' - \tfrac{1}{4\epsilon'}|S|^2\}\chi_{\nu/2,2\nu'} \geq 0.$$

By factorizing

$$S = (T_1 + T_2)\sqrt{r\Theta'};\ T_1 = \big(|\mathrm{d}r|^2 + \tfrac{1}{2}B\big)\sqrt{\Theta'/r},\ \ T_2 = r|\mathrm{d}r|^2\Theta''/\sqrt{r\Theta'},$$

it suffices to show that

$$\chi_{\nu/2,2\nu'}\{4\epsilon' - 2|T_1|^2 - 2|T_2|^2\}\chi_{\nu/2,2\nu'} \geq 0.$$

This is valid for all $\nu \geq 1$ sufficiently large (uniformly in parameters). Hence, in conclusion,

$$\chi_{\nu,\nu'}\big(-\operatorname{Im}(Bp_\Theta) + 2\operatorname{Re}(Ap_\Theta)\big)\chi_{\nu,\nu'} \geq -\kappa'. \tag{C.10}$$

By (C.6)–(C.10) the asserted inequality (C.3) is valid uniformly in the parameters $\nu' \geq 2\nu \geq 2\nu_0$, $\delta \in (0, \delta_0]$ and $m \geq 1$. □

Proof of Proposition C.1. We let E and ϕ be as in the proposition. Set

$$\sigma_0 = \sup\{\sigma \geq 0 |\ \mathrm{e}^{\sigma r}\phi \in \mathcal{H}\},$$

and assume $\sigma_0 < \infty$. We fix (large) $R > 1$ and (small) $\epsilon, \delta_0 > 0$ in agreement with Lemma C.2 (with this σ_0 as input). If $\sigma_0 > 0$, we choose $\sigma \in [0, \sigma_0)$ and $\delta \in (0, \delta_0]$ such that $\sigma + \delta > \sigma_0$. If $\sigma_0 = 0$, we set $\sigma = 0$ and choose any $\delta \in (0, \delta_0]$. In both cases we have $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$, and we fix B and $\nu \geq 1$ in agreement with Lemma C.2. We then have the estimate (C.3) at our disposal for all $\nu' \geq 2\nu$ and $m \geq 1$.

We shall apply this estimate in the state

$$\phi_\Theta = \mathrm{e}^{\Theta}\phi = \mathrm{e}^{\Theta_m^{\sigma,\delta}}\phi.$$

Due to Lemma A.10 we have $\phi_\Theta \in \mathcal{H}^1$.

We note, putting $R_\nu = \mathrm{i}[H, \chi_\nu] = \mathrm{Re}\,(\chi'_\nu p^r)$,

$$\begin{aligned} \mathrm{i}(H_\Theta - E)\chi_{\nu,\nu'}\phi_\Theta &= \mathrm{i}\chi_{\nu,\nu'}\mathrm{e}^{\Theta}(H - E)\phi + \mathrm{e}^{\Theta}(R_\nu - R_{\nu'})\phi \\ &= \mathrm{e}^{\Theta}(R_\nu - R_{\nu'})\phi \end{aligned} \tag{C.11}$$

In particular $\chi_{\nu,\nu'}\phi_\Theta \in \mathcal{D}(H) \cap \mathcal{D}(A)$. Due to Lemma A.12 and (C.3) we therefore have

$$\epsilon\|\chi_{\nu,\nu'}\phi_\Theta\|^2 \leq 2\,\mathrm{Im}\langle A\chi_{\nu,\nu'}\phi_\Theta, (H_\Theta - E)\chi_{\nu,\nu'}\phi_\Theta\rangle - \mathrm{Re}\langle B(H_\Theta - E)\rangle_{\chi_{\nu,\nu'}\phi_\Theta}. \tag{C.12}$$

Let us estimate the right hand side. For the first term of (C.12) we use (C.11) and obtain

$$\begin{aligned} &2\,\mathrm{Im}\langle A\chi_{\nu,\nu'}\phi_\Theta, (H_\Theta - E)\chi_{\nu,\nu'}\phi_\Theta\rangle \\ &= -\langle A\phi_\Theta, \mathrm{e}^{\Theta}(R_\nu - R_{\nu'})\phi\rangle + \text{h.c.} \\ &\leq C_\nu(\|\chi_{\nu/2}\phi\|^2 + \|\chi_{\nu/2}p\phi\|^2) \\ &\quad + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\phi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\phi\|^2). \end{aligned}$$

Here and below the constants C_ν and C_m are independent of $\nu' \geq 2\nu$, and in addition C_ν is independent of $m \geq 1$. Similarly (using the boundedness of B) we can estimate the second term of (C.12) as

$$\begin{aligned} -\mathrm{Re}\langle B(H_\Theta - E)\rangle_{\chi_{\nu,\nu'}\phi_\Theta} &\leq \tfrac{\epsilon}{2}\|\chi_{\nu,\nu'}\phi_\Theta\|^2 + C_\nu(\|\chi_{\nu/2}\phi\|^2 + \|\chi_{\nu/2}p\phi\|^2) \\ &\quad + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\phi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\phi\|^2). \end{aligned}$$

Hence we have proved (with new constants)

$$\begin{aligned} \tfrac{\epsilon}{2}\|\chi_{\nu,\nu'}\phi_\Theta\|^2 &\leq C_\nu(\|\chi_{\nu/2}\phi\|^2 + \|\chi_{\nu/2}p\phi\|^2) \\ &\quad + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\phi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\phi\|^2). \end{aligned}$$

Now let $\nu' \to \infty$ in this estimate invoking Lebesgue's dominated convergence theorem (and Lemma A.10). This makes the second term disappear, and consequently we are left with the bound

$$\|\chi_\nu \mathrm{e}^{\Theta_m^{\sigma,\delta}}\phi\|^2 \leq \tfrac{2C_\nu}{\epsilon}(\|\chi_{\nu/2}\phi\|^2 + \|\chi_{\nu/2}p\phi\|^2). \tag{C.13}$$

By letting $m \to \infty$ in (C.13) invoking Lebesgue's monotone convergence theorem we conclude that $\chi_\nu\mathrm{e}^{(\sigma+\delta)r}\phi \in \mathcal{H}$. This is a contradiction since $\sigma + \delta > \sigma_0$. □

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