Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Functional Analysis 247 (2007) 1–94
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Spiraling attractors and quantum dynamics for a class of long-range magnetic fields

Horia D. Cornean [a], Ira Herbst [b], Erik Skibsted [c,*]

[a] *Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, 9220 Aalborg, Denmark*
[b] *Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA*
[c] *Institut for Matematiske Fag, Aarhus Universitet, Ny Munkegade, 8000 Aarhus C, Denmark*

Received 9 September 2003; accepted 11 April 2006

Communicated by Richard B. Melrose

Abstract

We consider the long time behavior of a quantum particle in a 2D magnetic field which is homogeneous of degree -1. If the field never vanishes, above a certain energy the associated classical dynamical system has a globally attracting periodic orbit in a reduced phase space. For that energy regime, we construct a simple approximate evolution based on this attractor, and prove that it completely describes the quantum dynamics of our system.

Keywords: Classical and quantum dynamics; Long range magnetic fields; Asymptotic completeness

Contents

* Corresponding author.
E-mail addresses: cornean@math.aau.dk (H.D. Cornean), iwh@virginia.edu (I. Herbst), skibsted@imf.au.dk (E. Skibsted).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.04.032

1. Introduction

1.1. Two-dimensional purely magnetic Hamiltonians

A classical particle in a magnetic field is described by the Hamiltonian

$$h(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{2}\big(\boldsymbol{\xi} - \mathbf{a}(\mathbf{x})\big)^2, \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}. \tag{1.1}$$

The magnetic field $\mathbf{B}(\mathbf{x})$ is obtained from the vector potential $\mathbf{a}$ by exterior differentiation, $\mathbf{B}(\mathbf{x}) = d\mathbf{a}(\mathbf{x})$. In this paper we study the classical and quantum dynamics of a two-dimensional particle in a magnetic field of the form

$$\mathbf{B}(\mathbf{x}) = \frac{b(\theta)}{r}\, dx_1 \wedge dx_2, \quad \mathbf{x} = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2. \tag{1.2}$$

We are interested in orbits $(\mathbf{x}(t), \boldsymbol{\xi}(t))$ for which

$$\lim_{t\to\infty} r(t) = \infty \tag{1.3}$$

and hence in scattering theory. The decay rate $\langle\mathbf{x}\rangle^{-1}$ in (1.2) seems to be the borderline rate of decay for which we can be assured of (1.3) (at least for some range of energies). For if we take $\mathbf{B}(\mathbf{x}) = (b/r^\gamma)\, dx_1 \wedge dx_2$ with b a nonzero constant and $0 < \gamma < 1$, a vector potential satisfying $\mathbf{B}(\mathbf{x}) = d\mathbf{a}(\mathbf{x})$ is readily found and leads to the conservation laws

$$l = \frac{\partial h}{\partial \theta} = Const, \qquad E = h = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\left(\frac{l}{r} - cr^{1-\gamma}\right)^2 = Const.$$

Here $c = \frac{b}{2-\gamma}$, l is the angular momentum and E is the energy. It follows that all orbits are confined to a bounded region of phase space. Indeed, Miller and Simon analyzed the corresponding quantum Hamiltonian and showed that its spectrum is pure point, dense in $[0,\infty)$ (for more details, see [2, Theorem 6.2]).

On the other hand, much work over the last twenty years has been done in analyzing the quantum problem with $|\mathbf{B}(\mathbf{x})| = \mathcal{O}(\langle\mathbf{x}\rangle^{-1-\epsilon})$ with $\epsilon > 0$ (in any dimension $\geqslant 2$). We briefly review known results in this case. Firstly, the existence part of wave operators for $1 < \gamma < \infty$ is covered by general results of Hörmander (see [9]) which hold in combination with a long-range scalar potential. The comparison dynamics used in [9] to construct a wave operator preserves the momentum (it is a refined Dollard-type dynamics). Asymptotic completeness was proved by Hörmander (using stationary methods) in [10, Chapter 30]. In addition we mention here the work of Robert (see [18]) which also includes long-range scalar potentials. The wave operators in [18] are constructed using the stationary modifier of Isozaki–Kitada (see [11] for details). Very recently Roux and Yafaev revisited this problem in [19], and they also investigated the spectral properties of the corresponding scattering matrix S.

Secondly, the case $\gamma > 3/2$ was further investigated by Loss and Thaller in [12,13] for purely magnetic Schrödinger and Dirac operators, where they prove existence and completeness for the ordinary Møller operators employing Enss' time-dependent approach. Then Nicoleau and Robert in [16] treat the Schrödinger problem for $\gamma > 3/2$ by using stationary scattering theory; in addition, they allow short-range scalar potentials. Enss (see [5]) extended their Schrödinger result to include long-range scalar potentials, giving a simplified proof of existence and asymptotic completeness of the modified Dollard wave operators. The modification here only uses the scalar and not the vector potentials. We mention that these results for $\gamma > 3/2$ can now be recovered as particular cases of the more general results in [19].

Scattering theory with $\gamma = 1$ (no decay on the vector potential) does not appear to be treated in the literature. Similar problems with homogeneous of degree zero electric potential have been considered by two of the authors in [7,8]; see also a related work of Hassell, Melrose and Vasy in [6]. In those cases, the Hamiltonian is roughly $H = -\Delta + V(\mathbf{x}/|\mathbf{x}|)$, where V is defined on the unit sphere. The generic behavior for the classical orbits in this situation is that they are eventually trapped in the directions in which V has local extrema (in the quantum case the local maxima and saddle points are excluded), hence very roughly the trajectories are asymptotically straight.

The behavior for the two-dimensional magnetic case with $\gamma = 1$ turns out to be different (at least for the case treated in this paper). Assume that we are given a magnetic field which is homogeneous of degree -1 outside the unit disc, i.e. is given by $r^{-1}b(\theta)$ for $r \geqslant 1$. For the classical orbits staying outside the unit disc (for all large times) and with energy $E > E_b$, where

$$E_b = \max_{\theta \in [0,2\pi]} b^2(\theta)/2, \tag{1.4}$$

we have an "easy" Mourre estimate implying that their radial velocities eventually become positive, hence these orbits move to infinity. (We remark that when b is constant, E_b equals the mobility edge in the Miller–Simon model, i.e. that particular energy above which the spectrum is purely absolutely continuous while below it the spectrum is dense pure point.) A more detailed analysis under the additional condition that b is strictly negative (a similar analysis may be done for $b > 0$) shows that the asymptotic orbits are logarithmic spirals and not asymptotically straight as in the potential case. Moreover, we can even go below E_b with our considerations if b is not constant.

The main goal of the paper is to demonstrate analogous behavior in quantum mechanics. In a following paper [1] we will study the case in which b has zeros and in particular the zero flux case $\int_0^{2\pi} b(\theta)\,d\theta = 0$. For the latter case the classical scattering orbits approach a direction in which $b(\theta) = 0$. When b changes sign but the flux is different from zero both types of behavior may occur: some trajectories are drawn toward the half-lines defined by the zeroes of b while others will spiral.

There are indications of somewhat similar results in dimensions higher than two, although the geometry and analysis are more complicated.

1.2. Classical mechanics: preliminaries and main results

Our system consists here of a classical particle (with charge -1) confined to a plane and subjected to a magnetic field $\mathbf{B}$ which is assumed to be homogeneous of degree -1. As usual, $\mathbf{B}$ is "orthogonal" to the plane in which the particle moves so it has only one nonzero component (the "third" one) which is of the form $B(\mathbf{x}) = b(\theta)/r$. We assume that b is smooth and negative. The associated transverse magnetic vector potential is $\mathbf{a}(\mathbf{x}) = (-\sin(\theta), \cos(\theta))b(\theta)$.

The corresponding classical Hamiltonian function is (in polar coordinates)

$$h(r,\theta;\rho,l) := \frac{1}{2}\rho^2 + \frac{1}{2}\left(\frac{l}{r} - b(\theta)\right)^2, \tag{1.5}$$

where $\rho = \frac{\mathbf{x}}{|\mathbf{x}|} \cdot (\boldsymbol{\xi} - \mathbf{a})$ is the radial velocity and $l = x_1 p_2 - x_2 p_1$ is the canonical angular momentum. The Hamilton equations for r and θ are

$$\frac{dr}{dt} = \frac{\partial h}{\partial \rho} = \rho, \qquad \frac{d\theta}{dt} = \frac{\partial h}{\partial l} = \frac{1}{r}\left(\frac{l}{r} - b\right). \tag{1.6}$$

The Hamilton equations for ρ and l are

$$\frac{d\rho}{dt} = -\frac{\partial h}{\partial r} = \left(\frac{l}{r} - b(\theta)\right)\frac{l}{r^2} \tag{1.7}$$

and

$$\frac{dl}{dt} = -\frac{\partial h}{\partial \theta} = \left(\frac{l}{r} - b(\theta)\right) b'(\theta). \tag{1.8}$$

Let us introduce the transverse velocity $\xi := \frac{l}{r} - b$, which obeys the equations

$$\frac{d\theta}{dt} = \frac{\xi}{r}, \qquad \frac{d\xi}{dt} = -\frac{\xi + b}{r}\rho. \tag{1.9}$$

Since h in (1.5) does not depend on t, the energy is conserved; that is, on a given trajectory one has

$$\rho^2(t) + \xi^2(t) = 2E. \tag{1.10}$$

1.2.1. An attractive Lagrangian manifold

We will now discuss various results obtained in the classical framework. Apart from their own intrinsic interest they serve as motivation for our main result in quantum mechanics (see Theorem 1.1 below).

Define the extended configuration space

$$\mathcal{A} := \left\{\bar{x} = (t, r, \theta)\colon\ t > 0,\ r > 0,\ \theta \in \mathbb{T}\right\},$$

and consider the function

$$\bar{h}(\bar{x}, \bar{\eta}) := \tau + h(r, \theta; \rho, l), \quad \bar{x} \in \mathcal{A},\ \bar{\eta} = (\tau, \rho, l).$$

We introduce a symplectic form on $T^*\mathcal{A}$ given by

$$d\mathbf{x} \wedge d\boldsymbol{\xi} + dt \wedge d\tau = d\theta \wedge dl + dr \wedge d\rho + dt \wedge d\tau.$$

We construct a solution S (defined on $\mathcal{A}$) to the equation $\bar{h}(\bar{x}, \nabla S) = 0$, which is nothing but the Hamilton–Jacobi equation (see Section 3 for details)

$$\partial_t S + h(r, \theta; \partial_r S, \partial_\theta S) = 0.$$

Consider the associated Lagrangian manifold

$$\mathcal{L} := \{\bar{z} = (\bar{x}, \bar{\eta}) \colon \bar{x} \in \mathcal{A},\ \bar{\eta} = \nabla S\} \subseteq T^*\mathcal{A}.$$

Then $\mathcal{L}$ is invariant under the flow corresponding to $\bar{h}$, which when restricted to $\mathcal{L}$ can be written

$$\frac{d\tilde{r}}{dt} = \partial_r S(t, \tilde{r}, \tilde{\theta}), \qquad \frac{d\tilde{\theta}}{dt} = \tilde{r}^{-1}\big[\tilde{r}^{-1}\partial_\theta S(t, \tilde{r}, \tilde{\theta}) - b(\tilde{\theta})\big] \tag{1.11}$$

(with the momenta satisfying $\bar{\eta} = \nabla S$ of course). It is natural to ask how closely an orbit originating off $\mathcal{L}$ is approximated by solutions of Eqs. (1.11). We show in Section 3 that $\mathcal{L}$ is attractive for all energies above a certain threshold $E_d \leqslant E_b$. More precisely, assume that $(r, \theta; \rho, l)$ is a solution for the symbol h with energy $E > E_d$ which exists for all $t > 0$. Then for any $\delta > 0$, the quantities $E + \partial_t S$, $\rho - \partial_r S$, $(l - \partial_\theta S)/r$ are all $\mathcal{O}(t^{-1+\delta})$ as $t \to \infty$. Here and henceforth $t \to \infty$ means $t \to +\infty$.

Even though (1.11) may seem somewhat complicated at first glance, we obtain in Section 3 that $\tilde{\theta}(t)$ is strictly increasing and grows logarithmically in time. This allows us to consider the radius as a function of the angle $r(\tilde{\theta}) = \tilde{r}(t(\tilde{\theta}))$ and eventually prove that

$$\ln r = C(E)\tilde{\theta} + R(E, \tilde{\theta}), \tag{1.12}$$

where $C(E)$ is a positive increasing function and R is 2π-periodic in $\tilde{\theta}$. Moreover, $C(E) \searrow 0$ when $E \searrow E_d$, and one can prove that $C(E)/\sqrt{2E}$ approaches a positive constant when E increases to infinity. With $R(E, \tilde{\theta}) = R(E)$ independent of $\tilde{\theta}$, (1.12) is the equation for the so-called logarithmic spiral. On the other hand, if the energy equals E_d, we can construct closed classical orbits which can be put arbitrarily far away from the origin. This indicates that even if there still exist scattering states below E_d, the mechanism through which they go to infinity is different. What we know is that in the constant b case, according to the Miller–Simon model, the spectrum below $E_d = E_b = b^2/2$ is pure point, thus no scattering is possible. For further comments see Section 10.

An example of a typical integral curve first defined for the full Hamiltonian vector field and then projected to the rectangular configuration space is given in Fig. 1 ($E > E_d$).

1.2.2. Classical comparison dynamics and asymptotic completeness

Motivated by the above considerations, we introduce

$$\gamma_1 = \rho - \partial_r S, \qquad \gamma_2 = \frac{l - \partial_\theta S}{r}, \qquad h_a = h - \frac{1}{2}\big(\gamma_1^2 + \gamma_2^2\big), \qquad \bar{h}_a(\bar{x}, \bar{\eta}) = \tau + h_a. \tag{1.13}$$

It is easy to see that the Hamilton equations for (r, θ) obtained from h_a coincide with the system in (1.11). Moreover, the dynamics of $\bar{h}$ and $\bar{h}_a$ coincide on $\mathcal{L}$, hence recalling the result of Lemma 3.14 mentioned above, the dynamics generated by $\bar{h}_a$ should approximate the real one.

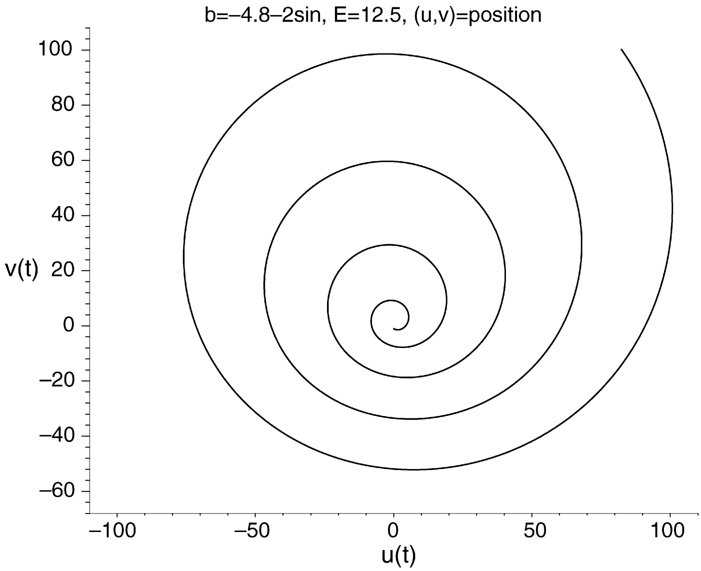

Fig. 1.

To fix notation, denote by $\mathbf{q}_{a,t} = (\rho_a(t), l_a(t))$ the "momenta" generated by h_a; the flow generated by h_a is denoted with $\mathbf{V}_{a,t}$ and acts as $\mathbf{V}_{a,t}(r_1, \theta_1; \mathbf{q}_{a,1}) = (\mathbf{v}_t; \mathbf{q}_{a,t})$. The flow generated by h is denoted by $\mathbf{V}_t$ and $\mathbf{V}_t(r(1), \theta(1); \rho(1), l(1)) = (r(t), \theta(t); \rho(t), l(t))$ gives the classical solutions of the true dynamics.

Let $\mathbf{W}_{a,t} = \mathbf{V}_{a,t}^{-1}$ denote the inverse flow (explicitly, writing the equations for $\mathbf{V}_{a,t}$ as $d\mathbf{x}/dt = \mathbf{F}(t, \mathbf{x})$, the equations for $\mathbf{W}_{a,t}$ are $d\mathbf{z}/ds = -\mathbf{F}(t + 1 - s, \mathbf{z})$, where $\mathbf{z}(1) = \mathbf{V}_{a,t}(\mathbf{x})$ and $\mathbf{W}_{a,t}(\mathbf{z}(1)) = \mathbf{z}(t) = \mathbf{x}$). The obvious interpretation of $\mathbf{W}_{a,t}$ is that it gives back the initial conditions used in computing $\mathbf{V}_{a,t}$.

Classical asymptotic completeness would be the existence of

$$\Omega_+ := \lim_{t\to\infty} \mathbf{W}_{a,t} \circ \mathbf{V}_t, \tag{1.14}$$

and the limit represents the initial data one should put into the dynamics $\mathbf{V}_{a,t}$ generated by h_a in order to get a good approximation to any true orbit $\mathbf{V}_t$.

Although we do not prove the existence of the above limit, we do prove for energies larger than E_d the existence of

$$\Pi\Omega_+ := \lim_{t\to\infty} \Pi\mathbf{W}_{a,t} \circ \mathbf{V}_t, \tag{1.15}$$

where Π projects on the configuration space of (r, θ)'s. Note for the flow $\mathbf{V}_{a,t}$ that the equations for the configuration space part $\mathbf{v}_t$ in (1.11) are completely decoupled from the momenta $\mathbf{q}_{a,t}$, and that $\mathbf{W}_{a,t}$ consequently enjoys the same property. In fact denoting the inverse of $\mathbf{v}_t$ by $\mathbf{w}_t$, we obtain $\mathbf{w}_t = \Pi\mathbf{W}_{a,t}$ simplifying the right-hand side of (1.15). We denote by (r_+, θ_+) the limit in (1.15) and call the entries the asymptotic radius and angle, respectively. Intuitively, when put into the direct flow $\mathbf{v}_t$ this limit provides us with a good approximation of the configuration space component of the true orbit at time t. See Proposition 3.13 for details.

1.3. Quantum mechanics: preliminaries and main results

Motivated by its classical counterpart, we choose a magnetic vector potential (we denote $\mathbf{x} = (r, \theta)$)

$$\mathbf{a}(\mathbf{x}) = (-\sin\theta, \cos\theta) b(\theta) m_+(r) \in C^\infty\big(\mathbb{R}^2\big), \tag{1.16}$$

where $0 \leqslant m_+ \leqslant 1$ is a smooth cut-off function equal to zero if $r \leqslant \frac{1}{4}$ and equal to one if $r \geqslant \frac{1}{2}$. Notice that $\mathbf{a}$ is homogeneous of degree zero outside the unit disc while the corresponding magnetic field is homogeneous of degree -1.

The classical Hamiltonian from (1.5) now becomes an operator

$$\begin{aligned} H &= \frac{1}{2}(\mathbf{p} - \mathbf{a})^2 = \frac{1}{2}\mathbf{p}^2 - \frac{1}{2}(\mathbf{p}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{p}) + \frac{1}{2}m_+^2 b^2 \\ &= -\frac{1}{2}\frac{\partial^2}{\partial r^2} - \frac{1}{2r}\frac{\partial}{\partial r} + \frac{1}{2}\left(\frac{L}{r} - m_+(r)b(\theta)\right)^2, \end{aligned} \tag{1.17}$$

which is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ with the domain $H^2(\mathbb{R}^2)$ (we denoted the angular momentum $L = -i\partial_\theta$).

We will often identify $L^2(\mathbb{R}^2)$ with $L^2(\mathbb{R}_+ \times \mathbb{T})$ through the unitary transformation

$$L^2\big(\mathbb{R}^2\big) \ni f(r,\theta) \to r^{1/2} f(r,\theta) \in L^2(\mathbb{R}_+ \times \mathbb{T}). \tag{1.18}$$

As an operator on $L^2(\mathbb{R}_+ \times \mathbb{T})$ the Hamiltonian H takes the form

$$H = -\frac{1}{2}\frac{\partial^2}{\partial r^2} - \frac{1}{8r^2} + \frac{1}{2}\left(-\frac{i}{r}\frac{\partial}{\partial\theta} - m_+(r)b(\theta)\right)^2. \tag{1.19}$$

We will see that there is an "easy" Mourre estimate with the generator of dilations $A = 1/2(\mathbf{p}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{p})$ as conjugate operator,

$$i[H, A] \geqslant H - b^2(\theta)/2 + K, \tag{1.20}$$

with K being relatively compact to H, see (2.2) for the classical counterpart. Indeed, after easy computations employing polar coordinates one obtains

$$i[\mathbf{p}\cdot\mathbf{a}, \mathbf{p}\cdot\mathbf{x}] = i[\mathbf{p}\cdot\mathbf{a}, \mathbf{x}\cdot\mathbf{p}] = \mathbf{p}\cdot(\mathbf{a} + \mathbf{a}_c),$$

where $\mathbf{a}_c(\mathbf{x}) = (\sin(\theta), -\cos(\theta)) b(\theta) r m_+'(r)$ is smooth and compactly supported. Then

$$\begin{aligned} i[H, A] &= \mathbf{p}^2 - (1/2)\mathbf{p}\cdot(\mathbf{a} + \mathbf{a}_c) - (1/2)(\mathbf{a} + \mathbf{a}_c)\cdot\mathbf{p} + \mathbf{a}\cdot\mathbf{a}_c \\ &= H - \frac{(\mathbf{a} - \mathbf{a}_c)^2}{2} + \frac{1}{2}(\mathbf{p} - \mathbf{a}_c)^2 \\ &\geqslant H - \frac{(\mathbf{a} - \mathbf{a}_c)^2}{2}. \end{aligned} \tag{1.21}$$

Thus we obtain (1.20).

This computation indicates that above $E_b = \max|b|^2/2$ things are somewhat "easier," just as they are in the classical case (see (2.2)). We will see in Section 3 that if $b < 0$ is not constant, we can go below E_b down to the critical energy E_d by using a more involved conjugate operator; notice though that $E_d = E_b$ when b is constant (see Proposition 2.10). For the classical version of this conjugate operator see (2.33).

Hence, according to Mourre (see [15]), the interval (E_d,∞) is a subset of the absolutely continuous spectrum and does not contain singular continuous spectrum. Possible embedded eigenvalues in this interval are discrete and may at most accumulate at E_d.

We introduce a comparison dynamics roughly generated by the quantization of the symbol h_a of (1.13). A similar approximate dynamics was used in [8]. This type of dynamics is motivated by a related one introduced by Yafaev [20]; see also [3,19]. We can define a family of isometries

$$L^2\big((E_d,\infty)\times\mathbb{T}\big)\ni f\mapsto U_0(t)f\in L^2(\mathbb{R}_+\times\mathbb{T}),\quad t\geqslant 1,$$

by (see (3.56) and (3.57))

$$\big[U_0(t)f\big](r,\theta)=e^{iS(t,r,\theta)}J_t^{1/2}(r,\theta)f\big(-(\partial_t S)\big(1,\mathbf{w}_t(r,\theta)\big),\theta_1(t,r,\theta)\big),\tag{1.22}$$

where J_t is the Jacobian determinant arising from the various changes of variables which makes $U_0(t)$ an isometry.

With this comparison dynamics we have existence of the direct wave operator (denoted by Ω_+^d) and completeness (see the remark after the statement of Theorem 4.2 for a more general result including short-range perturbations).

Theorem 1.1. *Denote by* $\mathcal{H}_{E_d} := \mathbf{1}_{(E_d,\infty)\setminus\sigma_{\rm pp}(H)}(H)L^2(\mathbb{R}_+\times\mathbb{T})$. *Then the following limits exist and define unitary operators which are mutually inverse*:

$$\begin{aligned}\Omega_+^d&=\operatorname*{s-lim}_{t\to\infty}e^{itH}U_0(t):L^2\big((E_d,\infty)\times\mathbb{T}\big)\mapsto\mathcal{H}_{E_d},\\ \Omega_+&=\operatorname*{s-lim}_{t\to\infty}U_0^*(t)e^{-itH}:\mathcal{H}_{E_d}\mapsto L^2\big((E_d,\infty)\times\mathbb{T}\big).\end{aligned}\tag{1.23}$$

We have the existence of the asymptotic observables defined on $\mathcal{H}_{E_d}$ (see (3.56)):

$$\begin{aligned}r_+&:=\operatorname*{s.r.-lim}_{t\to\infty}e^{itH}M\big(r_1(t,\cdot,\cdot)\big)e^{-itH},\\ e^{i\theta_+}&:=\operatorname*{s.r.-lim}_{t\to\infty}e^{itH}M\big(e^{i\theta_1(t,\cdot,\cdot)}\big)e^{-itH},\end{aligned}\tag{1.24}$$

where the notation $M(\cdot)$ signifies multiplication operator and s.r.-lim means strong resolvent limit. These operators can be expressed in terms of the wave operators of Theorem 1.1; they represent quantum analogs of the classical asymptotic radius and angle, r_+ and θ_+, discussed previously. For details, see Theorem 4.2.

In the case, where $b < 0$ does not depend on θ we have the result $E_d = E_b = b^2/2$ in (1.23) and (1.24), cf. the Miller–Simon result, and the formula for $U_0(t)$ reads

$$\big[U_0(t)f\big](r,\theta)=\exp\left\{i\frac{r^2}{2t}-i\frac{b^2t}{2}\right\}\sqrt{\frac{r}{t^2}}\,f\left(\frac{r^2}{2t^2}+\frac{b^2}{2},\theta+b\frac{t}{r}\ln(t)\right).\tag{1.25}$$

Moreover in this case the term $R(E,\tilde{\theta})$ in (1.12) is indeed constant in $\tilde{\theta}$.

Heuristically, our comparison dynamics moves the support of an initial state of sufficiently high energy along the integral curves of (1.11) which are spirals moving counter-clockwise to infinity with the radius proportional to t and the angle proportional to $\ln t$, cf. (1.12) (see also (3.65) and (3.66)). (Clearly this picture is confirmed by (1.25) in the constant b case.) Asymptotic completeness means that any state with high enough energy can be thought of (asymptotically in time) as a superposition of translates along these logarithmic spirals.

1.4. Gauge covariance of the wave operators and unitarity of the S-matrix

Using an argument similar to the one which led to our "outgoing" wave operators (see (1.23)), we can also give an approximate dynamics at negative times (see Section 9 for details), and consequently define some "incoming" wave operators (denoted by Ω_-^d and Ω_-). It can be shown that Ω_-^d maps unitarily $L^2((E_d,\infty)\times\mathbb{T})$ onto $\mathcal{H}_{E_d}$ and Ω_- is its inverse.

The non-trivial fact that needs to be shown (see Proposition 9.2) is that the E_d for negative times is the same as that for positive times. One might feel that this follows from time reversal invariance. But it does not seem to. In fact time invariance is different with a magnetic field. If the time reversed orbit is $\mathbf{x}_r(t) := \mathbf{x}(-t)$, then the time reversed velocity is $\mathbf{v}_r(t) := \dot{\mathbf{x}}_r(t) = -\dot{\mathbf{x}}(-t)$. If the force is given only by a scalar potential, then $(\mathbf{x}_r,\mathbf{v}_r)$ satisfy Newton's equations. But if there is a magnetic field $\mathbf{B}(\mathbf{x})$ this also needs to be changed to $-\mathbf{B}(\mathbf{x})$.

Hence the S-matrix defined as $(\Omega_+^d)^*\Omega_-^d$ is unitary on $L^2((E_d,\infty)\times\mathbb{T})$. Our wave operators have a simple transformation law under a time independent gauge transformation. If

$$\mathbf{a}\to\mathbf{a}+\nabla f,\qquad H\to e^{if}He^{-if}\quad\text{and}\quad S\to S+f,$$

then

$$\Omega_\pm^d\to e^{if}\Omega_\pm^d.$$

It follows that the S-matrix is gauge invariant.

In the gauge we use here there is no asymptotic momentum and we conjecture that there does not exist a gauge, where an asymptotic momentum exists. This would mean that no momentum preserving approximate dynamics is available. This partially motivates our choice of approximate dynamics (1.22) which is not momentum preserving. For a comparative discussion of various types of wave operators as used in [9,11,20], see [3].

2. Classical mechanics: the globally attracting periodic orbit

Our main interest in this section is to determine the so called "scattering energies" and to study the long time behavior of any classical orbit corresponding to such an energy. We will see that $E_b=\max|b|^2/2$ is a threshold above which things are "simpler" while below it they are "more complicated."

We assume that the orbits stay away from the origin at all positive times, i.e. $r(t)\neq 0$ for $t\geqslant 0$. We assume that $b<0$; in this section we take the magnetic field $B(\mathbf{x})=b(\theta)/r$ for all $\mathbf{x}\neq 0$, cf. Section 1.2.

Let us first notice that we always have a maximal velocity bound as an immediate consequence of energy conservation (1.10) which gives $|\dot{r}| \leqslant \sqrt{2E}$ and yields

$$r/t \leqslant \sqrt{2E} + \epsilon, \quad t \geqslant T_\epsilon. \tag{2.1}$$

2.1. $E > E_b$ leads to an "easy" classical Mourre estimate

Next we shall show that if $E > E_b$ then not only does the particle leave the origin, but we also have a positive lower bound for its velocity. More precisely, we show that $\dot{r}(t) = \rho(t) \geqslant \rho_0 > 0$ if t is sufficiently large (see (2.4)).

To achieve that, compute the time derivative of ρr using (1.6)–(1.10) and get

$$\frac{d}{dt}(\rho r) = \dot{\rho} r + \rho^2 = \xi(b + \xi) + 2E - \xi^2 = 2E + \xi b \geqslant E - \frac{b^2}{2}. \tag{2.2}$$

After integration we obtain

$$(\rho r)(t) \geqslant (\rho r)(t_0) + (E - E_b)(t - t_0)$$

which leads to the conclusion that for sufficiently large t we must have at least $\rho(t) > 0$. Knowing the sign of ρ, we can express it as $\sqrt{2E - \xi^2}$ and introduce it in (1.9) obtaining for the transverse velocity ξ,

$$\frac{d\xi}{dt} = -\frac{\xi + b}{r}\sqrt{2E - \xi^2}. \tag{2.3}$$

We have the following localization at large times.

Lemma 2.1. *Fix $0 < \epsilon < \min|b| = -\max b$. Then for every trajectory of energy $E > E_b$ there exists $T > 0$ sufficiently large such that*

$$\xi(t) \in \big[\min|b| - \epsilon, \max|b| + \epsilon\big], \quad t \geqslant T.$$

Proof. We already know that for t large enough, the radial velocity ρ is nonzero, hence $|\xi(t)| < \sqrt{2E}$, for every $t \geqslant T_1$. Fix $\epsilon > 0$ and assume that either $\xi(T_1) \in (\max|b| + \epsilon, \sqrt{2E}\,)$ or $\xi(T_1) \in (-\sqrt{2E}, \min|b| - \epsilon)$. We first prove that there exists $T_2 > T_1$ such that $\xi(T_2) \in [\min|b| - \epsilon, \max|b| + \epsilon]$. Indeed, if we assume the contrary it means that either $\xi(t) \geqslant \max|b| + \epsilon$ or $\xi(t) \leqslant \min|b| - \epsilon$ for every $t \geqslant T_1$. In the first situation, (2.3) implies that $\xi(t)$ is decreasing and

$$\frac{d\xi}{dt}(t) \leqslant -\frac{\epsilon}{r}\sqrt{2E - \xi^2(T_1)}, \quad t \geqslant T_1,$$

while in the second situation, (2.3) implies that $\xi(t)$ is increasing and if we define M as the maximum between $\xi^2(T_1)$ and $(\min|b| - \epsilon)^2$ then

$$\frac{d\xi}{dt}(t) \geqslant \frac{\epsilon}{r}\sqrt{2E - M}, \quad t \geqslant T_1.$$

Using (2.1), we see that in both cases the variation in ξ is logarithmic in t so ξ cannot be bounded.

Thus in any case we can find T_2 with $\xi(T_2) \in [\min|b| - \epsilon, \max|b| + \epsilon]$. Let us prove that $\xi(t)$ will stay in the same closed interval for all times $t \geqslant T_2$. Assume on the contrary that there exists $T_3 > T_2$ such that $\xi(T_3) > \max|b| + \epsilon$. Then the intermediate value theorem gives exactly one point $T_4 \in [T_2, T_3)$, where

$$\xi(T_4) = \max|b| + \epsilon, \qquad \max|b| + \epsilon < \xi(t), \quad T_4 < t < T_3.$$

The mean value theorem would lead to the existence of $\tau \in (T_4, T_3)$, where $\frac{d\xi}{dt}(\tau) > 0$ which contradicts (2.3) which says that $\frac{d\xi}{dt}(\tau)$ has the same sign as $-\xi(\tau) + |b| < -\epsilon < 0$.

In a similar way, one proves that $\xi(t)$ cannot become less than $\min|b| - \epsilon$ for any $t \geqslant T_2$ and we are done. □

We may now conclude that we also have a minimal velocity bound, that is for every sufficiently small ϵ, there exists $T = T_\epsilon$ large enough such that (remember that we imposed the condition $E > E_b$)

$$\rho(E,t) = \sqrt{2E - \xi^2(E,t)} \geqslant \sqrt{2E - \left(\max|b| + \epsilon\right)^2}, \quad t \geqslant T. \tag{2.4}$$

Another important consequence of Lemma 2.1 is that the transverse velocity $\xi(t)$ eventually will be positive. Using this fact we may express the time variable from the first equation in (1.9) as a function of θ and introduce it in (2.3), obtaining one equation for $\xi(\theta)$:

$$\frac{d\xi}{d\theta} = -\frac{b(\theta) + \xi(\theta)}{\xi(\theta)}\sqrt{2E - \xi^2(\theta)}, \quad \theta \geqslant \theta_0,\ 0 < \xi(\theta_0) < \sqrt{2E}. \tag{2.5}$$

More generally, motivated by these considerations we transform (1.7) and (1.9) into a system of equations with θ as variable and $E > 0$:

$$\begin{cases} \partial_\theta \rho = b + \xi, \\ \partial_\theta \xi = -\dfrac{b+\xi}{\xi}\rho, \quad \rho(\theta_0) \in (-\sqrt{2E}, \sqrt{2E}\,) \text{ and } \xi(\theta_0) \in (0, \sqrt{2E}\,). \\ \xi^2 + \rho^2 = 2E, \end{cases} \tag{2.6}$$

Notice that (2.6) is derived under the assumption that $\xi > 0$, while no sign assumption on ρ is imposed. Although (2.6) have maximal solutions that are not globally defined (depending on the initial conditions), we shall only be interested in globally defined in fact periodic solutions. As we will see shortly the above system may admit periodic solutions even for some energies $E \leqslant E_b$.

Clearly the single equation

$$\partial_\theta \rho = b + \sqrt{2E - \rho^2}, \quad \rho(\theta_0) \in (-\sqrt{2E}, \sqrt{2E}\,), \tag{2.7}$$

is equivalent to (2.6) if we only consider solutions to (2.7) with $\xi := \sqrt{2E - \rho^2}$ strictly positive.

2.2. Existence of a periodic solution to (2.5)

With the standing hypotheses $E > E_b$ and $b < 0$ we look at (2.5) with the initial angle θ_0 chosen to be zero. For any $a \in (0, \sqrt{2E})$, denote by $\xi(E, a; \cdot)$ the maximal solution to (2.5) with specified initial data $\xi(E, a; 0) = a$.

Lemma 2.2. *For every $a \in [\min|b|/2, \sqrt{E + E_b}]$, the maximal solution is global and stays in the same interval for all $\theta \in \mathbb{R}_+$.*

Proof. We reason as in Lemma 2.1. Let us assume on the contrary that there exists $\theta_1 > 0$ such that either $0 < \xi(E, a; \theta_1) < \min|b|/2$ or $\xi(E, a; \theta_1) > \sqrt{E + E_b}$. Denote by θ_2 the largest argument in $[0, \theta_1)$, where we have that $\xi(E, a; \theta_2) = \min|b|/2$ (or $\sqrt{E + E_b}$, respectively). Then (2.5) implies that

$$\frac{\partial \xi}{\partial \theta}(E, a; x) > 0 \ (< 0), \quad (\forall)\theta_2 < x < \theta_1,$$

which clearly contradicts $\xi(E, a; \theta_1) < (>)\ \xi(E, a; \theta_2)$. □

We may now conclude that the mapping

$$g(E; \cdot) : \left[\min|b|/2, \sqrt{E + E_b}\,\right] \mapsto \left[\min|b|/2, \sqrt{E + E_b}\,\right],$$
$$g(E; a) = \xi(E, a; 2\pi)$$

has at least one fixed point so there exists at least one 2π-periodic solution.

Remark. An analysis similar to the one we did before shows that for any choice of $a_1 \in (0, \min|b|]$ and $a_2 \in [\max|b|, \sqrt{2E}\,)$ we again have that the interval $[a_1, a_2]$ is left invariant by the above mapping g. In fact, the next result says there is only one periodic solution which does not get outside $[\min|b|, \max|b|]$. Moreover, all other global (non-periodic) solutions will be drawn to the periodic one.

Denote by ξ_E any 2π-periodic solution to Eq. (2.5), with the conditions $\xi_E(\theta) \in (0, \sqrt{2E})$ and $\theta \in [0, \infty)$.

Lemma 2.3. *There is exactly one such solution, and it obeys*

$$\mathrm{Ran}(\xi_E) \subseteq I := \left[\min|b|, \max|b|\right].$$

Proof. The existence of a solution ξ_E with values in I follows from the previous considerations. Consider any other solution to the same equation (which may not be periodic), starting at $\xi(0) \in (0, \sqrt{2E})$. Assume that $\xi(0) \neq \xi_E(0)$, otherwise $\xi(\theta) = \xi_E(\theta)$ everywhere.

If $\rho_E(\theta) = \sqrt{2E - \xi_E^2(\theta)}$ and $\rho = \sqrt{2E - \xi^2(\theta)}$ then introduce $F = \rho - \rho_E$. Since Eq. (2.5) can be rewritten as

$$\frac{d}{d\theta}\sqrt{2E - \xi^2(\theta)} = b(\theta) + \xi(\theta), \tag{2.8}$$

we get

$$\frac{dF}{d\theta} = \xi - \xi_E = -hF; \quad h = \frac{\xi_E - \xi}{\rho - \rho_E} = \frac{\rho + \rho_E}{\xi + \xi_E}. \tag{2.9}$$

Notice that we used the energy conservation in order to obtain the second expression for h.

Then by integration:

$$F(\theta) = F(0)\exp\left[-\int_0^\theta h(\phi)\,d\phi\right]. \tag{2.10}$$

Next using the bound

$$h \geqslant \frac{\rho_E}{\sqrt{2E} + \xi_E} \geqslant \frac{\sqrt{2E - \max \xi_E^2}}{2\sqrt{2E}} =: C(E,b) \quad (>0),$$

we get

$$\left|F(\theta)\right| \leqslant \left|F(0)\right| \exp\bigl[-C(E,b)\theta\bigr], \quad (\forall)\theta \geqslant 0,$$

and conclude that

$$\lim_{\theta\to\infty}\left|\xi(\theta) - \xi_E(\theta)\right| = 0. \tag{2.11}$$

In particular, this will prove the lemma. Indeed, if there were a different periodic orbit $\tilde{\xi}_E$ we would have

$$\left|\xi_E(2n\pi) - \tilde{\xi}_E(2n\pi)\right| = \left|\xi_E(0) - \tilde{\xi}_E(0)\right| > 0, \quad \forall n \in \mathbb{N},$$

and this would contradict (2.11). □

2.3. Periodic solutions below E_b

This is the first place, where we are going to allow E to go below E_b. First we prove that there exists $E_d \in [\min|b|^2/2, E_b]$ such that (2.6) admits global periodic solutions for every $E > E_d$. If b is not constant then we show that $E_d \in (\min|b|^2/2, E_b)$.

Fix $E_0 > 0$ and consider (2.7) with the notation extended to include the energy and initial value of ρ

$$\partial_\theta \rho(E_0, a, \theta) = b(\theta) + \sqrt{2E_0 - \rho^2(E_0, a, \theta)},$$
$$\rho(E_0, a, 0) = a \in \left(-\sqrt{2E_0}, \sqrt{2E_0}\right), \tag{2.12}$$

or equivalently

$$\rho(E_0, a, \theta) = a + \int_0^\theta \bigl(b(\phi) + \sqrt{2E_0 - \rho^2(E_0, a, \phi)}\bigr)\, d\phi.$$

Proposition 2.4. *Assume that for some $E_0 > 0$ Eq.* (2.12) *admits a global, C^1 and 2π-periodic solution denoted by ρ_{E_0}. Then it satisfies*

$$c := \inf_\theta \sqrt{2E_0 - \rho_{E_0}^2(\theta)} \geqslant \min|b| > 0. \tag{2.13}$$

If in addition we have

$$I(E_0, \rho_{E_0}) := \int_0^{2\pi} \frac{\rho_{E_0}}{\sqrt{2E_0 - \rho_{E_0}^2}}\, d\varphi > 0, \tag{2.14}$$

then there exists $\epsilon > 0$ such that for every $E \in (E_0 - \epsilon, E_0 + \epsilon)$ we have a periodic solution ρ_E obeying $I(E, \rho_E) > 0$ as in (2.14).

Proof. Before starting the proof, let us explain the meaning of (2.14). Assuming that we have ρ_E, define $\xi_E := \sqrt{2E - \rho_E^2}$. Consider the initial value problem ($t \geqslant 1$):

$$\frac{d\tilde{r}}{dt} = \rho_E(\tilde{\theta}), \qquad \frac{d\tilde{\theta}}{dt} = \frac{\xi_E(\tilde{\theta})}{\tilde{r}}, \qquad \bigl(\tilde{r}(1), \tilde{\theta}(1)\bigr) = (1, 0). \tag{2.15}$$

It is easy to check that at least for t close to 1 the above system admits a solution $(\tilde{r}, \tilde{\theta})$ which also solves the Hamilton equations (1.6)–(1.9), hence it corresponds to a real orbit at energy E. We notice that the above system gives

$$\tilde{r}(t) = \tilde{r}\bigl(\tilde{\theta}(t)\bigr) = \exp\left\{ \int_0^{\tilde{\theta}(t)} (\rho_E/\xi_E)(\varphi)\, d\varphi \right\} \tag{2.16}$$

hence the solution is global, $\mathrm{Ran}(\tilde{\theta}) = [0, \infty)$ and $\tilde{r}(t)$ increases "in mean" after each complete revolution around the origin and escapes to infinity. It is important to remark though that $\tilde{r}(t)$ may not be strictly increasing as a function of time as it was the case for energies above E_b.

Now let us start the proof of the proposition. If ρ_{E_0} is periodic and C^1, then it cannot equal $\pm\sqrt{2E_0}$ at any point θ_1 since otherwise θ_1 would be an extremum point and thus $0 = \partial_\theta \rho_{E_0}(\theta_1) = b(\theta_1)$ which contradicts $b < 0$ everywhere. Next, let us show that $c \geqslant \min|b|$. Indeed, assume that the minimum of $\xi_{E_0} := \sqrt{2E_0 - \rho_{E_0}^2}$ is attained at some point θ_1. Then $\rho_{E_0}(\theta_1)\partial_\theta \rho_{E_0}(\theta_1) = 0$. If $\rho_{E_0}(\theta_1) = 0$ then $\xi_{E_0} = \sqrt{2E_0}$, ρ_{E_0} vanishes identically and $b = -\sqrt{2E_0}$. Obviously $c = \min|b|$ in this case. If $\rho_{E_0}(\theta_1) \neq 0$ then $\partial_\theta \rho_{E_0}(\theta_1) = 0$ thus $\xi_{E_0}(\theta_1) = c = -b(\theta_1)$ and (2.13) follows.

Next, denote by $a_0 = \rho_{E_0}(0)$. General results in O.D.E. ensure the existence of $\delta_1, \delta_2 > 0$ such that $\rho(E, a, \theta)$ exists and is smooth in all arguments for $E \in (E_0 - \delta_1, E_0 + \delta_1)$, $a \in (a_0 - \delta_2,$

$a_0 + \delta_2)$ and $\theta \in [0, 2\pi]$. We intend to apply the implicit function theorem to the equation $\rho(E, a, 2\pi) = a$; differentiating (2.12) with respect to a we get

$$\partial_\theta(\partial_a \rho)(E, a, \theta) = -\frac{\rho}{\sqrt{2E - \rho^2}} \partial_a \rho(E, a, \theta)$$

which by integration yields

$$\partial_a \rho(E_0, a_0, 2\pi) = \exp\bigl(-I(E_0, \rho_{E_0})\bigr) < 1.$$

Then the implicit function theorem gives a smooth solution $a(E)$ to the equation $\rho(E, a(E), 2\pi) = a(E)$, in a small open interval centered at E_0. The proposition is concluded by putting $\rho_E(\theta) = \rho(E, a(E), \theta)$. □

For every $E_0 > 0$ denote by $P(E_0)$ the statement "there is a C^1 periodic solution to (2.12) satisfying (2.14)." Then define

$$\mathcal{E} := \bigl\{E_0 > 0\colon\ P(E_0) \text{ is true}\bigr\}, \qquad E_d := \inf \mathcal{E}. \tag{2.17}$$

Lemma 2.5. *For every $E > E_b$ let ξ_E be the periodic solution to* (2.5) *and let $\rho_E := \sqrt{2E - \xi_E^2}$ (> 0). Then ρ_E satisfies the bound $I(E, \rho_E) > 0$ as in* (2.14). *Moreover (ρ_E, ξ_E) is the unique periodic solution to* (2.6) *which satisfies* (2.14).

Proof. The facts that ρ_E satisfies (2.14) and that (ρ_E, ξ_E) is a solution to (2.6) are trivial; we only need to prove that there are no other solutions to (2.6) which also satisfies (2.14).

Assume that $(\tilde{\rho}_E, \tilde{\xi}_E)$ is such a solution. First, $\tilde{\rho}_E$ cannot be strictly negative because in that case (2.14) cannot be fulfilled. Second, if $\tilde{\rho}_E(\theta)$ is strictly positive, then we can write $\tilde{\rho}_E = \sqrt{2E - \tilde{\xi}_E^2}$ and hence $\tilde{\xi}_E$ satisfies (2.5). Whence by Lemma 2.3 we conclude that $\tilde{\xi}_E = \xi_E$ and $\tilde{\rho}_E = \rho_E$. Third if $\tilde{\rho}_E(\theta_1) = 0$ we have $\tilde{\xi}_E(\theta_1) = \sqrt{2E}$ and hence the first equation of (2.6) leads to $\partial_\theta \tilde{\rho}_E(\theta_1) \geqslant \sqrt{2E} - \max|b| > 0$. From the periodicity of $\tilde{\rho}_E(\theta)$ we then conclude that indeed this function cannot have zeros. We have reduced to the previous case. □

However, it is not yet clear whether we can construct periodic solutions which obey (2.14) for all energies in (E_d, ∞). This question will be answered shortly.

But first let us prove a few (interesting) facts about $\rho_E(\theta)$. We start with its energy dependence:

Proposition 2.6. *Let $E_0 > 0$ and ρ_{E_0} be as in Proposition* 2.4 (*obeying also* (2.14)). *Denote by $\rho_E(\theta)$ the smooth and periodic solution to* (2.12) *constructed in the proof of Proposition* 2.4 *for every $E \in (E_0 - \epsilon, E_0 + \epsilon)$. Define $\xi_E = \sqrt{2E - \rho_E^2}$ (> 0). We then have*

$$\inf_\theta \partial_E \rho_E(\theta) > 0, \tag{2.18}$$

$$\sup_\theta \partial_E^2 \rho_E(\theta) < 0 \quad \textit{and} \tag{2.19}$$

$$\inf_\theta \partial_E \frac{\rho_E}{\xi_E}(\theta) > 0. \tag{2.20}$$

Proof. We start with some general results we use in the proof. Suppose $x:\mathbb{R}\mapsto\mathbb{R}$ is 2π-periodic and satisfies the equation

$$x'(\theta)=-h(\theta)x(\theta)+g(\theta), \tag{2.21}$$

where h and g are continuous, 2π-periodic and $\int_0^{2\pi} h(\theta)\,d\theta>0$.

The boundedness of x and the integral condition on h ensure that

$$\lim_{\theta\to-\infty} e^{-\int_\theta^0 h(\varphi)\,d\varphi}x(\theta)=0,$$

and after a standard computation we get

$$x(\theta)=\int_0^\infty g(\theta-\phi)e^{-\int_0^\phi h(\theta-\phi')\,d\phi'}\,d\phi. \tag{2.22}$$

We can rewrite (2.12) as

$$\partial_\theta\rho_E=b+\xi_E, \tag{2.23}$$

and obviously

$$\rho_E\partial_E\rho_E=1-\xi_E\partial_E\xi_E. \tag{2.24}$$

Now let us prove the monotonicity of ρ_E in (2.18). Differentiate (2.23) with respect to E and get $\partial_E\xi_E=\partial_\theta(\partial_E\rho_E)$, then introduce this in (2.24) to obtain

$$\partial_\theta(\partial_E\rho_E)=-\frac{\rho_E}{\xi_E}(\partial_E\rho_E)+\frac{1}{\xi_E}=-h(\partial_E\rho_E)+g. \tag{2.25}$$

Notice that according to (2.14) we have that $\int_0^{2\pi} h(\theta)\,d\theta>0$. Thus g is strictly positive and the equation has the same form as in (2.21). It follows according to (2.22) that $\partial_E\rho_E>0$.

The concavity in (2.19) is shown using the same idea. Write

$$\rho_E^3\bigl(\partial_E^2\rho_E\bigr)=-(1-\xi_E\partial_E\xi_E)^2-\rho_E^2\bigl[(\partial_E\xi_E)^2+\xi_E\partial_E^2\xi_E\bigr], \tag{2.26}$$

then isolate $\partial_E^2\xi_E$:

$$\partial_E^2\xi_E=-\frac{\rho_E}{\xi_E}\cdot\bigl(\partial_E^2\rho_E\bigr)-\frac{1}{\xi_E}\bigl[(\partial_E\rho_E)^2+(\partial_E\xi_E)^2\bigr]. \tag{2.27}$$

Differentiate (2.23) twice with respect to E to get $\partial_E^2\xi_E=\partial_\theta(\partial_E^2\rho_E)$ and introduce it in (2.27):

$$\partial_\theta\bigl(\partial_E^2\rho_E\bigr)=-h\bigl(\partial_E^2\rho_E\bigr)+g.$$

Since we again have the integral condition on h while g now is strictly negative, (2.22) implies $\partial_E^2\rho_E<0$ and we are done.

Finally, let us prove (2.20). Using $\xi_E = \sqrt{2E - \rho_E^2}$ we have

$$\partial_E(\rho_E/\xi_E) = (2E\partial_E\rho_E - \rho_E)/\xi_E^3,$$

so it is enough to prove that $x(\theta) = 2E\partial_E\rho_E(\theta) - \rho_E(\theta)$ is strictly positive. By direct computation we have (use (2.24))

$$\partial_\theta x = 2E\partial_E\xi - b - \xi_E = -\frac{\rho_E}{\xi_E}x - b$$

and since $-b > 0$ we are done. □

We can now claim the absence of gaps in $\mathcal{E} \cap (E_d, \infty)$.

Corollary 2.7. *For every $E > E_d$ there is a periodic solution $(\rho, \xi) = (\rho_E, \sqrt{2E - \rho_E^2})$ to* (2.6) *which obeys* (2.14). *Moreover, there is no other periodic solution with this property.*

Proof. Assume there exists $E_0 > E_d$ and $E_0 \notin \mathcal{E}$. Since E_d is the infimum of $\mathcal{E}$ then there must exist $E_1 \in \mathcal{E}$ so that $E_1 < E_0$. It follows that some $\rho_{E_1}(\theta)$ exists and satisfies (2.14). Then we can apply the local construction of Proposition 2.4 in order to obtain similar solutions for slightly larger energies. In fact, we claim that this construction can be continued up to E_0 and cannot stop before. Indeed, assume there exists $E_2 \in (E_1, \infty)$ such that the branch coming from ρ_{E_1} stops at E_2. We know that $\rho_E(\theta)$ is increasing as a function of E on (E_1, E_2) and bounded from above by $\sqrt{2E_0}$, hence admits a limit $\rho_{E_2}(\theta)$. We conclude that ρ_{E_2} is periodic, solves the integral version of (2.12) with $a = \rho_{E_2}(0)$, thus it is C^1.

Then define $\xi_{E_2} = \sqrt{2E_2 - \rho_{E_2}^2}$ and notice that according to (2.13) it cannot be zero. Hence ρ_{E_2} and ξ_{E_2} are smooth functions of θ which solve (2.6). Then they also obey (2.14) because (2.20) ensures that ρ_E/ξ_E is increasing in E at fixed θ. It follows that the construction of Proposition 2.4 can be repeated at E_2. Hence ρ_E does not stop there. Thus we must have $E_0 \in \mathcal{E}$ and therefore $\mathcal{E} = (E_d, \infty)$.

Now let us prove the uniqueness of such a solution. Assume that $(\tilde{\rho}_E, \tilde{\xi}_E)$ is another solution to (2.6) obeying (2.14). Then using the local construction of Proposition 2.4 together with the monotonicity in (2.20) we can uniquely construct the solution "branches" originating from ρ_E and $\tilde{\rho}_E$ as functions of E until we reach an energy $E' > E_b$, where we know that $\rho_{E'} = \tilde{\rho}_{E'}$ due to Lemma 2.5. Going backwards from E' using the uniqueness part of the implicit function theorem, cf. the proof of Proposition 2.4, we conclude that $\tilde{\rho}_E = \rho_E$ and we are done. □

Remark. The above uniqueness proof works only for solutions which we a priori know obey (2.14). If we drop (2.14) in Corollary 2.7 uniqueness is no longer true, see Section 9.

We know by now that if $E > E_b$ then ρ_E is positive. From Proposition 2.4 we see that if ρ_E is non-negative (but not identically zero), we can still go lower in energy with our construction. Hence we give here another definition related to the energy for which ρ_E can become zero:

$$E_c := \inf\bigl\{E_0 > 0\colon \inf_\theta \rho_{E_0}(\theta) > 0\bigr\}. \tag{2.28}$$

However, if b is a negative constant, then it is easy to see that $E_d = E_c = E_b$. Indeed, in this case we have $\xi_{E_0} = -b > 0$ independent of θ and E_0. Moreover, $\rho_{E_0} = \sqrt{2E_0 - b^2}$ and the result follows.

More interesting is the situation in which b is not constant. We start with a lemma.

Lemma 2.8. *If b is not constant then $E_d < E_c < E_b$.*

Proof. Since $\rho_E(\theta)$ is increasing in E and is bounded from below by zero if $E > E_c$, we can define the functions

$$\rho_c(\theta) = \lim_{E \searrow E_c} \rho_E(\theta), \qquad \xi_c(\theta) = \lim_{E \searrow E_c} \xi_E(\theta) = \sqrt{2E - \rho_c^2(\theta)}. \tag{2.29}$$

Notice that ρ_c and ξ_c are smooth and periodic solutions to (2.6), where $E = E_c$. By (2.13) we have $\xi_c(\theta) > 0$ for all θ while $\rho_c(\theta_1) = 0$ for some θ_1. But ρ_c cannot be identically zero, otherwise (2.12) would read as

$$\frac{d}{d\theta}\rho_c = \sqrt{2E_c} + b. \tag{2.30}$$

Since the right-hand side is not identically zero this leads to a contradiction. In particular, this means

$$\int_0^{2\pi} \frac{\rho_c}{\xi_c}\, d\theta > 0, \tag{2.31}$$

and according to Proposition 2.4 it follows that $E_d < E_c$.

Now let us prove that $E_c < E_b$. Assume on the contrary that $E_c = E_b$. Pick θ_1 such that $\rho_c(\theta_1) = 0$ and $\xi_c(\theta_1) = \sqrt{2E_b}$. Since $b + \sqrt{2E_b} \geqslant 0$ we learn from (2.5) that

$$\frac{d}{d\theta}\left(\xi_c - \sqrt{2E_b}\right) = -h\left(\xi_c - \sqrt{2E_b}\right) + g; \quad g \leqslant 0.$$

In particular

$$\frac{d}{d\theta}\left\{e^{\int h\, d\phi}\left(\xi_c - \sqrt{2E_b}\right)\right\} \leqslant 0,$$

which shows that $\xi_c(\theta) \geqslant \sqrt{2E_b}$ (and therefore $\xi_c(\theta) = \sqrt{2E_b}$) for $\theta < \theta_1$. This implies that $\rho_c(\theta) = 0$ for $\theta < \theta_1$, and the periodicity gives that $\rho_c(\theta)$ is identically zero, contradicting (2.31). □

Now let us focus on E_d, again in the case when b is not constant. Reasoning as we did for E_c, there exist two smooth and periodic functions ρ_d and ξ_d which solve (2.6) with $E = E_d$ and

$$\xi_d \geqslant c > 0, \qquad \int_0^{2\pi} \frac{\rho_d}{\xi_d}\, d\theta = 0. \tag{2.32}$$

It is interesting what happens when the energy decreases from E_c to E_d. At E_c, the radial velocity is zero somewhere but still non-negative; then for $E \in (E_d, E_c)$ the velocity ρ_E attains more and more negative values and the integral in (2.14) becomes smaller and smaller when E_d is approached.

Lemma 2.9. *If b is not constant then*

$$\frac{1}{2}\left[-\frac{1}{2\pi}\int_0^{2\pi} b(\theta)\,d\theta\right]^2 < E_d.$$

Proof. Since we have $\partial_\theta \rho_d = b + \sqrt{2E_d - \rho_d^2}$, then by integration we get

$$0 = \int_0^{2\pi} b(\theta)\,d\theta + \int_0^{2\pi} \sqrt{2E_d - \rho_d^2}\,d\theta.$$

Since ρ_d is not identically zero we get $-\int_0^{2\pi} b(\theta)\,d\theta < 2\pi\sqrt{2E_d}$ and we are done. □

Now let us summarize what we obtained in the last few lemmas.

Proposition 2.10. *If b is a negative constant then $E_d = E_c = E_b = b^2/2$. If b is not constant, then*

$$\frac{\min|b|^2}{2} < \frac{1}{2}\left[-\frac{1}{2\pi}\int_0^{2\pi} b(\theta)\,d\theta\right]^2 < E_d < E_c < E_b = \frac{\max|b|^2}{2}.$$

2.4. A classical Mourre estimate above E_d

We now investigate the long term behavior of an arbitrary classical orbit, defined for every $t \geqslant 0$, corresponding to an energy $E > E_d$. We will see that the periodic solution provided by Corollary 2.7 generates an attractor.

Consider an orbit $(r(t), \theta(t))$ solving (1.6)–(1.9) and obeying (1.10) with $E > E_d$. We use the notation (ρ_E, ξ_E) for the periodic solution provided by Corollary 2.7.

For every $C > 0$ define

$$A_C(t) = \left\{\partial_E \rho_E\big(\theta(t)\big) + C\big[\rho(t) - \rho_E\big(\theta(t)\big)\big]\right\} \cdot r(t). \tag{2.33}$$

Proposition 2.11. *For every $\epsilon > 0$, there exists $T > 1$ large enough (depending on the particular orbit) such that*

$$\rho(t) - \rho_E\big(\theta(t)\big) \geqslant -\epsilon, \quad \forall t \geqslant T. \tag{2.34}$$

Proof. We first compute the derivative of A_C with respect to t; using the Hamilton equations we get (also use (2.24))

$$\frac{d}{dt} r\partial_E\rho_E = \xi\partial_E\xi_E + \rho\partial_E\rho_E = 1 + (\xi - \xi_E)\partial_E\xi_E + (\rho - \rho_E)\partial_E\rho_E, \tag{2.35}$$

and

$$\frac{d}{dt} r(\rho - \rho_E) = (\xi - \xi_E)\xi + (\rho - \rho_E)\rho = \frac{1}{2}(\rho - \rho_E)^2 + \frac{1}{2}(\xi - \xi_E)^2, \tag{2.36}$$

where the last equality is a consequence of energy conservation and the identity

$$(\xi - \xi_E)\xi + (\rho - \rho_E)\rho = \frac{1}{2}\left(\rho^2 + \xi^2\right) - \frac{1}{2}\left(\rho_E^2 + \xi_E^2\right) + \frac{1}{2}(\rho - \rho_E)^2 + \frac{1}{2}(\xi - \xi_E)^2. \tag{2.37}$$

Combining (2.33), (2.35) and (2.36) we get

$$\frac{dA_C}{dt} = 1 + (\xi - \xi_E)\partial_E\xi_E + (\rho - \rho_E)\partial_E\rho_E + \frac{C}{2}\cdot\left[(\rho - \rho_E)^2 + (\xi - \xi_E)^2\right]. \tag{2.38}$$

Since $\partial_E\xi_E$ and $\partial_E\rho_E$ are bounded, it is easy to see that there exists $C(E) > 0$ (i.e. only depending on E) such that if $C \geqslant C(E)$

$$\frac{dA_C}{dt} \geqslant \frac{1}{2}, \quad t \geqslant 1. \tag{2.39}$$

Then integrating (2.33) we obtain that for T large enough:

$$A_C(t) \geqslant \frac{t}{4}, \quad t \geqslant T. \tag{2.40}$$

Using (2.33) we finally get

$$\rho(t) - \rho_E\big(\theta(t)\big) + \frac{1}{C}\partial_E\rho_E\big(\theta(t)\big) > 0, \quad t \geqslant T,$$

and the proof follows by choosing C large enough. □

3. Propagation estimates in classical mechanics

In this section we work with classical orbits that are defined for all times $t \geqslant 0$. We want to show that the radial and transverse velocity $\rho(t)$ and $\xi(t)$ corresponding to an arbitrary orbit $(r(t), \theta(t))$ at energy $E > E_d$, will get arbitrarily close to $\rho_E(\theta(t))$ and $\xi_E(\theta(t))$ for large enough times.

Let us introduce four real-valued cut-off functions $F \in C^\infty(\mathbb{R})$; they depend on real parameters $a < b < c < d$:

$$
\begin{aligned}
&F_+ = F_+^{a,b};\\
&F_+(x) = 0 \quad \text{for } x < a, \qquad F_+(x) = 1 \quad \text{for } x > b,\\
&F_+'(x) \geqslant 0, \qquad \sqrt{F_+},\ \sqrt{1-F_+},\ \sqrt{F_+'} \in C^\infty(\mathbb{R}).
\end{aligned} \tag{3.1}
$$

$$F_- = F_-^{c,d} = 1 - F_+^{c,d}. \tag{3.2}$$

$$F_{+-} = F_{+-}^{a,b,c,d} = F_+^{a,b} F_-^{c,d}. \tag{3.3}$$

$$F_{++}(x) = F_{++}^{a,b,c,d}(x) = \int_a^x F_{+-}^{a,b,c,d}(s)\, ds. \tag{3.4}$$

3.1. A minimal velocity bound: the particle leaves the origin

Proposition 3.1. *There exists $d > 0$ small enough such that for every $c < d$ and $F_- = F_-^{c,d}$ we have*

$$\lim_{t\to\infty} \bigl|F_-\bigl(r(t)/t\bigr)\bigr| = 0. \tag{3.5}$$

In particular, fixing $c = d/2$ implies the existence of T large enough such that $r(t)/t \geqslant d/2$ for all $t \geqslant T$.

Proof. The methods we use here will from now on be employed throughout the paper for both classical and quantum mechanics. We first need the following lemma.

Lemma 3.2. *There exists a $d_1 > 0$ small enough such that for all $c_1 < d_1$ and $F_- = F_-^{c_1,d_1}$ we have*

$$\int_1^\infty \frac{1}{t} |F_-'|(A_C/t)\, dt \leqslant Const, \tag{3.6}$$

where the above constant does not depend on the particular orbit. Moreover, for all $c_1 \in (0, d_1)$

$$F_-^{c_1/3, c_1/2}(A_C/t) \to 0, \tag{3.7}$$

but the rate of convergence depends on the particular orbit.

Proof. We start with (3.6). Differentiate the bounded propagation observable $\Phi(t) = F_-(A_C(t)/t)$ and get (the derivative of F_- is negative)

$$\partial_t \Phi(t) = -|F_-'|\bigl(A_C(t)/t\bigr) \cdot \left(\frac{1}{t}\partial_t A_C(t) - \frac{1}{t^2} A_C(t)\right),$$

or equivalently (here $t \geqslant 1$)

$$\frac{1}{2t}|F_-'|\big(A_C(t)/t\big) \leqslant \partial_t A_C(t)\frac{1}{t}|F_-'|\big(A_C(t)/t\big) = -\partial_t \Phi(t) + |F_-'|\big(A_C(t)/t\big) \cdot \big(A_C(t)/t\big)\frac{1}{t},$$

where the first inequality is given by (2.39). Since F_- is supported in $(-\infty, d_1]$, then $|F_-'|(A_C(t)/t) \cdot (A_C(t)/t) \leqslant |F_-'|(A_C(t)/t) \cdot d_1$ hence

$$\left(\frac{1}{2} - d_1\right)\frac{1}{t}|F_-'|\big(A_C(t)/t\big) \leqslant -\partial_t \Phi(t).$$

Because $\Phi \leqslant 1$, choosing $d_1 \in (0, 1/2)$ and after integration (3.6) follows.

A few words about (3.7). We could say that due to (2.40) the existence of the limit would immediately follow. But we use another argument which is based on (3.6) and can be generalized later on to quantum mechanics. We prove two things: first, that the limit exists; second, the limit equals zero.

Now consider $F_-^{c_1/3,c_1/2}(A_C(t)/t)$. To prove the existence of a limit when $t \to \infty$ we employ a Cook type argument, that is we show the absolute integrability of its time derivative:

$$\left|\partial_t F_-^{c_1/3,c_1/2}\big(A_C(t)/t\big)\right| \in L^1\big((1,\infty)\big).$$

But we see that on the given orbit, the above derivative may be bounded by (we omit the superscripts)

$$Const \cdot |F_-'|\big(A_C(t)/t\big)\frac{1}{t},$$

where the constant depends on the orbit; here we used the boundedness of the support of F_-', and that $\partial_t A_C$ is bounded. Now since $c_1/2 < d_1$, the integrability is ensured by (3.6).

Now let us prove that the limit is zero. First, note that $\sup_{t \geqslant 1} |A_C(t)/t| < \infty$ because of the energy conservation and maximal velocity bound (2.1). Second, choose $a < b < 0$ negative enough such that $\sup_{t \geqslant 1} |A_C(t)/t| < |b|$. Take $F_{++} = F_{++}^{a,b,c_1/2,c_1}$ like in (3.4). We will now mimic the proof of (3.6) and show that

$$\int_1^\infty \frac{1}{t}|F_{++}'|(A_C/t)\,dt \leqslant Const. \tag{3.8}$$

First, since A_C/t cannot enter the interval $(-\infty, b)$ we have $F_{++}'(A_C/t) = F_-^{c_1/2,c_1}(A_C/t)$. Second, differentiating $\Phi(t) := F_{++}(A_C/t)$ we obtain:

$$\partial_t \Phi(t) = F_-^{c_1/2,c_1}(A_C/t) \cdot \left(\frac{1}{t}\partial_t A_C - \frac{1}{t^2}A_C\right) \geqslant \frac{1}{t}F_-^{c_1/2,c_1}(A_C/t)(1/2 - c_1),$$

and we can now integrate and obtain (3.8).

We then can write

$$F_-^{c_1/3,c_1/2}\big(A_C(t)/t\big)F_{++}'\big(A_C(t)/t\big) = F_-^{c_1/3,c_1/2}\big(A_C(t)/t\big). \tag{3.9}$$

Because of (3.8) we conclude that there exists a sequence of diverging times $\{t_n\}$ such that

$$F'_{++}\big(A_C(t_n)/t_n\big) \to 0.$$

Then (3.9) implies the same thing for $F_-(A_C(t_n)/t_n)$. Since we have proven the existence of the limit, it must be zero. □

We are now ready to finish the proof of (3.5). Write

$$F_-^{c,d}(r/t) = F_-^{c,d}(r/t)\big\{F_-^{c_1/3,c_1/2}(A_C/t) + F_+^{c_1/3,c_1/2}(A_C/t)\big\},$$

where we choose $0 < c < d \ll c_1$. The contribution from the first term is handled by (3.7); the second will eventually equal zero because $r/t \leqslant d$ and $A_C/t \geqslant c_1$ are not simultaneously true if d is small enough. □

3.2. ξ moves away from zero

We now want to prove that the transverse velocity ξ will eventually have a sign after waiting long enough. That is, we want to first exclude the possibility that the particle stops rotating around the origin.

Lemma 3.3. *There exists $d > 0$ sufficiently small such that for all $F_{++} = F_{++}^{-d,-d/2,d/2,d}$* (*see* (3.4))

$$\int_1^\infty \frac{1}{t}|F'_{++}|\big(\xi(t)\big)\,dt \leqslant Const. \tag{3.10}$$

Moreover, (*see* (3.3)) *if* $0 < d_1 < d/2$ *is also sufficiently small then for any F_{+-} with support in* $[-d_1, d_1]$ *we have*

$$\lim_{t\to\infty} F_{+-}\big(\xi(t)\big) = 0. \tag{3.11}$$

Proof. We start with (3.10). Differentiate the bounded propagation observable $\Phi(t) = F_{++}(\xi(t))$ and get

$$\partial_t\Phi(t) = F'_{++}\big(\xi(t)\big)\frac{d\xi}{dt} = -F'_{++}\big(\xi(t)\big)\frac{\xi(t)\rho}{r} + F'_{++}\big(\xi(t)\big)\frac{(-b)\rho}{r}. \tag{3.12}$$

A consequence of Proposition 2.11 is the inequality

$$\liminf_{t\to\infty}\big[\rho(t) - \rho_E\big(\theta(t)\big)\big] \geqslant 0,$$

and by (2.13) there exists $\epsilon > 0$ (only depending on the location of E) and $T > 1$ large enough (depending on the orbit) such that

$$\rho(t) \geqslant -\sqrt{2E} + \epsilon, \quad t \geqslant T. \tag{3.13}$$

But $F'_{++}(\xi(t))$ is zero unless $|\xi|$ is less than d. Energy conservation will force ρ to stay either above $\sqrt{2E-d^2}$ or below $-\sqrt{2E-d^2}$. If we choose d small enough (depending on ϵ in (3.13)) we can rule out the second alternative so we have

$$F'_{++}\big(\xi(t)\big)\frac{(-b)\rho}{r} \geqslant F'_{++}\big(\xi(t)\big)\frac{(\min|b|)\sqrt{2E-d^2}}{r}, \quad t \geqslant T.$$

Introducing this in (3.12) we obtain another T large enough such that with $C_1 > 0$ and $t \geqslant T$

$$\partial_t \Phi(t) \geqslant \frac{1}{r}F'_{++}\big(\xi(t)\big)\cdot\big(-d\sqrt{2E} + \big(\min|b|\big)\sqrt{2E-d^2}\big) \geqslant C_1\cdot\frac{1}{t}F'_{++}\big(\xi(t)\big),$$

where in the last inequality we used the maximal velocity bound (2.1) and we took d small enough in order to get a positive lower bound. We then integrate and get the result.

The proof of the limit in (3.11) is similar to the one we gave for (3.7). First, with a Cook type argument we reduce the existence of the limit to the absolute integrability of

$$\left|F'_{+-}\big(\xi(t)\big)\frac{d\xi}{dt}\right| \leqslant Const\frac{1}{t}\big|F'_{+-}\big(\xi(t)\big)\big|.$$

We can write

$$F'_{+-}\big(\xi(t)\big)F'_{++}\big(\xi(t)\big) = F'_{+-}\big(\xi(t)\big)$$

because of their support properties, so the integrability problem can be reduced to (3.10).

Second, notice that we also may write

$$F_{+-}\big(\xi(t)\big)F'_{++}\big(\xi(t)\big) = F_{+-}\big(\xi(t)\big)$$

and repeat the sub-sequence argument we first employed right after (3.9). It follows that the limit must be zero. □

3.3. *ξ is positive for large times*

We saw in the previous subsection that ξ could not return to zero for large times; now we prove that ξ becomes positive.

Lemma 3.4. *If d_1 is the same as in* (3.11)*, consider $-d_1 < c_2 < d_2 < 0$. Then for $F_- = F_-^{c_2,d_2}$ we have*

$$\lim_{t\to\infty} F_-\big(\xi(t)\big) = 0. \tag{3.14}$$

Proof. The existence of the above limit again follows after using a Cook argument; notice that F'_- is supported in $(-d_1, d_1)$ so we can apply the propagation estimate in (3.10).

We now prove that the limit is zero. We know from energy conservation that $\rho(t) \in [-\sqrt{2E}, \sqrt{2E}]$. Fix some $a < b < -\sqrt{2E}$ and $\sqrt{2E} < c < d$ and consider $F_{+-} = F_{+-}^{a,b,c,d}$ (see (3.3)); we have

$$F_{+-}\big(\rho(t)\big) = 1. \tag{3.15}$$

Consider also that particular F_{++} whose derivative gives back the above F_{+-}. If we can prove the propagation estimate

$$\int_1^\infty \frac{1}{t} F'_{++}\big(\rho(t)\big) F_-\big(\xi(t)\big)\, dt \leqslant Const, \tag{3.16}$$

then because of (3.15) we conclude that $F_-(\xi(t))$ admits a divergent sequence of times along which it tends to zero and this would prove the lemma.

We now prove (3.16). Consider the observable $\Phi(t) = F_{++}(\rho(t))F_-(\xi(t))$ and compute its time derivative:

$$\partial_t \Phi(t) = \frac{\xi \cdot (\xi + b)}{r} F'_{++} F_- + F_{++} F'_- \frac{d\xi}{dt}.$$

Now notice that on the support of F_- we have $\xi < d_2 < 0$ hence

$$\frac{\xi \cdot (\xi + b)}{r} F'_{++} F_- \geqslant \frac{|d_2|^2 + |d_2| \min|b|}{r} F'_{++} F_- \geqslant Const \frac{1}{t} F'_{++} F_-,$$

where in the second inequality we used the maximal velocity bound. Finally,

$$Const \frac{1}{t} F'_{++} F_- \leqslant F_{++} |F'_-| \cdot \left| \frac{d\xi}{dt} \right| + \partial_t \Phi(t)$$

and the integrability of the right-hand side finishes the proof. □

Remark. If we put together the previous two lemmas, we obtain that $\xi(t) \geqslant d_1 > 0$ for all $t \geqslant T$, where T is large enough.

3.4. ρ gets trapped near ρ_E and ξ near ξ_E

We already know from energy conservation and Proposition 2.11 that the radial velocity is localized somewhere in the interval

$$\rho(t) \in \big[\rho_E\big(\theta(t)\big) - \epsilon, \sqrt{2E}\,\big], \quad t \geqslant T, \tag{3.17}$$

where T is large enough and depends on everything. We now intend to show that $\rho(t)$ can only spend a finite amount of time outside an interval of width ϵ centered at $\rho_E(\theta(t))$.

For every $E' \geqslant E$, define

$$B_{E'}(t) := \rho(t) - \rho_{E'}\big(\theta(t)\big).$$

Proposition 3.5. *Denote by $F_+ = F_+^{\epsilon/2,\epsilon}$. Then*

$$\lim_{t \to \infty} F_+\big(B_E(t)\big) = 0.$$

Moreover, there exists T large enough such that

$$\left|\rho(t) - \rho_E\big(\theta(t)\big)\right| \leqslant \epsilon, \quad t \geqslant T. \tag{3.18}$$

Proof. Notice that $F_+(B_E(t))$ is zero unless $\rho(t) \in [\rho_E(\theta(t)) + \epsilon/2, \sqrt{2E}\,]$. We now try to break this interval into smaller pieces which can be more easily treated using propagation estimates.

Lemma 3.6. *Define $\epsilon_0 := \frac{\epsilon}{4\sup_\theta(\partial_E \rho_E(\theta))}$. Then there exists M large enough such that uniformly in $\theta \in [0, 2\pi]$ we have*

$$\left[\rho_E(\theta) + \epsilon/2, \sqrt{2E}\,\right] \subset \left\{\rho_{E'}(\theta)\colon\ E' \in [E + \epsilon_0, M]\right\}. \tag{3.19}$$

If $N \geqslant 1$ is an integer, define $E_0' := E + \epsilon_0$ and $E_k' := E_0' + \frac{k}{N}(M - E_0')$. Then for every $\epsilon_2 > 0$ there exists N large enough such that

$$\left\{\rho_{E'}(\theta)\colon\ E' \in [E + \epsilon_0, M]\right\} \subset \bigcup_{k=1}^{N} \left[\rho_{E_k'}(\theta) - \epsilon_2, \rho_{E_k'}(\theta) + \epsilon_2\right]. \tag{3.20}$$

Proof. We know that ξ_E cannot exceed $\max|b|$ for all energies, hence energy conservation gives $\rho_{E'} \sim \sqrt{2E'}$ when $E' \to \infty$. This is how we get the existence of M. Now since ρ_E increases with E, in order to get (3.19) we only need to verify that uniformly in θ

$$\rho_{E_0'}(\theta) \leqslant \rho_E(\theta) + \epsilon/2.$$

The concavity in energy and the definition of ϵ_0 then gives

$$\rho_{E_0'}(\theta) - \rho_E(\theta) \leqslant (E_0' - E) \cdot \big(\partial_E \rho_E(\theta)\big) \leqslant \epsilon/4,$$

and (3.19) is proved.

As for (3.20), it is sufficient that uniformly in θ

$$\rho_{E_{k+1}'}(\theta) \leqslant \rho_{E_k'}(\theta) + \epsilon_2, \quad k \in \{0, 1, \ldots, N-1\}.$$

Since $\rho_{E_{k+1}'}(\theta) - \rho_{E_k'}(\theta) \leqslant Const \cdot (E_{k+1}' - E_k') = Const/N$, we can make this difference smaller than ϵ_2 and we are done. □

The next step is to rule out the possibility for $\rho(t)$ to be located in small intervals like those in (3.20).

For every $\epsilon_2 > 0$ consider

$$F_{+-} = F_{+-}^{-5\epsilon_2, -4\epsilon_2, 4\epsilon_2, 5\epsilon_2}.$$

Denote by F_{++} precisely that function of the type (3.4) whose derivative gives back the above F_{+-}.

Lemma 3.7. *Uniformly in $E' \in [E+\epsilon_0, M]$ there exists ϵ_2 small enough such that*

$$\int_1^\infty \frac{1}{t} F'_{++}\big(B_{E'}(t)\big)\,dt = \int_1^\infty \frac{1}{t} F_{+-}\big(B_{E'}(t)\big)\,dt < \infty. \tag{3.21}$$

Moreover,

$$\lim_{t\to\infty} F_{+-}^{-4\epsilon_2,-3\epsilon_2,3\epsilon_2,4\epsilon_2}\big(B_{E'}(t)\big) = 0. \tag{3.22}$$

Proof. We only prove (3.21), since the limit follows from the usual Cook type argument.

We introduce the bounded observable $\Phi(t) = -F_{++}(B_{E'}(t))$ and compute its time derivative:

$$\partial_t \Phi(t) = F_{+-}\big(B_{E'}(t)\big)\frac{\xi\cdot(\xi_{E'}(\theta(t)) - \xi(t))}{r}. \tag{3.23}$$

Since we know that $\xi(t) \geqslant d_1 > 0$ after some time, we can write using energy conservation that

$$\begin{aligned}\xi_{E'}\big(\theta(t)\big) - \xi(t) &= \frac{2E' - \rho_{E'}^2(\theta(t)) - 2E + \rho^2(t)}{\xi_{E'}(\theta(t)) + \xi(t)} \\ &= \frac{2(E'-E)}{\xi_{E'}(\theta(t)) + \xi(t)} + \frac{\rho(t) + \rho_{E'}(\theta(t))}{\xi_{E'}(\theta(t)) + \xi(t)} B_{E'}(t).\end{aligned} \tag{3.24}$$

Since $E' - E \geqslant \epsilon_0$, we get that for $t \geqslant T_1$ we have

$$F_{+-}\big(B_{E'}(t)\big)\cdot\big(\xi_{E'}\big(\theta(t)\big) - \xi(t)\big) \geqslant (C_1\epsilon_0 - C_2\epsilon_2)\cdot F_{+-}\big(B_{E'}(t)\big), \tag{3.25}$$

where C_1 and C_2 are positive constants not depending on ϵ_2. Moreover, they can be uniformly chosen if E' is restricted to compact sets. Hence if ϵ_2 is small enough, we have the desired positivity and we can integrate in (3.23) thus yielding (3.21). □

Now let us get back to the proposition. With ϵ_2 provided by the above lemma, construct the intervals from (3.20) and see that there exists a constant $c > 0$ such that

$$c\cdot\chi_{[\rho_E(\theta)+\epsilon/2,\sqrt{2E}]}(x) \leqslant \sum_{k=1}^N F_{+-}^{-2\epsilon_2,-\epsilon_2,\epsilon_2,2\epsilon_2}\big(x - \rho_{E'_k}(\theta)\big) \tag{3.26}$$

which leads to

$$F_+\big(B_E(t)\big) \leqslant \frac{1}{c}\cdot F_+\big(B_E(t)\big)\cdot\sum_{k=1}^N F_{+-}^{-2\epsilon_2,-\epsilon_2,\epsilon_2,2\epsilon_2}\big(B_{E'_k}(t)\big). \tag{3.27}$$

Then the use of (3.22) finishes the proof. □

Remark. An immediate consequence of the above proposition and energy conservation is a sharp localization for ξ, too. Namely, for every $\epsilon > 0$ there exists T large enough such that

$$\left|\xi(t) - \xi_E\big(\theta(t)\big)\right| \leqslant \epsilon, \quad t \geqslant T. \tag{3.28}$$

3.5. The eikonal and Hamilton–Jacobi equation

Let (ρ_E, ξ_E) be as in Corollary 2.7. Define $\tilde{S} : (E_d, \infty) \times (0, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ by

$$\tilde{S}(E, r, \theta) := r\rho_E(\theta). \tag{3.29}$$

By direct computation we can show that $\tilde{S}$ solves the eikonal equation

$$(\partial_r \tilde{S})^2 + \big(\partial_\theta \tilde{S}/r - b(\theta)\big)^2 = 2E.$$

Define $r(E, t, \theta) := t/(\partial_E \rho_E)(\theta) > 0$. We then have

Lemma 3.8. *For all $t > 0$, $E > E_d$ and $\theta \in \mathbb{R}$*

$$\partial_E r(E, t, \theta) > 0, \qquad \lim_{E\to\infty} r(E, t, \theta) = \infty, \qquad \lim_{E \searrow E_d} r(E, t, \theta) = 0. \tag{3.30}$$

Proof. Clearly, $r(E, t, \theta)$ increases with E because of the properties of ρ_E, see Proposition 2.6. The only nontrivial thing is proving that $\operatorname{Ran} r(\cdot, t, \theta) = (0, \infty)$. But this is equivalent to proving that

$$\lim_{E \searrow E_d} \partial_E \rho_E(\theta) = +\infty, \qquad \lim_{E\to\infty} \partial_E \rho_E(\theta) = 0. \tag{3.31}$$

For, let us introduce (2.25) into (2.22) and obtain

$$\partial_E \rho_E(\theta) = \int_0^\infty \frac{1}{\xi_E(\theta - \phi)} \exp\left\{-\int_0^\phi \frac{\rho_E}{\xi_E}(\theta - \phi')\, d\phi'\right\} d\phi. \tag{3.32}$$

Since $\xi_E(\theta)$ is continuous on both arguments and strictly positive, for any $E_1 > E_d$ we can find $c > 0$ such that

$$\inf_{E\in[E_d, E_1]} \inf_{\theta\in\mathbb{T}} \frac{1}{\xi_E(\theta)} \geqslant c.$$

Choose M to be arbitrarily large and see that for $E \in [E_d, E_1]$ we have

$$\partial_E \rho_E(\theta) \geqslant c \cdot \int_0^M \exp\left\{-\int_0^\phi \frac{\rho_E}{\xi_E}(\theta - \phi')\, d\phi'\right\} d\phi. \tag{3.33}$$

Hence we get (see (2.32))

$$\liminf_{E\searrow E_d} \partial_E \rho_E(\theta) \geqslant c \cdot \int_0^M \exp\left\{-\int_0^\phi \frac{\rho_d}{\xi_d}(\theta-\phi')\,d\phi'\right\} d\phi.$$

Because the integral in (2.32) is zero, we get the existence of another constant $c' > 0$ such that for every θ and ϕ

$$\exp\left\{-\int_0^\phi \frac{\rho_d}{\xi_d}(\theta-\phi')\,d\phi'\right\} \geqslant c',$$

and since M was arbitrary, we get $\lim_{E\searrow E_d} \partial_E \rho_E(\theta) = \infty$.

The other limit is proven using (3.32) again. When E becomes large, we know that ξ_E is trapped inside $[\min|b|, \max|b|]$ while $\rho_E \sim \sqrt{2E}$. It follows that the right-hand side in (3.32) can be bounded by a constant times $E^{-1/2}$ and we are done. □

Now consider the partial Legendre transform of $\tilde{S}$:

$$S:(0,\infty)\times(0,\infty)\times\mathbb{R}\mapsto\mathbb{R}, \quad S(t,r,\theta) := \sup_{E\geqslant E_d}\left\{\tilde{S}(E,r,\theta) - tE\right\}. \tag{3.34}$$

The previous lemma stated that

$$\operatorname{Ran} r(\cdot,t,\theta) = (0,\infty). \tag{3.35}$$

Since $\partial_E r(\cdot,t,\theta) > 0$, one can express E as a function $E(t,\cdot,\theta)$ of $r\in(0,\infty)$, obeying

$$E(t,\cdot,\theta):(0,\infty)\mapsto(E_d,\infty), \quad t = (\partial_E\tilde{S})\big(E(t,r,\theta),r,\theta\big). \tag{3.36}$$

With this definition, the function S in (3.34) becomes

$$S(t,r,\theta) = r\rho_{E(t,r,\theta)}(\theta) - tE(t,r,\theta). \tag{3.37}$$

This will be our Hamilton–Jacobi function. Using (2.8), (3.29) and (3.34) one readily verifies the following identities:

$$\partial_t S(t,r,\theta) = -E(t,r,\theta), \tag{3.38}$$

$$\partial_r S(t,r,\theta) = \rho_{E(t,r,\theta)}(\theta), \tag{3.39}$$

$$\partial_\theta S(t,r,\theta) = r\big[b(\theta) + \xi_{E(t,r,\theta)}(\theta)\big]. \tag{3.40}$$

Therefore, S is C^∞ on its domain and obeys the Hamilton–Jacobi equation

$$0 = \partial_t S + \frac{1}{2}\left(\frac{\partial_\theta S}{r} - b\right)^2 + \frac{1}{2}(\partial_r S)^2. \tag{3.41}$$

Proposition 3.9. *The function $S(t,r,\theta)$ is homogeneous of degree 1 in the variables (r,t), i.e. $S(t,r,\theta) = tS(1, r/t, \theta)$. Suppose $K \subset (0,\infty)$ is compact. Then for every $n \geqslant 0$ and $m \geqslant 0$ we can find a constant $C = C_{m,n,K}$ such that*

$$\sup_{\theta} \sup_{r/t \in K} \left| \partial_r^n \partial_\theta^m S(t,r,\theta) \right| \leqslant \frac{C}{t^{n-1}}, \quad t > 1. \tag{3.42}$$

Proof. Let us see why S has the stated homogeneity property. First, notice that $E(t,r,\theta)$ only depends on r/t and θ since it was obtained from the equation $t/r = \partial_E \rho_E(E,\theta)$ (see (3.36)). Second, apply (3.37).

In order to prove (3.42), we use the scaling property and the fact that $E(t,r,\theta)$ is restricted to some compact in (E_d,∞) only depending on K. Further details are omitted. □

3.6. A priori localization for $E(t, r(t), \theta(t))$

Consider again a trajectory corresponding to an energy $E > E_d$; we know that (3.18) and (3.28) hold and we now want a similar localization for the difference between $E(t, r(t), \theta(t))$ and E.

We again use propagation estimates which can and will be generalized later to quantum mechanics. For simplifying notation, replace $E(t, r(t), \theta(t))$ by E_t. We take ϵ in (3.18) and (3.28) very small. Define

$$D(t) := 1 - \frac{r(t)}{t} \cdot \partial_E \rho_E\bigl(\theta(t)\bigr). \tag{3.43}$$

Then we have

Proposition 3.10. *Let $F_{+-} = F_{+-}^{\epsilon_1/2, \epsilon_1, c, d}$, $\epsilon_1 > 0$, and consider the corresponding F_{++}. Then we have:*

$$\int_1^\infty \frac{1}{t} F'_{++}\bigl(D(t)\bigr)\, dt = \int_1^\infty \frac{1}{t} F_{+-}\bigl(D(t)\bigr)\, dt < \infty. \tag{3.44}$$

Moreover, there exists T large enough such that

$$\left| E + \partial_t S\bigl(t, r(t), \theta(t)\bigr) \right| \leqslant \epsilon_1, \quad t \geqslant T. \tag{3.45}$$

Proof. Differentiating D and using the identity $1 - \rho_E \partial_E \rho_E = \xi_E \partial_E \xi_E$ we obtain:

$$\frac{dD}{dt} = -\frac{D}{t} - \frac{1}{t} \cdot \bigl[\bigl(\rho - \rho_E\bigl(\theta(t)\bigr)\bigr) \cdot \partial_E \rho_E + \bigl(\xi - \xi_E\bigl(\theta(t)\bigr)\bigr) \cdot \partial_E \xi_E\bigr]. \tag{3.46}$$

Consider the bounded propagation observable $\Phi(t) = -F_{++}(D(t))$, differentiate it and use (3.46), (3.18) and (3.28):

$$\Phi' = -F_{+-}(D) \cdot \frac{dD}{dt} \geqslant \frac{1}{t} F_{+-}(D) \cdot (\epsilon_1/2 - Const \cdot \epsilon), \quad t \geqslant T.$$

Hence if ϵ is small enough, we have the desired positivity and we can integrate in order to get (3.44). Now using the usual procedure, we can prove that

$$\lim_{t\to\infty} F_{+-}\big(D(t)\big)=0$$

and because D is bounded and the above limit did not depend on the choice of c and d in F_{+-}, we can write

$$\lim_{t\to\infty} F_{+}^{\epsilon_1/2,\epsilon_1}\big(D(t)\big)=0.$$

Reasoning in a very similar way, we can also prove that for $a,b<-\epsilon_1$ and $F_{+-}=F_{+-}^{a,b,-\epsilon_1,-\epsilon_1/2}$ we have

$$\int_1^\infty \frac{1}{t}F_{+-}\big(D(t)\big)\,dt<\infty,\qquad \lim_{t\to\infty}F_{+-}\big(D(t)\big)=0,\qquad \lim_{t\to\infty}F_{-}^{-\epsilon_1,-\epsilon_1/2}\big(D(t)\big)=0. \tag{3.47}$$

We conclude that

$$\big|D(t)\big|\leqslant \epsilon_1,\quad t\geqslant T. \tag{3.48}$$

Using the equation defining $E(t,r(t),\theta(t))$ and (3.43) we obtain

$$t/r(t)=\partial_E\rho_{E_t}\big(\theta(t)\big),\qquad D=1-\frac{\partial_E\rho_E(\theta(t))}{\partial_E\rho_{E_t}(\theta(t))}. \tag{3.49}$$

Then

$$D(t)=\frac{\partial_E^2\rho_{x(t)}(\theta(t))}{\partial_E\rho_{E_t}(\theta(t))}\cdot(E_t-E), \tag{3.50}$$

where $x(t)$ is somewhere between E and E_t. Because of the minimal and maximal velocity cut-offs in (2.1) and (3.5), we know that E_t thus $x(t)$ varies in a time independent compact interval provided t is large enough. This means that we can write

$$|E_t-E|\leqslant Const\cdot\big|D(t)\big|,\quad t\geqslant T,$$

hence (3.45) follows and the proposition is proven. □

3.7. *The classical comparison dynamics*

We already know that given an energy $E>E_d$, the momenta of any real orbit corresponding to E tend to get closer and closer to the periodic ones. We would also like to have a similar property for the trajectory itself, but we first need a comparison orbit, which is constructed in what follows.

For every $\theta_1 \in \mathbb{R}$ and every $E > E_d$, denote by (see Lemma 3.8):

$$r_1 := r(E, 1, \theta_1) = \frac{1}{\partial_E \rho_E(\theta_1)} > 0, \tag{3.51}$$

Consider the following system of equations

$$\begin{cases} \dfrac{d\tilde{r}}{dt}(E, t) = \rho_E\big(\tilde{\theta}(E, t)\big), \\ \dfrac{d\tilde{\theta}}{dt}(E, t) = \dfrac{\xi_E(\tilde{\theta}(E, t))}{\tilde{r}(E, t)} \end{cases} \tag{3.52}$$

with $\tilde{r}(E, 1) = r_1$ and $\tilde{\theta}(E, 1) = \theta_1$. Notice that in rectangular coordinates this is equivalent to

$$\frac{d\tilde{\mathbf{x}}}{dt} = \nabla\tilde{S}(E, \tilde{\mathbf{x}}) - \mathbf{a}(\tilde{\mathbf{x}}), \qquad \tilde{\mathbf{x}}(1) = r_1\big(\cos(\theta_1), \sin(\theta_1)\big). \tag{3.53}$$

With reference to (3.36) we have the following.

Lemma 3.11. *For all $t \geqslant 1$ we have $E(t, \tilde{r}(E, t), \tilde{\theta}(E, t)) = E = E(1, r_1, \theta_1)$.*

Proof. It suffices to show the identity $\tilde{r}(E, t) = r(E, t, \tilde{\theta}(E, t))$.

Differentiating the equation $r(E, t, \tilde{\theta}(E, t)) = \frac{t}{\partial_E \rho_E(\tilde{\theta}(E,t))}$ with respect to t and using (2.23) and (2.24) one obtains

$$\begin{aligned} \frac{dr}{dt}\big(E, t, \tilde{\theta}(E, t)\big) &= \frac{1}{\partial_E \rho_E(\tilde{\theta}(E, t))} - \frac{t\partial_E(\partial_\theta \rho_E)}{[\partial_E \rho_E]^2}\big(E, \tilde{\theta}(E, t)\big) \cdot \frac{d\tilde{\theta}}{dt}(E, t) \\ &= \frac{1}{\partial_E \rho_E}\left[1 - \xi_E(\partial_E \xi_E)\frac{r}{\tilde{r}}\right] = \rho_E - \frac{\xi_E(\partial_E \xi_E)(r - \tilde{r})}{(\partial_E \rho_E)\tilde{r}}, \end{aligned}$$

or using (3.52) again

$$\begin{aligned} &\frac{d}{dt}\big[r\big(E, t, \tilde{\theta}(E, t)\big) - \tilde{r}(E, t)\big] = -\frac{\xi_E(\partial_E \xi_E)}{(\partial_E \rho_E)\tilde{r}}[r - \tilde{r}], \\ &r\big(E, 1, \tilde{\theta}(E, 1)\big) - \tilde{r}(E, 1) = r_1 - r_1 = 0. \end{aligned}$$

Solving this initial value problem shows that the difference must be zero at all times, therefore the identity follows. □

We define the mapping:

$$\mathbf{v}_t : \mathbb{R}_+ \times \mathbb{T} \mapsto \mathbb{R}_+ \times \mathbb{T}, \quad \mathbf{v}_t(r_1, \theta_1) := \big(\tilde{r}\big(E(1, r_1, \theta_1), t\big), \tilde{\theta}\big(E(1, r_1, \theta_1), t\big)\big). \tag{3.54}$$

Notice first that the system of equations defining $\mathbf{v}_t$ in (3.52) can be more compactly written as (see (3.38)–(3.40) and Lemma 3.11)

$$\frac{d\mathbf{v}_t}{dt} = \left((\partial_r S)(t,\mathbf{v}_t), \left(\frac{1}{r^2}\partial_\theta S - \frac{b}{r}\right)(t,\mathbf{v}_t)\right) = \mathbf{X}(t,\mathbf{v}_t),$$
$$\mathbf{v}_1 = (r_1,\theta_1) \in \mathbb{R}_+ \times \mathbb{T}, \tag{3.55}$$

where we introduced

$$\mathbf{X}(t,\mathbf{x}) = \left((\partial_r S)(t,\mathbf{x}), \left(\frac{1}{r^2}\partial_\theta S - \frac{b}{r}\right)(t,\mathbf{x})\right), \quad \mathbf{x} = (r,\theta).$$

Then $\mathbf{v}_t$ admits an inverse denoted by $\mathbf{w}_t$:

$$\mathbf{w}_t : \mathbb{R}_+ \times \mathbb{T} \mapsto \mathbb{R}_+ \times \mathbb{T}, \quad \mathbf{w}_t(r,\theta) = \big(r_1(t,r,\theta), \theta_1(t,r,\theta)\big). \tag{3.56}$$

If we denote by $\mathbf{u}_\tau(r,\theta)$ the solution to the equation

$$\frac{d\mathbf{u}_\tau}{d\tau} = -\mathbf{X}(t-\tau+1,\mathbf{u}_\tau), \qquad \mathbf{u}_1 = (r,\theta) \in \mathbb{R}_+ \times \mathbb{T}, \quad \tau \in (1,t), \tag{3.57}$$

we have $\mathbf{w}_t = \mathbf{u}_t$.

We point out that $\mathbf{v}_t(r_1,\theta_1)$ also solves two of the Hamilton equations (corresponding to the configuration space) for the symbol h_a we introduced in (1.13). The other two equations give a solution we denoted by $\mathbf{q}_{a,t} = (\rho_a(t), l_a(t))$ in Section 1.2, and we defined the total direct flow to be $\mathbf{V}_{a,t} = (\mathbf{v}_t; \mathbf{q}_{a,t})$ corresponding to a set of initial data $(r_1,\theta_1; \rho_a(1), l_a(1))$. The inverse flow denoted by $\mathbf{W}_{a,t}$ has as the "configuration space part" the flow $\mathbf{w}_t$ described above.

In order to get an idea about how $\mathbf{w}_t$ depends on the initial conditions, we first look at the case when $b(\theta) = b < 0$ is a negative constant. We then know that $E_d = E_c = E_b$ and the periodic solution is $\xi_E(\theta) = -b > 0$, which from (3.29) and (3.34) leads to

$$\tilde{S}(E,r,\theta) = r\sqrt{2E - b^2}, \qquad S(t,r,\theta) = \frac{r^2}{2t} - \frac{b^2 t}{2}. \tag{3.58}$$

The energy function $E(t,r,\theta)$ in (3.36) is

$$E(t,r,\theta) = \frac{r^2}{2t^2} + \frac{b^2}{2}, \qquad E(1,r_1,\theta_1) = \frac{r_1^2}{2} + \frac{b^2}{2}. \tag{3.59}$$

We now can explicitly solve (3.52) obtaining

$$r(t,r_1,\theta_1) = tr_1, \qquad \theta(t,r_1,\theta_1) = -\frac{b}{r_1}\ln(t) + \theta_1, \tag{3.60}$$

while the inverse flow is

$$r_1(t,r,\theta) = \frac{r}{t}, \qquad \theta_1(t,r,\theta) = \theta + b\frac{t}{r}\ln(t) \tag{3.61}$$

or in a more compact form

$$\mathbf{w}_t(r,\theta) = \left(\frac{r}{t}, \theta + b\frac{t}{r}\ln(t)\right). \tag{3.62}$$

The Jacobian matrix (with respect to the initial conditions) is then

$$\mathbf{w}_t'(r,\theta) = \begin{pmatrix} \frac{1}{t} & 0 \\ -b\frac{t}{r^2}\ln(t) & 1 \end{pmatrix}. \tag{3.63}$$

The remaining part of this subsection is devoted to showing a similar behavior for the inverse flow even when $b(\theta)$ is not constant. That is, we look again at the Jacobian matrix $\mathbf{w}_t'(r,\theta)$ and prove that given any small $\delta > 0$, then by performing a derivative with respect to r we introduce a decay of order $t^{\delta-1}$ while differentiating with respect to θ we introduce a growth of at most t^{δ}. These estimates (and extensions to higher order derivatives) will play an important role in the rest of the paper.

We first study the direct flow (3.52), since we intend to use the inverse function theorem. An immediate consequence of Lemma 3.11 is

$$\tilde{r}\big(E(1,r_1,\theta_1),t\big) = \frac{t}{\partial_E \rho_{E(1,r_1,\theta_1)}(\tilde{\theta}(E(1,r_1,\theta_1),t))}. \tag{3.64}$$

Moreover, if we impose the condition that $E(1,r_1,\theta_1) \in K$, where K is a compact subset of (E_d,∞), then there exist two positive constants $C_1(K) < C_2(K)$ independent of θ_1 such that

$$t \cdot C_1 \leqslant \tilde{r}\big(E(1,r_1,\theta_1),t\big) \leqslant t \cdot C_2, \quad t \geqslant 1. \tag{3.65}$$

Knowing that the periodic solution ξ_E is always trapped between $\min|b|$ and $\max|b|$, using (3.65) and (3.52) we get that if $E(1,r_1,\theta_1) \in K$ then there exist two other positive constants $C_3(K) < C_4(K)$ independent of θ_1 such that

$$C_3 \cdot \ln(t) \leqslant \tilde{\theta}(t,r_1,\theta_1) - \theta_1 \leqslant C_4 \cdot \ln(t). \tag{3.66}$$

In a similar way, imposing the condition $E(t,r,\theta) \in K$ we have the positive constants $C < C'$ only depending on K such that

$$\begin{gathered} C \cdot \frac{r}{t} \leqslant r_1(t,r,\theta) \leqslant C' \cdot \frac{r}{t}, \\ -C' \cdot \ln(t) \leqslant \theta_1(t,r,\theta) - \theta \leqslant -C \cdot \ln(t). \end{gathered} \tag{3.67}$$

These estimates give some information about the location of r_1 and θ_1 as functions of t, r and θ. It remains to study how their derivatives behave with respect to r and θ.

3.7.1. Dependence of the direct flow on r_1 and θ_1

We start by looking at the Jacobian determinant for the direct flow

$$J_{\mathbf{v}_t} = J_{\mathbf{v}_t}(t,r_1,\theta_1) = \begin{vmatrix} \partial_{r_1}\tilde{r} & \partial_{\theta_1}\tilde{r} \\ \partial_{r_1}\tilde{\theta} & \partial_{\theta_1}\tilde{\theta} \end{vmatrix}. \tag{3.68}$$

We shall prove that it grows precisely like t. With the notations from (3.55) we have $J_{\mathbf{v}_t} = \det(\frac{\partial \mathbf{v}_t}{\partial \mathbf{v}_1})$ which according to general results obeys the equation (we denote by $E_t := E(t,\mathbf{v}_t) = -\partial_t S(t,\mathbf{v}_t)$):

$$\frac{d}{dt}J_{\mathbf{v}_t}=(\nabla_{\mathbf{x}}\cdot\mathbf{X})(t,\mathbf{v}_t)J_{\mathbf{v}_t}=\left[\partial_r^2 S(t,\mathbf{v}_t)+\frac{1}{\tilde r}\left(\frac{\partial_\theta^2 S}{\tilde r}-b'(\tilde\theta)\right)\right]J_{\mathbf{v}_t}$$
$$=\left[\partial_r^2 S(t,\mathbf{v}_t)+\frac{1}{\tilde r}\partial_{\tilde\theta}\big[\xi_{E_t}(\tilde\theta)\big]\right]J_{\mathbf{v}_t}. \tag{3.69}$$

We integrate (3.69) and obtain

$$J_{\mathbf{v}_t}=\exp\left\{\int_1^t\left[\partial_r^2 S(\tau,\mathbf{v}_\tau)+\frac{1}{\tilde r}\left(\frac{\partial_\theta^2 S}{\tilde r}-b'(\tilde\theta)\right)\right]d\tau\right\}$$
$$=\exp\left\{\int_1^t\left[\partial_r^2 S(\tau,\mathbf{v}_\tau)+\frac{1}{\tilde r}\big((\partial_E\xi_{E_t})(\tilde\theta)\cdot(\partial_\theta E)(t,\tilde r,\tilde\theta)\big)\right]d\tau\right\}$$
$$\times\exp\left\{\int_1^t\frac{1}{\tilde r}(\partial_\theta\xi_{E_t})(\tilde\theta)\,d\tau\right\}. \tag{3.70}$$

We may simplify the second exponential realizing that one can express t as $t(\tilde\theta,r_1,\theta_1)$ by inverting the function $\tilde\theta(t,r_1,\theta_1)$ given by the direct flow. Consequently we may introduce

$$\tilde{\tilde r}(\tilde\theta,r_1,\theta_1)=\tilde r\big(t(\tilde\theta,r_1,\theta_1),r_1,\theta_1\big), \tag{3.71}$$

and (3.52) reduces to (denote by $E_1=E(1,r_1,\theta_1)$)

$$\frac{\partial\tilde{\tilde r}}{\partial\theta}(\theta,r_1,\theta_1)=\frac{\rho_{E_1}(\theta)}{\xi_{E_1}(\theta)}\cdot\tilde{\tilde r}(\theta,r_1,\theta_1) \tag{3.72}$$

which in turn leads to

$$\tilde{\tilde r}(\theta,r_1,\theta_1)=r_1\exp\left\{\int_{\theta_1}^{\theta}\frac{\rho_{E_1}(\phi)}{\xi_{E_1}(\phi)}\,d\phi\right\}. \tag{3.73}$$

With the same change of variables and keeping in mind that $E_t=E_1$ (see Lemma 3.11):

$$\int_1^t\frac{1}{\tilde r}(\partial_\theta\xi_{E_t})(\tilde\theta)\,d\tau=\int_{\theta_1}^{\tilde\theta}\frac{1}{\xi_{E_1}}(\partial_\phi\xi_{E_1})(\phi)\,d\phi=\ln\left[\frac{\xi_{E_1}(\tilde\theta)}{\xi_{E_1}(\theta_1)}\right], \tag{3.74}$$

which introduced in (3.70) yields

$$J_{\mathbf{v}_t}=\frac{\xi_{E_1}(\tilde\theta)}{\xi_{E_1}(\theta_1)}\exp\left\{\int_1^t\left(\partial_r^2 S(\tau,\mathbf{v}_\tau)+\frac{1}{\tilde r}\big(\partial_E\xi_{E_1}(\tilde\theta)\cdot(\partial_\theta E)(\tau,\mathbf{v}_\tau)\big)\right)d\tau\right\}. \tag{3.75}$$

We now treat the integral in (3.75). We compute using (3.39), (3.52) and the identity $\partial_E \rho_{E(t,r,\theta)}(\theta) = t/r$:

$$\begin{aligned}
&\partial_r^2 S(\tau, \mathbf{v}_\tau) + \frac{1}{\tilde{r}}\bigl(\partial_E \xi_{E_\tau}(\tilde{\theta}) \cdot (\partial_\theta E)(\tau, \mathbf{v}_\tau)\bigr) - \frac{\rho_{E_\tau}(\tilde{\theta})}{\tilde{r}} \\
&\quad = -\frac{d\tilde{\theta}}{d\tau}\left[\frac{1}{\xi_{E_\tau}\partial_E^2 \rho_{E_\tau}}\bigl\{(\partial_E \rho_{E_\tau})^2 + (\partial_E \xi_{E_\tau})^2\bigr\} + \frac{\rho_{E_\tau}}{\xi_{E_\tau}}\right](\tilde{\theta}) \\
&\quad = -\frac{d\tilde{\theta}}{d\tau} \cdot \partial_\theta \ln\bigl|\partial_E^2 \rho_{E_\tau}\bigr|,
\end{aligned} \tag{3.76}$$

where the last equality comes from (2.27). We conclude that

$$\int_0^{2\pi} \left[\frac{1}{\xi_{E_\tau}\partial_E^2 \rho_{E_\tau}}\bigl\{(\partial_E \rho_{E_\tau})^2 + (\partial_E \xi_{E_\tau})^2\bigr\} + \frac{\rho_{E_\tau}}{\xi_{E_\tau}}\right] d\tilde{\theta} = 0.$$

We also compute (using (3.52) again)

$$\left\{\int_1^t \frac{\rho_{E_\tau}(\tilde{\theta}(\tau))}{\tilde{r}(\tau)}\right\} d\tau = \ln \frac{\tilde{r}(t)}{\tilde{r}(1)}.$$

Consequently we conclude that there exists $C = C(r_1, \theta_1) > 0$ such that (see also (3.65))

$$C^{-1} \cdot t \leqslant J_{\mathbf{v}_t}(t, r_1, \theta_1) \leqslant C \cdot t. \tag{3.77}$$

We now estimate the individual derivatives of $\tilde{r}(t, r_1, \theta_1)$ and $\tilde{\theta}(t, r_1, \theta_1)$ with respect to r_1 and θ_1. In conjunction with (3.77) those will be useful for estimating derivatives of the inverse flow.

Using (3.71) in the second equation of (3.52) after separating variables and integrating we get (remember the abbreviation $E_1 = E(1, r_1, \theta_1)$)

$$\int_{\theta_1}^{\tilde{\theta}(t, r_1, \theta_1)} \frac{\tilde{\tilde{r}}(\phi, r_1, \theta_1)}{\xi_{E_1}(\phi)}\, d\phi = t - 1. \tag{3.78}$$

Differentiating with respect to r_1 we obtain

$$\frac{\partial \tilde{\theta}}{\partial r_1} \cdot \frac{\tilde{\tilde{r}}(\tilde{\theta}, r_1, \theta_1)}{\xi_{E_1}(\tilde{\theta})} = -\int_{\theta_1}^{\tilde{\theta}} \partial_{r_1}\left[\frac{\tilde{\tilde{r}}(\phi, r_1, \theta_1)}{\xi_{E_1}(\phi)}\right] d\phi. \tag{3.79}$$

Let us prove a very rough estimate of the above derivative (here and below the constants are uniformly bounded on compact energy intervals $K \subset (E_d, \infty)$):

$$\left|\frac{\partial \tilde{\theta}}{\partial r_1}\right| \leqslant Const \cdot \bigl[1 + (\tilde{\theta} - \theta_1)^2\bigr] \leqslant Const \cdot \bigl[1 + \ln^2(t)\bigr], \quad t \geqslant 1. \tag{3.80}$$

Indeed, differentiating the integrand on the right-hand side of (3.79) with respect to r_1 we get (see (3.73)):

$$\frac{\tilde{\tilde{r}}(\phi, r_1, \theta_1)}{\xi_{E_1}(\phi)}\left\{1/r_1 + \int_{\theta_1}^{\phi} \partial_{r_1} \frac{\rho_{E_1}(\eta)}{\xi_{E_1}(\eta)}\, d\eta\right\} - \frac{\tilde{\tilde{r}}(\phi, r_1, \theta_1)}{\xi_{E_1}^2(\phi)} \cdot (\partial_E \xi_{E_1})(\phi) \cdot (\partial_r E)(1, r_1, \theta_1). \tag{3.81}$$

This together with the fact that $\tilde{\tilde{r}}$ increases with ϕ over a period, leads us to

$$\left|\partial_{r_1}\left[\frac{\tilde{\tilde{r}}(\phi, r_1, \theta_1)}{\xi_{E_1}(\phi)}\right]\right| \leqslant Const \cdot \tilde{\tilde{r}}(\tilde{\theta}, r_1, \theta_1)\big[1 + (\phi - \theta_1)\big],$$

which introduced in (3.79) and using (3.66) leads to the desired estimate in (3.80).

One can iterate this procedure by performing higher order derivatives with respect to θ_1 and r_1 in (3.79) and isolating the term having all the derivatives acting on $\tilde{\theta}$. Hence reasoning by induction (we skip the details), one obtains logarithmic type bounds for all derivatives of $\tilde{\theta}$:

$$\left|\frac{\partial^i \tilde{\theta}}{\partial r_1^j \partial \theta_1^k}\right|(t, r_1, \theta_1) \leqslant Const \cdot \big[1 + (\tilde{\theta} - \theta_1)^{i+1}\big] \leqslant Const \cdot \big[1 + \ln^{i+1}(t)\big], \quad t \geqslant 1, \tag{3.82}$$

for $i, j, k \in \{0, 1, 2, 3\}$, $i > 0$, $j + k = i$.

Let us now investigate the derivatives of $\tilde{r}$. Using (3.71), (3.73), (3.66), (3.82) and reasoning by induction one infers:

$$\left|\frac{\partial^i \tilde{r}}{\partial r_1^j \partial \theta_1^k}\right|(t, r_1, \theta_1) \leqslant Const \cdot t\big[1 + \ln^{i+1}(t)\big], \quad t \geqslant 1, \tag{3.83}$$

for $i, j, k \in \{0, 1, 2, 3\}$, $i > 0$, $j + k = i$.

3.7.2. Dependence of the inverse flow on r and θ

The following lemma (and similar bounds for some higher order derivatives; see the remark after the lemma) will be important in the next sections. It gives a precise meaning to the statement saying that when one differentiates the inverse flow with respect to r one gains a decay of almost t^{-1} while differentiating with respect to θ one gets back something almost bounded.

Given a compact energy interval $K \subset (E_d, \infty)$ we introduce for $t \geqslant 1$ the sets

$$\mathcal{A}_t(K) = \big\{(r, \theta) \in (0, \infty) \times \mathbb{T}\colon\ E(t, r, \theta) \in K\big\}. \tag{3.84}$$

We recall, cf. (3.56) and (3.57),

$$E(t, r, \theta) = E\big(1, \mathbf{w}_t(r, \theta)\big), \quad \mathbf{w}_t = \big(r_1(t, r, \theta), \theta_1(t, r, \theta)\big), \tag{3.85}$$

$$r = \tilde{r}\big(t, r_1(t, r, \theta), \theta_1(t, r, \theta)\big) \tag{3.86}$$

and

$$\theta = \tilde{\theta}\big(t, r_1(t, r, \theta), \theta_1(t, r, \theta)\big). \tag{3.87}$$

Lemma 3.12. *Fix $\delta \in (0,1)$. Then for any compact interval $K \subset (E_d, \infty)$ and any initial data for the inverse flow located in $\mathcal{A}_t(K)$ we have*

$$\left|\frac{\partial \theta_1}{\partial r}\right| = \left|-J_{\mathbf{V}_t}^{-1}\frac{\partial \tilde{\theta}}{\partial r_1}\right| \leqslant C \cdot t^{\delta-1}, \qquad \left|\frac{\partial \theta_1}{\partial \theta}\right| = \left|J_{\mathbf{V}_t}^{-1}\frac{\partial \tilde{r}}{\partial r_1}\right| \leqslant C \cdot t^{\delta}, \tag{3.88}$$

and

$$\left|\frac{\partial r_1}{\partial r}\right| = \left|J_{\mathbf{V}_t}^{-1}\frac{\partial \tilde{\theta}}{\partial \theta_1}\right| \leqslant C \cdot t^{\delta-1}, \qquad \left|\frac{\partial r_1}{\partial \theta}\right| = \left|-J_{\mathbf{V}_t}^{-1}\frac{\partial \tilde{r}}{\partial \theta_1}\right| \leqslant C \cdot t^{\delta}, \tag{3.89}$$

where $C > 0$ only depends on K and δ.

Proof. The result comes after a straightforward application of the inverse function theorem and by using (3.77), (3.82), (3.83); we replace the logarithms by $t^{\delta/2}$. □

We will later on need similar bounds for some higher-order derivatives; those are obtained along the same line, that is by applying the inverse function theorem in conjunction with the bounds obtained for derivatives of the direct flow.

3.8. Investigating classical asymptotic completeness

We now are in a position to prove (1.15), which as we have already stated is only a first step in showing classical asymptotic completeness.

Proposition 3.13. *Consider an arbitrary classical orbit defined for all positive times*:

$$\mathbf{V}_t = \big(r(t), \theta(t); \rho(t), l(t)\big)$$

corresponding to an energy $E > E_d$. For such an orbit the asymptotic radius and angle defined as entries of the limit (1.15)*, and denoted r_+ and θ_+, respectively, exist. Moreover the energy of the orbit is related to the asymptotic quantities by $E = -\partial_t S(1, r_+, \theta_+)$.*

Proof. We start by fixing further notation. Denote by (see (1.5) and (1.13))

$$\begin{aligned} \mathbf{F}(r,\theta,\rho,l) &= (\partial_\rho h, \partial_l h, -\partial_r h, -\partial_\theta h), \\ \mathbf{F}_a(r,\theta,\rho,l) &= (\partial_\rho h_a, \partial_l h_a, -\partial_r h_a, -\partial_\theta h_a). \end{aligned} \tag{3.90}$$

We now explicitly need the dependence on the initial conditions. For example, $\mathbf{V}_t(\mathbf{x})$ means the particular orbit which equals $\mathbf{x}$ at $t = 1$, i.e. $\mathbf{V}_1 = \mathbf{x} \in \mathbb{R}^4$.

The Hamilton equations for the true and comparison dynamics may be written in the form

$$\frac{d\mathbf{V}_t}{dt}(\mathbf{x}) = \mathbf{F}\big(\mathbf{V}_t(\mathbf{x})\big), \qquad \frac{d\mathbf{V}_{a,t}}{dt}(\mathbf{x}) = \mathbf{F}_a\big(\mathbf{V}_{a,t}(\mathbf{x})\big). \tag{3.91}$$

Since $\mathbf{W}_{a,t}$ denotes the inverse for $\mathbf{V}_{a,t}$, we have

$$0 = \frac{d}{dt}\big(\mathbf{W}_{a,t}\big(\mathbf{V}_{a,t}(\mathbf{x})\big)\big) = \frac{d\mathbf{W}_{a,t}}{dt}\big(\mathbf{V}_{a,t}(\mathbf{x})\big) + \big[\mathbf{W}'_{a,t}\big(\mathbf{V}_{a,t}(\mathbf{x})\big)\big]\mathbf{F}_a\big(\mathbf{V}_{a,t}(\mathbf{x})\big), \tag{3.92}$$

where $\mathbf{W}'_{a,t}$ denotes the total derivative of the vector valued function $\mathbf{y} \mapsto \mathbf{W}_{a,t}(\mathbf{y})$.

We shall prove the existence of the limit (1.15) by a Cook type argument, that is by showing that the time derivative is in $L^1((1,\infty))$. Hence we choose an initial condition $\mathbf{y}$ for the true orbit at energy $E > E_d$ and compute

$$\frac{d}{dt}\big(\Pi \mathbf{W}_{a,t}\big(\mathbf{V}_t(\mathbf{y})\big)\big) = \Pi \frac{d\mathbf{W}_{a,t}}{dt}\big(\mathbf{V}_t(\mathbf{y})\big) + \Pi\big[\mathbf{W}'_{a,t}\big(\mathbf{V}_t(\mathbf{y})\big)\big]\mathbf{F}\big(\mathbf{V}_t(\mathbf{y})\big). \tag{3.93}$$

Using (3.92) with $\mathbf{x} = \mathbf{W}_{a,t} \circ \mathbf{V}_t(\mathbf{y})$ in (3.93) we get

$$\frac{d}{dt}\big(\Pi \mathbf{W}_{a,t}\big(\mathbf{V}_t(\mathbf{y})\big)\big) = \Pi\big[\mathbf{W}'_{a,t}\big(\mathbf{V}_t(\mathbf{y})\big)\big]\big\{\mathbf{F}\big(\mathbf{V}_t(\mathbf{y})\big) - \mathbf{F}_a\big(\mathbf{V}_t(\mathbf{y})\big)\big\}. \tag{3.94}$$

An important feature of the 4×4 Jacobian matrix $\mathbf{W}'_{a,t}(\mathbf{x})$ is that it looks like ($\mathbf{x} = (x_1, x_2, x_3, x_4)$)

$$\mathbf{W}'_{a,t}(\mathbf{x}) = \begin{pmatrix} \mathbf{w}'_t(x_1, x_2) & \mathbf{0}_2 \\ A_{21}(\mathbf{x}) & A_{22}(\mathbf{x}) \end{pmatrix},$$

where $\mathbf{0}_2$ is the 2×2 zero matrix. This is a consequence of the decoupling of the equations for the comparison evolution.

With the notation $E_t = E(t, r(t), \theta(t))$ we introduce

$$\gamma_1(t) := \rho(t) - (\partial_r S)\big(t, r(t), \theta(t)\big) = \rho(t) - \rho_{E_t}\big(\theta(t)\big), \quad t \geqslant T, \tag{3.95}$$

and

$$\gamma_2(t) := \frac{l(t) - (\partial_\theta S)(t, r(t), \theta(t))}{r(t)} = \xi(t) - \xi_{E_t}\big(\theta(t)\big), \quad t \geqslant T, \tag{3.96}$$

where T is sufficiently large such that both the maximal and minimal velocity estimates hold (see (2.1) and (3.5)).

Hence (3.94) reads

$$\frac{d}{dt}\big(\Pi \mathbf{W}_{a,t}\big(\mathbf{V}_t(\mathbf{y})\big)\big) = \big[\mathbf{w}'_t\big(r(t), \theta(t)\big)\big]\left(\gamma_1(t), \frac{\gamma_2(t)}{r(t)}\right). \tag{3.97}$$

Proving that the right-hand side of (3.97) is in L^1 is what we do in the rest of this subsection. First, let us see that we can use Lemma 3.12; fix $\epsilon > 0$ small enough such that $K := [E - 2\epsilon, E + 2\epsilon] \subset (E_d, \infty)$. Then (3.45) implies that $E_t = E(t, r(t), \theta(t)) \in K$ if t is large enough and hence $(r(t), \theta(t)) \in \mathcal{A}_t(K)$. It means that the estimates in (3.88) and (3.89) hold for $t \geqslant T$, where T is large enough, therefore showing the integrability of the right-hand side of (3.97) is reduced to proving the following result.

Lemma 3.14. *Fix $\delta \in (0, 1)$. Then there exist a sufficiently large T and a positive constant C such that*

$$\max\big\{\big|\gamma_1(t)\big|, \big|\gamma_2(t)\big|\big\} \leqslant C \cdot t^{\delta - 1}, \quad t \geqslant T.$$

Proof. To motivate our somewhat complicated analysis we first replace γ_1 and γ_2 by the fixed energy quantities

$$g_1(t) := \rho(t) - \rho_E\big(\theta(t)\big), \qquad g_2(t) := \xi(t) - \xi_E\big(\theta(t)\big), \tag{3.98}$$

and give an easy proof that

$$g_j(t) = \mathcal{O}\big(t^{-1}\big). \tag{3.99}$$

Let $L := g_1^2 + g_2^2$ and note that energy conservation gives an "almost linear dependence" for g_1 and g_2. Namely from the equality

$$2E = \big[g_1(t) + \rho_E\big(\theta(t)\big)\big]^2 + \big[g_2(t) + \xi_E\big(\theta(t)\big)\big]^2$$

we obtain

$$g_1(t) \cdot \rho_E\big(\theta(t)\big) + g_2(t) \cdot \xi_E\big(\theta(t)\big) = -L/2. \tag{3.100}$$

Using the Hamilton equations and (2.6) it follows that

$$\begin{aligned} dg_1/dt &= (\xi/r) \cdot g_2, \\ dg_2/dt &= -(b+\xi)/r \cdot g_1 + (b\rho)/(\xi r) \cdot g_2 + \mathcal{O}(L/r), \end{aligned} \tag{3.101}$$

and thus (also using (3.100))

$$dL/dt = (2b\rho)/(\xi r) \cdot L + \mathcal{O}\big(L^{3/2}/r\big). \tag{3.102}$$

Note that $(d/dt)\ln(\xi r)^2 = -2b\rho/(\xi r)$ so that for t large

$$(d/dt)\big[(\xi r)^2 \cdot L\big] = \big[(\xi r)^2 \cdot L/t\big] \cdot \mathcal{O}\big(L^{1/2}\big), \tag{3.103}$$

where we have used the minimal velocity bound $r/t \geqslant c > 0$ for large t. According to (3.18) and (3.28), $L(t) \to 0$ as $t \to \infty$, so integrating (3.103) gives $L(t) = \mathcal{O}(t^{-2+\delta})$. Then integrating (3.103) again with this new information gives (3.99).

To bridge the gap between (ρ_E, ξ_E) and (ρ_{E_t}, ξ_{E_t}) (remember our notation $E_t = -\partial_t S(t, r(t), \theta(t))$), we make a Taylor expansion and keep only terms up to the first order in $E - E_t$. Thus we define

$$\begin{aligned} \hat\gamma_1(t) &:= \rho(t) - \big\{\rho_{E_t}\big(\theta(t)\big) + \partial_E \rho_{E_t}\big(\theta(t)\big) \cdot (E - E_t)\big\}, \\ \hat\gamma_2(t) &:= \xi(t) - \big\{\xi_{E_t}\big(\theta(t)\big) + \partial_E \xi_{E_t}\big(\theta(t)\big) \cdot (E - E_t)\big\}. \end{aligned} \tag{3.104}$$

We will also need to show that $E - E_t$ is small so we introduce a third γ and a third $\hat\gamma$:

$$\gamma_3(t) = \hat\gamma_3(t) := E + \partial_t S\big(t, r(t), \theta(t)\big) = E - E_t. \tag{3.105}$$

We can then rewrite (3.104) as

$$\hat{\gamma}_1(t) = \gamma_1(t) - (\partial_E \rho_{E_t})\big(\theta(t)\big) \cdot \gamma_3(t), \qquad \hat{\gamma}_2(t) = \gamma_2 - (\partial_E \xi_{E_t})\big(\theta(t)\big) \cdot \gamma_3(t). \tag{3.106}$$

We have previously shown some a priori smallness in (3.18), (3.28) and (3.45) for these quantities but just for the record we write again that for every $\epsilon > 0$, there exists T large enough such that

$$\max\big\{\big|\hat{\gamma}_j(t)\big|, \big|\gamma_k(t)\big|:\ j,k \in \{1,2,3\}\big\} \leqslant \epsilon, \quad t \geqslant T. \tag{3.107}$$

We split the proof of Lemma 3.14 into several pieces.

I. We start with a few constraints we have on the γ's coming from energy conservation. The first one is

$$\begin{aligned} 2E &= \rho^2(t) + \xi^2(t) = (\gamma_1 + \rho_{E_t})^2 + (\gamma_2 + \xi_{E_t})^2 \\ &= 2E_t + 2\gamma_1 \cdot \rho_{E_t} + 2\gamma_2 \cdot \xi_{E_t} + \big(\gamma_1^2 + \gamma_2^2\big) \end{aligned} \tag{3.108}$$

or equivalently

$$2\gamma_3(t) = 2\gamma_1(t) \cdot \rho_{E_t}\big(\theta(t)\big) + 2\gamma_2(t) \cdot \xi_{E_t}\big(\theta(t)\big) + \gamma_1^2(t) + \gamma_2^2(t). \tag{3.109}$$

Rewriting (3.109) with $\hat{\gamma}$'s we get a linear dependence (up to quadratic terms) between $\hat{\gamma}_1$ and $\hat{\gamma}_2$, similar to (3.100); we again employ the identity $1 - \rho_E \partial_E \rho_E = \xi_E \partial_E \xi_E$:

$$2\hat{\gamma}_1(t) \cdot \rho_{E_t}\big(\theta(t)\big) + 2\hat{\gamma}_2(t) \cdot \xi_{E_t}\big(\theta(t)\big) = \mathcal{O}\big(\hat{\gamma}^2\big). \tag{3.110}$$

II. We continue with the time derivative of $\hat{\gamma}_3$. The key equation is

$$t/r(t) = \partial_E \rho_{E_t}\big(\theta(t)\big). \tag{3.111}$$

Performing the derivative we get

$$1/r - \big(t/r^2\big)\rho = -\big(\partial_E^2 \rho_{E_t}\big) \cdot (\partial_t \hat{\gamma}_3) + (\partial_E \xi_{E_t}/r) \cdot \xi \tag{3.112}$$

and using again $1 - \rho_E \partial_E \rho_E = \xi_E \partial_E \xi_E$ we are led to

$$\partial_t \hat{\gamma}_3 = \partial_t \gamma_3 = \frac{1}{r \partial_E^2 \rho_{E_t}} \big\{(\partial_E \rho_{E_t}) \cdot \gamma_1 + (\partial_E \xi_{E_t}) \cdot \gamma_2\big\}, \tag{3.113}$$

and finally, using (3.111) in order to get rid of $r(t)$ on the right-hand side:

$$\partial_t \hat{\gamma}_3 = \frac{\partial_E \rho_{E_t}}{t \partial_E^2 \rho_{E_t}} \big\{(\partial_E \rho_{E_t}) \cdot \hat{\gamma}_1 + (\partial_E \xi_{E_t}) \cdot \hat{\gamma}_2\big\} + \frac{\partial_E \rho_{E_t}}{t \partial_E^2 \rho_{E_t}} \big[(\partial_E \rho_{E_t})^2 + (\partial_E \xi_{E_t})^2\big] \cdot \hat{\gamma}_3. \tag{3.114}$$

III. Next comes the time derivative of $\hat{\gamma}_1$. Compute first the derivative of γ_1 using the Hamilton equations and (2.6):

$$\begin{aligned}\partial_t\gamma_1(t) &= \frac{\xi(t)}{r(t)}\big[b\big(\theta(t)\big)+\xi(t)\big]\\ &\quad-\frac{\xi(t)}{r(t)}\big[b\big(\theta(t)\big)+\xi_{E_t}\big(\theta(t)\big)\big]+\partial_E\rho_{E_t}\big(\theta(t)\big)\cdot\partial_t\gamma_3(t)\\ &= \frac{\xi(t)}{r(t)}\gamma_2(t)+\partial_E\rho_{E_t}\big(\theta(t)\big)\cdot\partial_t\gamma_3(t).\end{aligned}\tag{3.115}$$

We then obtain (using (2.6), (1.9), and (3.111)):

$$\begin{aligned}\partial_t\hat{\gamma}_1 &= \partial_t\gamma_1-(\partial_E\xi_{E_t})\cdot(\xi/r)\cdot\gamma_3-(\partial_E\rho_{E_t})\cdot\partial_t\gamma_3+\gamma_3\cdot\big(\partial_E^2\rho_{E_t}\big)\partial_t\gamma_3\\ &= (\xi_{E_t}/r)\cdot\hat{\gamma}_2+\mathcal{O}\big(\hat{\gamma}^2/t\big)\\ &= -\big[\rho_{E_t}\cdot(\partial_E\rho_{E_t})/t\big]\cdot\hat{\gamma}_1+\mathcal{O}\big(\hat{\gamma}^2/t\big),\end{aligned}\tag{3.116}$$

where the last equality came from (3.110), (3.111), and (3.114).

Define

$$f_1(t):=\frac{1}{\partial_E\rho_{E_t}}\big(\theta(t)\big)>0.\tag{3.117}$$

We see that its time derivative gives (use (3.111) and (3.114)):

$$\begin{aligned}\partial_t f_1 &= \big(\partial_E^2\rho_{E_t}\big)/(\partial_E\rho_{E_t})^2\cdot(\partial_t\hat{\gamma}_3)-(f_1/t)\cdot(\partial_E\xi_{E_t})\cdot\xi\\ &= -f_1\cdot(1-\rho_{E_t}\partial_E\rho_{E_t})/t+\mathcal{O}(\hat{\gamma}/t).\end{aligned}\tag{3.118}$$

Combining this with (3.116) we get

$$\partial_t(f_1\hat{\gamma}_1)=-(1/t)(f_1\hat{\gamma}_1)+\mathcal{O}\big(\hat{\gamma}^2/t\big).\tag{3.119}$$

IV. Now we deal with the time derivative of $\hat{\gamma}_2$. The computations are more involved and we only give the relevant equation:

$$\partial_t\hat{\gamma}_2=-\frac{b+\xi_{E_t}}{r}\hat{\gamma}_1+\frac{b\rho_{E_t}}{r\xi_{E_t}}\hat{\gamma}_2+\mathcal{O}\big(\hat{\gamma}^2/t\big).\tag{3.120}$$

We remark that the right-hand side of the above equation contains $\hat{\gamma}_3$ only in the quadratic remainder. Then notice

$$\frac{b\rho_{E_t}}{r\xi_{E_t}}=\frac{b+\xi_{E_t}}{r\xi_{E_t}}\rho_{E_t}-\frac{\rho_{E_t}}{r}$$

and

$$\partial_t\big[\xi_{E_t}\big(\theta(t)\big)\big]=-\frac{b+\xi_{E_t}}{r}\rho_{E_t}+\mathcal{O}(\hat{\gamma}/t).$$

We introduce the second integrating factor

$$f_2(t) := (\xi_{E_t}/\partial_E \rho_{E_t})\big(\theta(t)\big). \tag{3.121}$$

Then

$$\partial_t(f_2\hat{\gamma}_2) = -\frac{b+\xi_{E_t}}{t}(\partial_E \rho_{E_t})\xi_{E_t}\cdot(f_1\hat{\gamma}_1) - (1/t)(f_2\hat{\gamma}_2) + \mathcal{O}\big(\hat{\gamma}^2/t\big). \tag{3.122}$$

Now we are ready to rewrite (3.114) in a more convenient form. Define

$$f_3(t) = -\big(\partial_E^2 \rho_{E_t}/\partial_E \rho_{E_t}\big)\big(\theta(t)\big). \tag{3.123}$$

Using the identity $(\partial_E \xi_{E_t})^2 + (\partial_E \rho_{E_t})^2 = -\rho_{E_t}\partial_E^2 \rho_{E_t} - \xi_{E_t}\partial_E^2 \xi_{E_t}$ together with the “linear” dependence (3.110) we obtain

$$\partial_t(f_3\hat{\gamma}_3) = \big[a_{31}(t)/t\big]\cdot(f_1\hat{\gamma}_1) - (1/t)(f_3\hat{\gamma}_3) + \mathcal{O}\big(\hat{\gamma}^2/t\big), \tag{3.124}$$

where $a_{31}(t)$ is a bounded scalar.

V. We are now ready to give a differential inequality involving all three $\hat{\gamma}$’s. First, rewrite (3.119), (3.122) and (3.124) in a more compact form:

$$\partial_t(f_j\hat{\gamma}_j) = \sum_{k=1}^{3}\big[a_{jk}(t)/t\big]\cdot(f_k\hat{\gamma}_k) + \mathcal{O}\big(\hat{\gamma}^2/t\big), \quad j \in \{1,2,3\}, \tag{3.125}$$

where $a_{jj} = -1$ for all j, and $a_{12} = a_{13} = a_{23} = a_{32} = 0$. In particular, the matrix $\{a\}$ is lower triangular. Notice also that we have been using the fact that when the energy is localized around $E > E_d$, one may get upper and lower bounds for f_j’s uniform in t; there exist upper bounds for a_{jk}’s too.

Define the Liapunov-type function

$$L_C := C\cdot(f_1\hat{\gamma}_1)^2 + (f_2\hat{\gamma}_2)^2 + (f_3\hat{\gamma}_3)^2, \tag{3.126}$$

where $C > 0$ is a very large positive constant only depending on the energy. Now let us see how we choose C. Compute

$$\begin{aligned}\partial_t L_C = {} & -(2/t)L_C + (2/t)\big[a_{21}\cdot(f_2\hat{\gamma}_2)(f_1\hat{\gamma}_1)\big] + (2/t)\big[a_{31}\cdot(f_3\hat{\gamma}_3)(f_1\hat{\gamma}_1)\big]\\ & + \mathcal{O}\big(\hat{\gamma}^3/t\big).\end{aligned} \tag{3.127}$$

We see that the cross terms can be bounded by

$$2\big|(f_j\hat{\gamma}_j)(f_1\hat{\gamma}_1)\big| \leqslant (1/\sqrt{C})\big[(f_j\hat{\gamma}_j)^2 + C(f_1\hat{\gamma}_1)^2\big] \leqslant (L_C/\sqrt{C}), \quad j \in \{2,3\}.$$

We conclude that for every $\delta > 0$, we can choose $C(\delta)$ sufficiently large such that for and for some large constant K_δ (we write L_δ instead of L_{C_δ}):

$$\partial_t L_\delta \leqslant -\frac{2-\delta/2}{t}L_\delta + K_\delta L_\delta^{3/2}/t. \tag{3.128}$$

Moreover, due to the a priori smallness of the $\hat{\gamma}$'s (see (3.107)) we get that given $\delta > 0$, there is T large enough such that:

$$K_\delta L_\delta^{1/2} \leqslant \delta/2, \quad t \geqslant T.$$

Combining this with (3.128) we finally obtain that for every $\delta > 0$ there is T sufficiently large such that

$$\partial_t L_\delta(t) \leqslant -\frac{2-\delta}{t} L_\delta(t), \quad t \geqslant T. \tag{3.129}$$

We are finally in position to end the proof of Lemma 3.14. Indeed, (3.129) implies that $t^{2-\delta} L_\delta(t)$ decreases if $t \geqslant T$, hence

$$|\hat{\gamma}_j|(t) \leqslant Const \cdot t^{\delta-1}, \quad j \in \{1,2,3\},\ t \geqslant T,$$

and by introducing this in (3.106) and the proof is complete. □

4. The main result

From now on we deal with quantum mechanics. This section contains the formulation of our main theorem.

4.1. A Mourre estimate above E_d

We know from (1.20) and (1.21) that the generator of dilations is a good conjugate operator for states with energy localization above E_b. We have already encountered this in the classical case (see (2.2)). Now the natural question is whether or not we can give a quantum counterpart to the quantity we defined in (2.33). The answer is positive and stated in what follows.

We introduce $\bar{r}_1(\mathbf{x}) := F_+^{1/2,1}(|\mathbf{x}|) \cdot |\mathbf{x}|$. For every $C > 0$ and $E > E_d$ define a "rotated" generator of dilations:

$$A_C(E) := \frac{C}{2} e^{-i\bar{r}_1 [\partial_E \rho_E(\theta)/C - \rho_E(\theta)]} [\mathbf{p} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{p}] e^{i\bar{r}_1 [\partial_E \rho_E(\theta)/C - \rho_E(\theta)]}, \tag{4.1}$$

where ρ_E is the periodic solution given by corollary (2.7).

Proposition 4.1. *Fix $E > E_d$. Then for every small enough $\epsilon > 0$, there exists $C = C(\epsilon, E)$ large enough and a compact operator K such that*

$$\mathbf{1}_{[E-\epsilon,E+\epsilon]}(H) i\big[H, A_C(E)\big] \mathbf{1}_{[E-\epsilon,E+\epsilon]}(H) \geqslant \frac{1}{2} \cdot \mathbf{1}_{[E-\epsilon,E+\epsilon]}(H) + K.$$

Proof. We can compute the commutator between H and $A_C(E)$ by reading off the classical computations we have done in (2.35)–(2.37) and making everything symmetric.

We first need a few definitions. For $a = 1/2$ and $b = 1$ we define a regularized "modulus" by

$$\bar{r}(\mathbf{x}) = \int_0^{|\mathbf{x}|} F_+(s)\, ds + \int_0^1 \big[1 - F_+(s)\big]\, ds \tag{4.2}$$

(notice that for $|\mathbf{x}| \geqslant 1$ we have $\bar{r}(\mathbf{x}) = |\mathbf{x}|$).

The radial velocity $\boldsymbol{\rho}$ is given by

$$\boldsymbol{\rho} := \frac{1}{2}\big\{(\mathbf{p}-\mathbf{a})\cdot(\nabla\bar{r}) + (\nabla\bar{r})\cdot(\mathbf{p}-\mathbf{a})\big\}. \tag{4.3}$$

It is easy to see that ρ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

Using polar coordinates in $L^2(\mathbb{R}^2)$ and (4.2), we have

$$\begin{aligned}\boldsymbol{\rho} &= -\frac{i}{2}\left\{\frac{\partial}{\partial r}\cdot F_+(r) + F_+(r)\cdot\frac{\partial}{\partial r} + \frac{1}{r}F_+(r)\right\}\\ &= -i\sqrt{F_+}\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)\sqrt{F_+}, \quad r = |\mathbf{x}|.\end{aligned} \tag{4.4}$$

As an operator on $L^2(\mathbb{R}_+\times\mathbb{T})$ the radial momentum takes the form:

$$\boldsymbol{\rho} = -i\sqrt{F_+}\,\partial_r\sqrt{F_+}. \tag{4.5}$$

Moreover there exists a smooth function $m_0(r)$ supported away from zero and with a decay of at least order r^{-2} such that

$$\boldsymbol{\rho}^2 = -F_+\partial_r^2 F_+ + m_0(r). \tag{4.6}$$

The transverse velocity ξ is given by

$$\xi := \frac{L}{\bar{r}} - m_+(r)b(\theta). \tag{4.7}$$

Finally we note (see Lemma 6.1 for related bounds) that for every $\beta > 0$ there exist $C, N \geqslant 1$ such that for all $z \in \mathbb{C}\setminus\mathbb{R}$

$$\begin{aligned}&\max\big\{\big\|\bar{r}^\beta(H-z)^{-1}(\bar{r})^{-\beta}\big\|, \big\|\bar{r}^\beta(\tilde{\boldsymbol{\rho}}-z)^{-1}(\bar{r})^{-\beta}\big\|\big\} \leqslant \frac{C}{|\Im(z)|}\left(\frac{\langle z\rangle}{|\Im(z)|}\right)^N,\\ &\langle z\rangle = \big(1+|z|^2\big)^{1/2}.\end{aligned} \tag{4.8}$$

We shall use (4.8) in the context of estimating commutators.

The various cut-offs will generate through commutation several terms which are relatively compact to H, while $\rho^2+\xi^2$ now equals $2H$ (up to relatively compact remainders) and not $2E$ as in the classical case. Hence we can write

$$\begin{aligned}i\big[H, A_C(E)\big] = 1 + H - E &+ \frac{1}{2}\big[(\xi-\xi_E)\partial_E\xi_E + (\rho-\rho_E)\partial_E\rho_E + \text{h.c.}\big]\\ &+ \frac{C}{2}\big[(\xi-\xi_E)^2 + (\rho-\rho_E)^2\big] + K_1,\end{aligned} \tag{4.9}$$

where h.c. means Hermitian conjugate and K_1 is a relatively compact remainder. In the form sense we have the inequality

$$-\frac{1}{2}\big[(\xi-\xi_E)\partial_E\xi_E+\mathrm{h.c.}\big]\leqslant\frac{C^{1/2}}{2}(\xi-\xi_E)^2+\frac{1}{2C^{1/2}}(\partial_E\xi_E)^2$$

while a similar one holds for $(\rho-\rho_E)\partial_E\rho_E$. We have $(H-E)\geqslant-\epsilon$ when restricted to the range of $\mathbf{1}_{[E-\epsilon,E+\epsilon]}(H)$; then choosing C large enough we obtain the desired positivity and the proposition is proven. □

Under the conditions we imposed, $1/\bar{r}^{\alpha}$ with $\alpha>1/2$ is a locally smooth perturbation for H, cf. Proposition 4.1, [15] and [17]. For any state like the one in (6.1) we have that

$$\int_1^\infty\big\langle\psi(t),(\bar{r})^{-2\alpha}\psi(t)\big\rangle\,dt=\int_1^\infty\big\|(\bar{r})^{-\alpha}\psi(t)\big\|^2\,dt\leqslant Const\|\psi\|^2. \tag{4.10}$$

We also learn that the point spectrum of H in (E_d,∞) is discrete with eigenvalues of finite multiplicity.

4.2. Construction of the approximate evolution

From now on we abbreviate $(0,\infty)$ by $\mathbb{R}_+$. Define for $t\geqslant1$ the operator

$$U_0(t):L^2\big((E_d,\infty)\times\mathbb{T}\big)\mapsto L^2(\mathbb{R}_+\times\mathbb{T}), \tag{4.11}$$

where (see also (3.56))

$$\big[U_0(t)f\big](r,\theta):=\exp\big\{iS(t,r,\theta)\big\}J_t^{1/2}(r,\theta)\cdot f\big(-\partial_tS\big(1,\mathbf{w}_t(r,\theta)\big),\theta_1(t,r,\theta)\big). \tag{4.12}$$

Here J_t is a Jacobian determinant which assures that $U_0(t)$ is unitary. More precisely, it equals the product between the Jacobian $J_{\mathbf{w}_t}$ of the inverse flow, and the Jacobian of the transformation $\mathbb{R}_+\ni(r,\theta)\mapsto(-\partial_tS(1,r,\theta),\theta)\in(E_d,\infty)$. We also introduce

$$\begin{aligned}&W(t):L^2\big((E_d,\infty)\times\mathbb{T}\big)\mapsto L^2(\mathbb{R}_+\times\mathbb{T}),\\&\quad\big[W(t)f\big](r,\theta):=\exp\big\{-iS(t,r,\theta)\big\}\big[U_0(t)f\big](r,\theta).\end{aligned} \tag{4.13}$$

For $f\in C_0^\infty((E_d,\infty)\times\mathbb{T})$ we have that $W(t)f$ is strongly differentiable and

$$-i\frac{d}{dt}W(t)f=-B(t)W(t)f. \tag{4.14}$$

Moreover,

$$B(t)W(t)f \in C_0^\infty(\mathbb{R}_+ \times \mathbb{T}), \tag{4.15}$$

$$\begin{aligned} 2B(t) &= -i(\partial_r S)(t,r,\theta)\partial_r - i\left(\frac{\partial_\theta S(t,r,\theta)}{r} - b(\theta)\right)\frac{1}{r}\partial_\theta + \text{h.c.} \\ &= \big(\nabla_{\mathbf{x}} S(t,\mathbf{x}) - \mathbf{a}(\mathbf{x})\big)\cdot \mathbf{p} + \text{h.c.}, \end{aligned} \tag{4.16}$$

where h.c. means Hermitian conjugate. The computation is fairly standard, and relies on (3.55) and (3.69).

We now want to determine the "generator of the free evolution," i.e. to describe the strong time derivative of $U_0(t)$.

Consider the following symmetric operators on $L^2(\mathbb{R}_+ \times \mathbb{T})$ (defined on smooth and compactly supported functions)

$$\gamma_1 = \boldsymbol{\rho} - \partial_r S(t,r,\theta) = \boldsymbol{\rho} - \boldsymbol{\rho}_{E(t,r,\theta)}(\theta) \tag{4.17}$$

and

$$\gamma_2 = \xi - \left(\frac{\partial_\theta S(t,r,\theta)}{r} - b(\theta)\right) = \xi - \xi_{E(t,r,\theta)}(\theta). \tag{4.18}$$

From (3.41) and (4.14) one infers that for any $f \in C_0^\infty((E_d,\infty)\times\mathbb{T})$ the mapping

$$(1,\infty) \ni t \mapsto U_0(t)f \in L^2(\mathbb{R}_+ \times \mathbb{T})$$

is differentiable and if t is large enough then

$$i\frac{d}{dt}U_0(t)f = H_0(t)U_0(t)f, \quad H_0(t) := H + \frac{1}{8r^2} - \frac{\gamma_1^2}{2} - \frac{\gamma_2^2}{2}. \tag{4.19}$$

4.3. Stating the main result of the paper

We are finally prepared to give the main theorem.

Theorem 4.2. *The limits in* (1.23) *exist and define mutually inverse unitary operators. Spelled out*:

I. (*Existence of scattering states.*) *For every ϕ in $L^2((E_d,\infty)\times\mathbb{T})$ there exists $\psi = \Omega_+^d\phi$ in the space $\mathcal{H}_{E_d} = \mathbf{1}_{(E_d,\infty)\setminus\sigma_{\mathrm{pp}}(H)}(H)L^2(\mathbb{R}_+\times\mathbb{T})$ such that*

$$\lim_{t\to\infty}\left\|e^{-itH}\Omega_+^d\phi - U_0(t)\phi\right\| = 0.$$

II. (*Asymptotic completeness.*) *For every ψ in the $\mathcal{H}_{E_d}$, there exists $\phi = \Omega_+\psi$ in $L^2((E_d,\infty)\times\mathbb{T})$ such that*

$$\lim_{t\to\infty}\left\|e^{-itH}\psi - U_0(t)\Omega_+\psi\right\| = 0.$$

III. (*Asymptotic observables.*) *The asymptotic radius and angle* (mod 2π) *as defined in* (1.24) *exist as operators on* $\mathcal{H}_{E_d}$; *moreover* ($M(\cdot)$ *denotes multiplication by the argument, see Lemma* 3.8 *for the definition of* $r(E,t,\theta)$):

$$e^{i\theta_+} = \Omega_+^d M\big(e^{i\theta}\big)\Omega_+, \qquad r_+ = \Omega_+^d M\big(r(E,1,\theta)\big)\Omega_+, \qquad H\mathbf{1}_{\mathcal{H}_{E_d}} = \Omega_+^d M(E)\Omega_+\mathbf{1}_{\mathcal{H}_{E_d}}.$$

Remark. We may readily add some short-range magnetic and scalar perturbations to H, and the above theorem remains true. Namely, we define (see also (1.16) and (1.17))

$$H_s := \frac{1}{2}(\mathbf{p}-\mathbf{a}-\mathbf{a}_s)^2 + V_s = H - \frac{1}{2}(\mathbf{p}-\mathbf{a})\cdot\mathbf{a}_s - \frac{1}{2}\mathbf{a}_s\cdot(\mathbf{p}-\mathbf{a}) + \frac{1}{2}\mathbf{a}_s^2 + V_s,$$

where $\mathbf{a}_s$ is C^1, V_s is relatively bounded with respect to $-\Delta$ with bound less than 1 and, in addition, for some $\epsilon > 0$ and as $|\mathbf{x}| \to \infty$,

$$V_s(\mathbf{x}),\ \mathbf{a}_s(\mathbf{x}) = \mathcal{O}\big(|\mathbf{x}|^{-(1+\epsilon)}\big);$$
$$D^\alpha V_s(\mathbf{x}),\ D^\alpha \mathbf{a}_s(\mathbf{x}) = \mathcal{O}\big(|\mathbf{x}|^{-2}\big); \quad |\alpha| = 2.$$

Then by standard Mourre theory and the theory of smooth perturbations, cf. (4.10), one first constructs the relative wave operator for the pair (H, H_s). Next invoking the stated theorem (for H) and the chain rule for wave operators one deduces the theorem with H replaced by H_s.

5. Proof of the existence of scattering states

Our proof of the first statement of Theorem 4.2 has two parts. Choose any $\phi \in C_0^\infty((E_d,\infty)\times\mathbb{T})$ and denote its support by I.

Firstly, we show that the following limit exists in $L^2(\mathbb{R}_+\times\mathbb{T})$:

$$\Omega_+^d\phi := \lim_{t\to\infty} e^{iHt}U_0(t)\phi. \tag{5.1}$$

Secondly, we show that this limit belongs to $\mathcal{H}_{E_d}$.

5.1. Existence of Ω_+^d

Employing the usual Cook argument, we can reduce the above limit to the "time integrability" of the "perturbation" (see (4.19)):

$$\int_1^\infty \left\| \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{4r^2}\right)\big[W(t)\phi\big](r,\theta)\right\|_{L^2(\mathbb{R}_+\times\mathbb{T})} dt < \infty. \tag{5.2}$$

In order to get an idea about why (5.2) should hold, we again look at the case when $b(\theta) = b < 0$ is a negative constant. From the definition (4.13) we get using (3.58), (3.59), (3.62) and (3.63) that

$$\big[W(t)\phi\big](r,\theta) = \sqrt{\frac{r}{t^2}}\,\phi\left(\frac{r^2}{2t^2} + \frac{b^2}{2}, \theta + b\frac{t}{r}\ln(t)\right), \tag{5.3}$$

in accordance with the formula (1.25) for $U_0(t)f$.

We may now see that on the support of $W(t)\phi$, r and t are effectively proportional; when we look at the integrand in (5.2), we realize that by differentiating with respect to r we obtain an extra decay of order either t^{-1} or $t^{-1}\ln(t)$. Two derivatives will make it integrable in t. As for the derivatives with respect to θ, we see that they are bounded; but $1/r^2$ is transformed into $1/t^2$ so (5.2) follows.

A similar argument can be carried out when b is not constant. Compared to Lemma 3.12 we need to go further, i.e. to investigate what happens when we differentiate twice with respect to r and θ. Taking two derivatives in (3.86) and (3.87) we get

$$\begin{aligned}
&\frac{\partial\tilde{r}}{\partial r_1}\frac{\partial^2 r_1}{\partial r^2}+\frac{\partial\tilde{r}}{\partial\theta_1}\frac{\partial^2\theta_1}{\partial r^2}=A_1,\\
&A_1=-\frac{\partial^2\tilde{r}}{\partial r_1^2}\left(\frac{\partial r_1}{\partial r}\right)^2-2\frac{\partial^2\tilde{r}}{\partial r_1\partial\theta_1}\frac{\partial r_1}{\partial r}\frac{\partial\theta_1}{\partial r}-\frac{\partial^2\tilde{r}}{\partial\theta_1^2}\left(\frac{\partial\theta_1}{\partial r}\right)^2
\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}
&\frac{\partial\tilde{\theta}}{\partial r_1}\frac{\partial^2 r_1}{\partial r^2}+\frac{\partial\tilde{\theta}}{\partial\theta_1}\frac{\partial^2\theta_1}{\partial r^2}=A_2,\\
&A_2=-\frac{\partial^2\tilde{\theta}}{\partial r_1^2}\left(\frac{\partial r_1}{\partial r}\right)^2-2\frac{\partial^2\tilde{\theta}}{\partial r_1\partial\theta_1}\frac{\partial r_1}{\partial r}\frac{\partial\theta_1}{\partial r}-\frac{\partial^2\tilde{\theta}}{\partial\theta_1^2}\left(\frac{\partial\theta_1}{\partial r}\right)^2.
\end{aligned}\tag{5.5}$$

We then have

$$\frac{\partial^2 r_1}{\partial r^2}=J_{\mathbf{v}_t}^{-1}\left(\frac{\partial\tilde{\theta}}{\partial\theta_1}\cdot A_1-\frac{\partial\tilde{r}}{\partial\theta_1}\cdot A_2\right)\tag{5.6}$$

and

$$\frac{\partial^2\theta_1}{\partial r^2}=J_{\mathbf{v}_t}^{-1}\left(-\frac{\partial\tilde{\theta}}{\partial r_1}\cdot A_1+\frac{\partial\tilde{r}}{\partial r_1}\cdot A_2\right).\tag{5.7}$$

Using (3.82), (3.83), (3.88) and (3.89) in (5.4) and (5.5) we get up to additional factors of t^δ (remember that δ is arbitrarily small; these terms appear when we replace the logarithms)

$$A_1\sim t^{-1},\qquad A_2\sim t^{-2},$$

which leads to

$$\frac{\partial^2 r_1}{\partial r^2},\ \frac{\partial^2\theta_1}{\partial r^2}\sim t^{-2}.\tag{5.8}$$

Similarly, one may show that (again forgetting about factors of t^δ)

$$\frac{\partial^2 r_1}{\partial\theta^2},\ \frac{\partial^2\theta_1}{\partial\theta^2}\sim Const,\qquad\frac{\partial^2 r_1}{\partial\theta\partial r},\ \frac{\partial^2\theta_1}{\partial\theta\partial r}\sim t^{-1}.\tag{5.9}$$

These estimates and Lemma 3.12 lead to the conclusion that the first and second order derivatives of the Jacobian corresponding to the inverse flow $J_{\mathbf{w}_t}(r,\theta) = J_{\mathbf{v}_t}^{-1}(r_1,\theta_1)$ behave like

$$\big[\partial_r J_{\mathbf{w}_t}^{1/2}\big] J_{\mathbf{w}_t}^{-1/2} = -\frac{1}{2}\partial_r(\ln J_{\mathbf{v}_t}\circ \mathbf{w}_t) \sim t^{-1}, \tag{5.10}$$

$$\big[\partial_\theta J_{\mathbf{w}_t}^{1/2}\big] J_{\mathbf{w}_t}^{-1/2} \sim Const, \tag{5.11}$$

$$\partial_r\big[\big[\partial_r J_{\mathbf{w}_t}^{1/2}\big] J_{\mathbf{w}_t}^{-1/2}\big] = -\frac{1}{2}\partial_r^2(\ln J_{\mathbf{v}_t}\circ \mathbf{w}_t) \sim t^{-2}, \tag{5.12}$$

$$\partial_\theta\big[\big[\partial_\theta J_{\mathbf{w}_t}^{1/2}\big] J_{\mathbf{w}_t}^{-1/2}\big] = -\frac{1}{2}\partial_\theta^2(\ln J_{\mathbf{v}_t}\circ \mathbf{w}_t) \sim Const, \tag{5.13}$$

$$\partial_\theta\big[\big[\partial_r J_{\mathbf{w}_t}^{1/2}\big] J_{\mathbf{w}_t}^{-1/2}\big] = -\frac{1}{2}\partial_\theta\partial_r(\ln J_{\mathbf{v}_t}\circ \mathbf{w}_t) \sim t^{-1}. \tag{5.14}$$

Up to factors of t^δ we conclude from Lemma 3.12, (5.8)–(5.14) that indeed

$$\big\|\partial_r^2\big[W(t)\phi\big](r,\theta)\big\|,\ \big\|r^{-2}\partial_\theta^2\big[W(t)\phi\big](r,\theta)\big\|,\ \big\|r^{-2}\big[W(t)\phi\big](r,\theta)\big\| \sim t^{-2}, \tag{5.15}$$

which finally leads to (5.2). □

5.2. *Proving the inclusion* $\mathrm{Ran}(\Omega_+^d) \subseteq \mathcal{H}_{E_d}$

We first show that $\mathrm{Ran}(\Omega_+^d) \subseteq \mathrm{Ran}(\mathbf{1}_{(E_d,\infty)}(H))$. The inclusion is based on the intertwining formula

$$H\Omega_+^d\phi = \Omega_+^d M(E)\phi, \tag{5.16}$$

where $\phi \in C_0^\infty((E_d,\infty)\times\mathbb{T})$. This formula implies a similar intertwining for resolvents and through functional calculus for any real and bounded function:

$$f(H)\Omega_+^d = \Omega_+^d M\big(f(E)\big), \tag{5.17}$$

which yields the result.

For every $\psi \in \mathrm{Dom}(H)$ and $t \geqslant 1$ we have

$$\big\langle H\psi, e^{itH}U_0(t)\phi\big\rangle = \int dr \int d\theta\, \overline{He^{-itH}\psi}(r,\theta)e^{iS(t,r,\theta)}\big[W(t)\phi\big](r,\theta). \tag{5.18}$$

We move H to the right and use (4.19):

$$\big\langle H\psi, e^{itH}U_0(t)\phi\big\rangle = \left\langle e^{-itH}\psi, \left(-\frac{1}{8r^2}+\frac{\gamma_1^2}{2}+\frac{\gamma_2^2}{2}\right)U_0(t)\phi\right\rangle + \big\langle e^{-itH}\psi, i\partial_t U_0(t)\phi\big\rangle.$$

When we differentiate $iU_0(t)\phi$ with respect to t, we get two terms: the first one contains the expression

$$-[\partial_t S](t,r,\theta)e^{iS(t,r,\theta)}\big[W(t)\phi\big](r,\theta) = \big[U_0(t)M(E)\phi\big](r,\theta),$$

cf. (3.38); the second one contains

$$i\partial_t\big[W(t)\phi\big](r,\theta)=B(t)\big[W(t)\phi\big](r,\theta),$$

cf. (4.14).

We then obtain

$$\big\langle H\psi, e^{itH}U_0(t)\phi\big\rangle=\big\langle\psi, e^{itH}U_0(t)M(E)\phi\big\rangle+\left\langle e^{-itH}\psi,\left(-\frac{1}{8r^2}+\frac{\gamma_1^2}{2}+\frac{\gamma_2^2}{2}\right)U_0(t)\phi\right\rangle$$
$$+\big\langle e^{-itH}\psi, e^{iS}B(t)W(t)\phi\big\rangle.$$

Now the second and the third term on the right-hand side of the above equality converge to zero as $t\to\infty$. This follows readily from the estimates of Section 7.1 together with the identity

$$\big(\gamma_1^2+\gamma_2^2\big)e^{iS(t,r,\theta)}\big[W(t)\phi\big](r,\theta)=-e^{iS(t,r,\theta)}\big(\partial_r^2+r^{-2}\partial_\theta^2\big)\big[W(t)\phi\big](r,\theta).$$

In conclusion

$$\big\langle H\psi,\Omega_+^d\phi\big\rangle=\big\langle\psi,\Omega_+^d M(E)\phi\big\rangle,$$

yielding (5.16).

Clearly $\Omega_+^d M(E)\phi\perp\mathcal{H}_{pp}$. We have proved $\mathrm{Ran}(\Omega_+^d)\subseteq\mathcal{H}_{E_d}$. □

We have now established I of Theorem 4.2. The remaining part of the paper is devoted to proving II. Given these results, III of Theorem 4.2 follows easily. For the last identity $H\mathbf{1}_{\mathcal{H}_{E_d}}=\Omega_+^d M(E)\Omega_+\mathbf{1}_{\mathcal{H}_{E_d}}$ we invoke (5.16).

6. Asymptotic completeness: propagation estimates in quantum mechanics

We now start the proof of local asymptotic completeness. In this section we are going to prove various propagation estimates for states $\psi(t)$ of the form (6.1), very similar to the ones we had for the classical problem.

The method we use is to show that for large t, $\psi(t)$ is "localized near the attractor" in the sense that $\psi(t)\sim\Pi_{i=1}^N F_i(A_i(t))\psi(t)$ with each $F_i(A_i(t))$ a "cut-off function" of some observable. It will be crucial that the Heisenberg derivative of the product of cut-off functions is essentially positive. For further motivation see the remarks at the beginning of the proof of Lemma 8.2.

6.1. Preliminaries and the Helffer–Sjöstrand formula

We start with various definitions and preliminary technical results. Let H be the Hamiltonian introduced in (1.17). We are going to study states of the form

$$\psi(t)=e^{-itH}f_E(H)\psi,\tag{6.1}$$

where $\tilde\epsilon>0$ is very small, $f_E\in C_0^\infty((E-\tilde\epsilon/2,E+\tilde\epsilon/2))$, $0\leqslant f_E\leqslant 1$ and $f_E=1$ on $(E-\tilde\epsilon/4,E+\tilde\epsilon/4)$ for $E\in(E_d,\infty)\setminus\sigma_{\mathrm{pp}}(H)$. The state $\psi\in L^2(\mathbb{R}^2)$.

We will later need two other cut-off functions. Namely, consider $f_{1,E} \in C_0^\infty((E - 4\tilde{\epsilon}/5, E + 4\tilde{\epsilon}/5))$, $0 \leqslant f_{1,E} \leqslant 1$ and equal to 1 on $(E - 3\tilde{\epsilon}/4, E + 3\tilde{\epsilon}/4)$. Moreover, consider $\tilde{f}_E \in C_0^\infty((E - \tilde{\epsilon}, E + \tilde{\epsilon}))$, $0 \leqslant \tilde{f}_E \leqslant 1$ and equal to 1 on $(E - 7\tilde{\epsilon}/8, E + 7\tilde{\epsilon}/8)$. We then have the property that

$$f_{1,E} f_E = f_E, \qquad \tilde{f}_E f_{1,E} = f_{1,E}.$$

Let us define the regularized radial and transverse velocities by

$$\tilde{\boldsymbol{\rho}} := \tilde{f}_E(H)\boldsymbol{\rho}\tilde{f}_E(H), \qquad \tilde{\xi} := \tilde{f}_E(H)\xi\tilde{f}_E(H). \tag{6.2}$$

Let us recall (see [4, Appendix C] or [14]) that for all bounded (or possibly unbounded) self-adjoint operators A and B (on the same Hilbert space) and for all $f \in C_0^\infty(\mathbb{R})$ we may represent ($\bar{\partial}$ denotes differentiation with respect to $\bar{z}$):

$$\bigl[A, f(B)\bigr] = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(B - z)^{-1}[A, B](B - z)^{-1}\,dx\,dy; \quad z = x + iy, \tag{6.3}$$

where $\tilde{f}$ is a smooth compactly supported almost analytic extension of f.

We shall need this and other commutator formulas for functions Γ of the type F_+ or type F_{++}. An almost analytic extension of Γ can be constructed obeying

$$\bigl|\bar{\partial}\tilde{\Gamma}(z)\bigr| \leqslant C_k\langle z\rangle^{-1-k}\bigl|\Im(z)\bigr|^k; \quad z \in \mathbb{C},\ k \in \mathbb{N}. \tag{6.4}$$

Then,

$$\begin{aligned}\bigl[A, \Gamma(B)\bigr] &= -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\Gamma}(z)(B - z)^{-1}[A, B](B - z)^{-1}\,dx\,dy \\ &= \frac{1}{2}\bigl\{\Gamma'(B)[A, B] + [A, B]\Gamma'(B)\bigr\} + R_1,\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}R_1 = -\frac{1}{2\pi}\int_{\mathbb{C}} &\bar{\partial}\tilde{\Gamma}(z)(B - z)^{-1} \\ &\times \bigl\{(B - z)^{-1}\bigl[B, [A, B]\bigr] - \bigl[B, [A, B]\bigr](B - z)^{-1}\bigr\}(B - z)^{-1}\,dx\,dy.\end{aligned} \tag{6.6}$$

Also,

$$\bigl[A, \Gamma(B)\bigr] = \sqrt{\Gamma'}(B)[A, B]\sqrt{\Gamma'}(B) + R_1 + R_2, \tag{6.7}$$

where

$$R_2 = \frac{1}{2}\bigl[[A, B], \sqrt{\Gamma'(B)}\,\bigr]\sqrt{\Gamma'(B)} - \frac{1}{2}\sqrt{\Gamma'(B)}\bigl[[A, B], \sqrt{\Gamma'(B)}\,\bigr]. \tag{6.8}$$

6.2. Non-commutativity and "integrable remainders"

To go from classical to quantum mechanics we have to deal with "errors" coming from the non-commutativity of certain operators. We will use the following technical lemmas to show that these "errors" are small. In order to simplify notation, we will often write r instead of the regularized modulus $\bar{r}$ (see (4.2)).

Lemma 6.1. *Suppose $F, G \in C_0^\infty(\mathbb{R}_+)$ and $F = 1$ on a neighborhood of the support of G. Let B be one of the operators H, A_C/t, $\tilde{\boldsymbol{\rho}}$ or $\tilde{\xi}$. Then for every integer $N \geqslant 1$, there exists $C > 0$ such that*

$$\big\|\big(1 - F(r/t)\big)(B - z)^{-1} G(r/t)\big\| \leqslant \frac{C}{|\Im(z)|}\left(\frac{\langle z\rangle}{|\Im(z)|}\right)^N t^{-N}.$$

Proof. One can find a function G_1 such that $G_1 G = G$ and $F G_1 = G_1$. Then for any $N \in \mathbb{N}$ we may write (abbreviating below $F = F(r/t)$, $G = G(r/t)$ and $G_1 = G_1(r/t)$)

$$(1 - F)(B - z)^{-1} G = (1 - F)(B - z)^{-1} G_1^N G.$$

Due to the support conditions we have

$$(1 - F)(B - z)^{-1} G = (1 - F)\mathrm{ad}_{G_1}^N\big((B - z)^{-1}\big)G, \tag{6.9}$$

where $\mathrm{ad}_{G_1}^0(B) = B$ and $\mathrm{ad}_{G_1}^k(B) = [\mathrm{ad}_{G_1}^{k-1}(B), G_1]$ for $k \geqslant 1$. Then

$$\mathrm{ad}_{G_1}^N\big((B - z)^{-1}\big) = \sum_{k_1+\dots+k_n=N} C_{k_1,\dots,k_n}(B - z)^{-1}\mathrm{ad}_{G_1}^{k_1}(B)\cdots(B - z)^{-1}\mathrm{ad}_{G_1}^{k_n}(B)(B - z)^{-1}. \tag{6.10}$$

It is then easy to see that each commutator brings a decay of order t^{-1}; more precisely, if B is either $\tilde{\boldsymbol{\rho}}$ or $\tilde{\xi}$ then

$$\big\|\mathrm{ad}_{G_1}^{k_j}(B)\big\| \leqslant Const \cdot t^{-k_j}.$$

If $B = H$ then

$$\big\|\mathrm{ad}_{G_1}^{k_j}(H) \cdot \langle H\rangle^{-1}\big\| \leqslant Const \cdot t^{-k_j}, \qquad \big\|\langle H\rangle(H - z)^{-1}\big\| \leqslant Const\frac{\langle z\rangle}{|\Im(z)|}.$$

If $B = A_C/t$ then the only nonzero contribution comes from $k_j = 1$ which gives

$$\mathrm{ad}_{G_1}^1(A_C/t) = \frac{1}{t} \cdot \frac{r}{t} G_1'(r/t) \sim \frac{1}{t}.$$

We then sum up the contributions coming from each commutator and conclude the lemma. □

Remark. Combining Lemma 6.1 with (6.3) we obtain for every function $f \in C_0^\infty(\mathbb{R})$ and B any of the operators H, A_C/t, $\tilde{\boldsymbol{\rho}}$ or $\tilde{\xi}$:

$$\left\| \bigl(1 - F(r/t)\bigr) f(B) G(r/t) \right\| = \mathcal{O}\bigl(t^{-\infty}\bigr). \tag{6.11}$$

We continue with another localization result needed later.

Lemma 6.2. *Assume F_{+-} has support in $\mathbb{R}_+$ and consider $F, G, L \in C_0^\infty(\mathbb{R})$ such that F equals 1 on a neighborhood of the support of G. Assume that B and D are either H, $\tilde{\boldsymbol{\rho}}$ or $\tilde{\xi}$. Then*

$$\left\| \bigl(1 - F(B)\bigr) \cdot L(D) \cdot G(B) \cdot F_{+-}(r/t) \right\| = \mathcal{O}\bigl(t^{-\infty}\bigr). \tag{6.12}$$

Proof. Clearly if $D = B$ there is nothing to be proven. First, assume that $B = \tilde{\boldsymbol{\rho}}$ and $D = \tilde{\xi}$; we can find a function G_1 as in the previous lemma such that $G_1 G = G$ and $F G_1 = G_1$. We then see that

$$\bigl(1 - F(B)\bigr) \cdot L(D) \cdot G(B) = \bigl(1 - F(B)\bigr) \cdot \mathrm{ad}^N_{G_1(B)}\bigl(L(D)\bigr) \cdot G(B), \quad N \in \mathbb{N}.$$

Looking at (4.5) and (4.7) we see that every time we commute $\tilde{\boldsymbol{\rho}}$ with $\tilde{\xi}$ we gain an "extra $1/\bar{r}$ decay." The same thing is true when we commute $G_1(\tilde{\boldsymbol{\rho}})$ with $L(\tilde{\xi})$ via the Helffer–Sjöstrand formula; hence N commutators provide us with N extra factors of $1/\bar{r}$. In order to transform them into $1/t$ we employ the previous lemma: multiply each $1/r$ with $1 = F^{(1)}_{+-}(r/t) + 1 - F^{(1)}_{+-}(r/t)$, where $F^{(1)}_{+-}$ has the property that $(1 - F^{(1)}_{+-})F_{+-} = 0$ and its support is included in $\mathbb{R}_+$. The previous lemma ensures that the resulting cross terms containing at least one $\frac{1}{r}(1 - F^{(1)}_{+-})$ are of order $\mathcal{O}(t^{-\infty})$ while $(1/\bar{r}) \cdot F^{(1)}_{+-}(r/t) \sim t^{-1}$. We then get

$$\bigl(1 - F(\tilde{\boldsymbol{\rho}})\bigr) \cdot \mathrm{ad}^N_{G_1(\tilde{\boldsymbol{\rho}})}\bigl(L(\tilde{\xi})\bigr) \cdot G(\tilde{\boldsymbol{\rho}}) \cdot F_{+-}(r/t) = \mathcal{O}\bigl(t^{-N}\bigr), \quad N \geqslant 1.$$

Finally, let us notice that we can follow the same argument for all possible choices for B and D, since every time we commute any two of the operators H, $\tilde{\boldsymbol{\rho}}$ and $\tilde{\xi}$ we gain the extra $1/r$ decay and we can repeat the same procedure as before. □

The last technical result we present here is a quantum version of energy conservation. Consider $f_1(\tilde{\boldsymbol{\rho}})$, $f_2(\tilde{\xi})$ and $f_E(H)$, where f_1 is supported in $[\rho_0 - \epsilon_1, \rho_0 + \epsilon_1]$, f_2 is supported in $[\xi_0 - \epsilon_1, \xi_0 + \epsilon_1]$ and $f_{1,E}$ is defined right after (6.1).

Lemma 6.3. *Assume F_{+-} is supported on $\mathbb{R}_+$ and assume that $|\rho_0^2 + \xi_0^2 - 2E| > 3\tilde{\epsilon}$. Then there exists $\epsilon_1 > 0$ small enough such that for every f_1 and f_2 as above we have*

$$\left\| f_1(\tilde{\boldsymbol{\rho}}) \cdot f_2(\tilde{\xi}) \cdot f_{1,E}(H) \cdot F_{+-}(r/t) \right\| = \mathcal{O}\bigl(t^{-\infty}\bigr). \tag{6.13}$$

Proof. Assume without loss that $\rho_0^2 + \xi_0^2 - 2E > 3\tilde{\epsilon}$. Choose $\phi \in L^2(\mathbb{R}^2)$ and define $\phi(t, \epsilon_1) = f_1(\tilde{\boldsymbol{\rho}}) \cdot f_2(\tilde{\xi}) \cdot f_{1,E}(H) \cdot F_{+-}(r/t)\phi$. Let us prove that for sufficiently small ϵ_1 we may write

$$\bigl\langle \phi(t, \epsilon_1), \bigl(\tilde{\boldsymbol{\rho}}^2 + \tilde{\xi}^2 - 2H\bigr)\phi(t, \epsilon_1) \bigr\rangle \geqslant \tilde{\epsilon} \left\| \phi(t, \epsilon_1) \right\|^2 + \mathcal{O}\bigl(t^{-\infty}\bigr) \|\phi\|^2. \tag{6.14}$$

Indeed, consider the expectation in the left-hand side of the above inequality. We can put near $\tilde{\boldsymbol{\rho}}^2$ some function $\tilde{f}_1(\tilde{\boldsymbol{\rho}})$ which equals 1 on the support of f_1 and is supported on $[\rho_0 - 2\epsilon_1, \rho_0 + 2\epsilon_1]$; then put near $\tilde{\xi}^2$ another function $\tilde{f}_2(\tilde{\xi})$ which equals 1 on the support of f_2 and is supported on $[\xi_0 - 2\epsilon_1, \xi_0 + 2\epsilon_1]$. According to the previous two lemmas we have that

$$\max\bigl\{\bigl\|\bigl[1 - \tilde{f}_2(\tilde{\xi})\bigr]\phi(t, \epsilon_1)\bigr\|,\ \bigl\|\bigl[1 - \tilde{f}_1(\tilde{\boldsymbol{\rho}})\bigr]\phi(t, \epsilon_1)\bigr\|\bigr\} \leqslant \mathcal{O}\bigl(t^{-\infty}\bigr)\|\phi\|. \tag{6.15}$$

In a similar way, we may put $\tilde{f}_E(H)$ near $2H$ at the expense of another $\mathcal{O}(t^{-\infty})\|\phi\|$ error. Then we see that in the form sense we have

$$\begin{aligned}
&\tilde{\boldsymbol{\rho}}^2 \tilde{f}_1(\tilde{\boldsymbol{\rho}}) + \tilde{\xi}^2 \tilde{f}_2(\tilde{\xi}) - 2H \tilde{f}_E(H) \\
&\quad \geqslant \bigl(|\boldsymbol{\rho}_0| - 2\epsilon_1\bigr)^2 \tilde{f}_1(\tilde{\boldsymbol{\rho}}) + \bigl(|\xi_0| - 2\epsilon_1\bigr)^2 \tilde{f}_2(\tilde{\xi}) - 2(E + \tilde{\epsilon}) \tilde{f}_E(H).
\end{aligned} \tag{6.16}$$

When we take the expectation of the right-hand side of (6.16) on $\phi(t, \epsilon_1)$ we can get rid of the cut-offs $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_E$ at the price of another $\mathcal{O}(t^{-\infty})\|\phi\|$ error. Then if ϵ_1 is small enough we have

$$\bigl(|\boldsymbol{\rho}_0| - 2\epsilon_1\bigr)^2 + \bigl(|\xi_0| - 2\epsilon_1\bigr)^2 - 2(E + \tilde{\epsilon}) \geqslant \tilde{\epsilon}$$

and (6.14) follows. Now let us prove that

$$\bigl\langle \phi(t, \epsilon_1), \bigl(\tilde{\boldsymbol{\rho}}^2 + \tilde{\xi}^2 - 2H\bigr)\phi(t, \epsilon_1)\bigr\rangle \leqslant \frac{\tilde{\epsilon}}{2}\bigl\|\phi(t, \epsilon_1)\bigr\|^2 + \mathcal{O}\bigl(t^{-\infty}\bigr)\|\phi\|^2, \quad t \geqslant T_{\tilde{\epsilon}}, \tag{6.17}$$

which together with (6.14) implies (6.13). Indeed, from (4.5), (4.7) and (1.19) we see that

$$\left\| \bigl[\tilde{\boldsymbol{\rho}}^2 + \tilde{\xi}^2 - 2H\bigr]\phi(t, \epsilon_1) - \frac{1}{4\bar{r}^2}\phi(t, \epsilon_1) \right\| = \mathcal{O}\bigl(t^{-\infty}\bigr)\|\phi\|.$$

Then if t is large enough we have

$$\left\langle \phi(t, \epsilon_1), \frac{1}{4\bar{r}^2}\phi(t, \epsilon_1) \right\rangle \leqslant \frac{\tilde{\epsilon}}{2}\bigl\|\phi(t, \epsilon_1)\bigr\|^2 + \mathcal{O}\bigl(t^{-\infty}\bigr)\|\phi\|^2$$

and we are done. □

6.3. A maximal velocity bound

The notation $\langle \cdot \rangle_t$ will be used to signify the expectation in a state like (6.1) at time t. We will often slightly abuse notation by abbreviating the notation $\bar{r}$ for the function in (4.2) as r. For example, we use the notation $\langle F'(r/t)\rangle_t$ in the integral in Lemma 6.4 stated below for the expectation of the operator of multiplication by $\mathbf{x} \to F'(\bar{r}(\mathbf{x})/t)$ in the state $\psi(t)$.

A standard computation will show that

$$\|\tilde{\boldsymbol{\rho}}\| \leqslant \sqrt{2E + 2\tilde{\epsilon}}. \tag{6.18}$$

Indeed the result follows from the definition (6.2), the fact that $|\nabla r(\mathbf{x})| \leqslant 1$ and the Cauchy–Schwarz inequality:

$$\|\tilde{\boldsymbol{\rho}}\| = \sup_{\|\phi\|=1} \left|\langle \phi, \tilde{\boldsymbol{\rho}}\phi\rangle\right| \leqslant \sup_{\|\phi\|=1} \sqrt{\left\langle \tilde{f}_E(H)\phi, 2H\tilde{f}_E(H)\phi\right\rangle} \leqslant (2E + 2\tilde{\epsilon})^{1/2}.$$

Lemma 6.4. *Let* $K = \sqrt{2E+1}$*,* $a = K$*,* $b = K + 1$ *and let* F *denote either* F_+ *or* F_{++} *(see* (3.1)–(3.4)*). Then*

$$\int_1^\infty t^{-1}\left\langle F'(r/t)\right\rangle_t\, dt \leqslant C\|\psi\|^2.$$

Proof. Consider the observable

$$\Phi(t) = -\tilde{f}_E(H)F(r/t)\tilde{f}_E(H). \tag{6.19}$$

By differentiating with respect to t we get

$$\partial_t\left\langle \Phi(t)\right\rangle_t = t^{-1}\left\langle r/t\,F'(r/t)\right\rangle_t - \left\langle \tilde{f}_E(H)i\left[H, F(r/t)\right]\tilde{f}_E(H)\right\rangle_t. \tag{6.20}$$

The above commutator may be written as

$$\begin{aligned} i\left[H, F(r/t)\right] &= (2t)^{-1}\left\{\mathbf{p}\cdot(\nabla r)F'(r/t) + \text{h.c.}\right\} \\ &= (2t)^{-1}\sqrt{F'}(r/t)\{\mathbf{p}\cdot\nabla r + \text{h.c.}\}\sqrt{F'}(r/t) \\ &= t^{-1}\sqrt{F'}(r/t)\boldsymbol{\rho}\sqrt{F'}(r/t). \end{aligned} \tag{6.21}$$

By (6.3)

$$\left\|\left[\tilde{f}_E(H), \sqrt{F'}(r/t)\right]\right\| \leqslant C\left\|(H-i)^{-1}i\left[H, \sqrt{F'}(r/t)\right](H-i)^{-1}\right\|. \tag{6.22}$$

Introducing (6.21) in (6.20), commuting $\tilde{f}_E(H)$ and $\sqrt{F'}(r/t)$ (using (6.3)), and invoking that $(r/t)F'(r/t) \geqslant KF'(r/t)$, we obtain

$$\partial_t\left\langle \Phi(t)\right\rangle_t \geqslant t^{-1}\left\langle \sqrt{F'}(r/t)(K - \tilde{\boldsymbol{\rho}})\sqrt{F'}(r/t)\right\rangle_t + R_1(t;\psi), \tag{6.23}$$

where the remainder $R_1(t;\psi)$ obeys

$$\left|R_1(t;\psi)\right| \leqslant Ct^{-2}\|\psi\|^2. \tag{6.24}$$

With our choice of K, $K - \tilde{\boldsymbol{\rho}} \geqslant \sqrt{2E+1} - \sqrt{2E+2\epsilon} > 0$ (for $\epsilon < 1/2$), cf. (6.18). Applied to the right-hand side of (6.23) we obtain after an integration that for every $T > 1$

$$\int_1^T t^{-1}\left\langle F'(r/t)\right\rangle_t\, dt \leqslant \left\langle \Phi(T)\right\rangle_T - \left\langle \Phi(1)\right\rangle_1 - \int_1^T R_1(t;\psi)\, dt. \tag{6.25}$$

Finally, using the fact that $\|\Phi(T)\| \leqslant C$ uniformly in $T > 1$ and (6.24), the right-hand side of (6.25) may be estimated independently of $T > 1$, and the lemma follows. □

Corollary 6.5. *Let $C > K + 1$ (with K as in Lemma 6.4) and suppose $F_1 \in L^\infty(\mathbb{R})$ with $\operatorname{supp}(F_1) \subseteq [K+1, C]$. Then*

$$\int_1^\infty t^{-1} \| F_1(r/t)\psi(t)\|^2 \, dt \leqslant Const \|\psi\|^2.$$

Proof. Using the inputs $a = K$, $b = K + 1$, $c = C$ and $d = C + 1$ in (3.3) and (3.4) we infer that $\sqrt{F'_{++}}\, F_1 = F_1$. Then

$$\| F_1(r/t)\psi(t)\|^2 \leqslant \|F_1\|_\infty^2 \cdot \left\| \sqrt{F'_{++}}\,(r/t)\psi(t)\right\|^2 = \|F_1\|_\infty^2 \langle F'_{++}(r/t)\rangle_t, \tag{6.26}$$

and we may use Lemma 6.4 to conclude the estimate. □

Corollary 6.6. *Let $C > K + 1$ and suppose $F_1 \in C_0^\infty(\mathbb{R})$ is real-valued and that $\operatorname{supp}(F_1) \subseteq [K+1, C]$. Then*

$$\lim_{t\to\infty} \| F_1(r/t)\psi(t)\| = 0.$$

Proof. We will prove that $\eta(t) := \|F_1(r/t)\psi(t)\|^2$ goes to zero as $t \to \infty$. From Corollary 6.5 we know that there exists a sequence $(t_n)_{n\geqslant 1}$ with $t_n \to \infty$ such that

$$\lim_{n\to\infty} \eta(t_n) = 0.$$

Hence we only need to prove that η has a limit at infinity; by a standard Cook type argument, this would be true if $\eta' \in L^1((1,\infty))$.

We compute

$$\eta'(t) = -2t^{-1}\langle r/t(F_1F_1')(r/t)\rangle_t + \langle i[H, F_1^2(r/t)]\rangle_t.$$

The first term may be estimated by

$$\left|-2t^{-1}\langle r/t(F_1F_1')(r/t)\rangle_t\right| \leqslant 2Ct^{-1}\left\|\sqrt{|F_1F_1'|}\,(r/t)\psi(t)\right\|^2,$$

which is in L^1 by Corollary 6.5.

The second term can be rewritten using that $\tilde{f}_E(H)\psi = \psi$ and a computation similar to (6.21):

$$\begin{aligned} &\tilde{f}_E(H) i\left[H, F_1^2(r/t)\right]\tilde{f}_E(H) \\ &\quad = t^{-1}\tilde{f}_E(H)F_1\mathbf{p}\cdot(\nabla r)F_1'\tilde{f}_E(H) + \text{h.c.} \\ &\quad = t^{-1}F_1\tilde{f}_E(H)\mathbf{p}\cdot(\nabla r)\tilde{f}_E(H)F_1' + \text{h.c.} + R_2(t). \end{aligned} \tag{6.27}$$

Using (6.3) one can show that $\|R_2(t)\| = \mathcal{O}(t^{-2})$, cf. (6.22), and hence integrable. Let us look at

$$t^{-1}\langle F_1(r/t)\tilde{f}_E(H)\mathbf{p}\cdot(\nabla r)\tilde{f}_E(H)F_1'(r/t)\rangle_t.$$

Taking the modulus we get an upper bound for it of the form

$$\frac{1}{2t}\|\tilde{f}_E(H)\mathbf{p}\cdot(\nabla r)\tilde{f}_E(H)\|\left(\|F_1(r/t)\psi(t)\|^2 + \|F_1'(r/t)\psi(t)\|^2\right) \tag{6.28}$$

which is integrable due to Corollary 6.5. In conclusion, η' is L^1 and we are done. □

Lemma 6.7. *Suppose $C > 1$ and $F_2 \in L^\infty(\mathbb{R})$ with $\operatorname{supp}(F_2) \subset (C, \infty)$ and $\|F_2\|_\infty \leqslant 1$. Then for all ψ in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$*

$$\sup_{t\geqslant 1}\|F_2(r/t)\psi(t)\|^2 \leqslant \frac{Const(\psi)}{C^2}.$$

Proof. First, notice that due to the support condition we have

$$C^2t^2\|F_2(r/t)\psi(t)\|^2 \leqslant \|rF_2(r/t)\psi(t)\|^2 \leqslant \||x|\psi(t)\|^2.$$

Second, since $\psi \in \mathcal{S}(\mathbb{R}^2)$ we have that $\psi(t) = e^{-itH}f_E(H)\psi$ is in the domain of multiplication with any power of $|x|$ and A (the dilation generator, cf. (1.20)). By integrating the second-order derivative (from $t = 0$) we get the estimate, cf. (1.21),

$$e^{itH}\mathbf{x}^2e^{-itH} \leqslant \mathbf{x}^2 + 2tA + Const\cdot(H+1)t^2$$

which leads to

$$\begin{aligned}\||x|\psi(t)\|^2 &\leqslant \langle f_E(H)\psi, \mathbf{x}^2 f_E(H)\psi\rangle + 2t\langle f_E(H)\psi, Af_E(H)\psi\rangle \\ &\quad + Const\cdot(E+1)t^2\|f_E(H)\psi\|^2.\end{aligned}$$

Now combine the two estimates and the lemma follows. □

Proposition 6.8. *Consider the function F_+ in* (3.1) *with $a = K+1$ and $b = K+2$* (*and with $K = \sqrt{2E+1}$*). *Then for all states $\psi(t)$ as in* (6.1) *we have*

$$\lim_{t\to\infty}\|F_+(r/t)\psi(t)\| = 0.$$

Proof. What follows is an $(\varepsilon/3)$-argument. Start by fixing $\varepsilon > 0$ (this is not the ϵ used to specify the state (6.1)). There are three steps.

I. Choose $\psi_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$ such that $\|\psi - \psi_\varepsilon\| < \varepsilon/3$.

II. Denote by $\mathbf{1}_\kappa$ the characteristic function for (κ, ∞). Employing Lemma 6.7, we find $C_\varepsilon > K + 2$ such that

$$\sup_{t \geqslant 1} \left\| \mathbf{1}_{C_\varepsilon}(r/t)\psi_\varepsilon(t) \right\| < \varepsilon/3.$$

III. Consider F_{+-} in (3.3) with the same a and b as above, and with $c = C_\varepsilon$ and $d = c + 1$. Then by Corollary 6.6 there exists $T_\varepsilon > 1$ such that

$$\sup_{t \geqslant T_\varepsilon} \left\| F_{+-}(r/t)\psi_\varepsilon(t) \right\| < \varepsilon/3.$$

We decompose

$$\begin{aligned} F_+(r/t)\psi(t) = {} & F_+(r/t)\big(\psi(t) - \psi_\varepsilon(t)\big) + \big[F_+(r/t) - \mathbf{1}_{C_\varepsilon}(r/t)\big]\psi_\varepsilon(t) \\ & + \mathbf{1}_{C_\varepsilon}(r/t)\psi_\varepsilon(t), \end{aligned} \tag{6.29}$$

use the triangle inequality, then the estimate

$$\left\| \big[F_+(r/t) - \mathbf{1}_{C_\varepsilon}(r/t)\big]\psi_\varepsilon(t) \right\| \leqslant \left\| F_{+-}(r/t)\psi_\varepsilon(t) \right\|$$

and the estimates from I–III, yielding

$$\sup_{t \geqslant T_\varepsilon} \left\| F_+(r/t)\psi(t) \right\| < \varepsilon,$$

and therefore the proposition. □

As a consequence of Proposition 6.8 we define

$$F_-^{\text{M.v.b.}} := F_-^{K+1,K+2}, \qquad \psi_1(t) := F_-^{\text{M.v.b.}}(r/t)\psi(t), \tag{6.30}$$

and notice that $\lim_{t\to\infty}(\psi_1(t) - \psi(t)) = 0$. We rewrite this as

$$\psi_1(t) \sim \psi(t).$$

6.4. A minimal velocity bound

We follow the same strategy as in the classical case and we use almost the same technique (with some complications due to non-commutativity). Since we have the maximal velocity bound, we can define a regularized conjugate operator as (here $\tilde{F}_- := F_-^{K+3,K+4}$, $K = \sqrt{2E+1}$)

$$\tilde{A}_C(E) = \tilde{F}_-(r/t)\tilde{f}_E(H)A_C(E)\tilde{f}_E(H)\tilde{F}_-(r/t). \tag{6.31}$$

Clearly this operator is bounded and grows at most linearly in t:

$$\left\| \tilde{A}_C(E) \right\| \leqslant Const \cdot t. \tag{6.32}$$

We start with the quantum equivalent of Lemma 3.2.

Lemma 6.9. *For every $F_{+-} = F_{+-}^{a,b,d/2,d}$ denote by F_{++} exactly that function as in* (3.4) *whose derivative gives back F_{+-}. Then there exists $d > 0$ small enough such that we have* (*see* (6.30))

$$\int_1^\infty \big\| \sqrt{F'_{++}}\big(\tilde{A}_C(E)/t\big)\psi_1(t)\big\|^2\,dt \leqslant Const \cdot \|\psi\|^2. \tag{6.33}$$

Moreover, for $F_- = F_-^{d/4,d/2}$ we have

$$\lim_{t\to\infty} F_-\big(\tilde{A}_C(E)/t\big)\psi_1(t) = 0. \tag{6.34}$$

Proof. Define the bounded observable

$$\Phi(t) = F_-^{\text{M.v.b.}}(r/t)F_{++}\big(\tilde{A}_C(E)/t\big)F_-^{\text{M.v.b.}}(r/t)$$

and differentiate $\langle \Phi(t)\rangle_{\psi(t)}$ with respect to t and get:

$$\partial_t\big\langle \Phi(t)\big\rangle_{\psi(t)} = R_1(t) + \big\langle D_{\mathrm{H}}\big[\tilde{A}_C(E)/t\big]\big\rangle_{\sqrt{F'_{++}}\psi_1(t)}, \tag{6.35}$$

where

$$D_{\mathrm{H}}X(t) := \partial_t X(t) + i\big[H, X(t)\big] \tag{6.36}$$

denotes the Heisenberg derivative; we also employed (6.4). The above remainder $R_1(t)$ can be treated with the same methods as before and shown to obey the estimate

$$\int_1^\infty \big|R_1(t)\big|\,dt \leqslant Const \cdot \|\psi\|^2.$$

Performing the Heisenberg derivative of $\tilde{A}_C(E)/t$, we obtain several terms (see (6.31)):

$$D_{\mathrm{H}}\big[\tilde{A}_C(E)/t\big] = R_2(t) + \tilde{F}_-(r/t)\tilde{f}_E(H)\big\{D_{\mathrm{H}}\big[A_C(E)/t\big]\big\}\tilde{f}_E(H)\tilde{F}_-(r/t).$$

Using Proposition 4.1 we can write

$$D_{\mathrm{H}}\big[\tilde{A}_C(E)/t\big] \geqslant R_3(t) + \frac{1}{3t}\tilde{F}_-(r/t)\tilde{f}_E(H)^2\tilde{F}_-(r/t) - \frac{1}{t}\big[\tilde{A}_C(E)/t\big].$$

The remainder $R_3(t)$ will also be integrable in the sense that

$$\int_1^\infty \big|\big\langle R_3(t)\big\rangle_{\sqrt{F'_{++}}\psi_1(t)}\big|\,dt \leqslant Const \cdot \|\psi\|^2.$$

Due to Lemmas 6.1 and 6.2 we may write

$$\big\langle \tilde{F}_-(r/t)\tilde{f}_E(H)^2\tilde{F}_-(r/t)\big\rangle_{\sqrt{F'_{++}}\psi_1(t)} = \big\|\sqrt{F'_{++}}\big(\tilde{A}_C(E)/t\big)\psi_1(t)\big\|^2 + \mathcal{O}\big(t^{-\infty}\big)\|\psi\|^2.$$

Putting everything back into (6.35) we obtain

$$\partial_t \langle \Phi(t) \rangle_{\psi(t)} \geqslant R_4(t) + \frac{1}{t} \left\| \sqrt{F'_{++}}\big(\tilde{A}_C(E)/t\big)\psi_1(t) \right\|^2 \cdot (1/3 - d).$$

We then integrate and (6.33) follows. The proof of (6.34) uses the same strategy as the one employed in Lemma 3.2 or Corollary 6.6. □

Define (here d is the one given by Lemma 6.9)

$$F_+^{\mathrm{dil}}\big(\tilde{A}_C(E)/t\big) := F_+^{d/8,d/4}\big(\tilde{A}_C(E)/t\big), \qquad \psi_2(t) := F_+^{\mathrm{dil}}\big(\tilde{A}_C(E)/t\big)\psi_1(t). \tag{6.37}$$

Clearly, (6.34) implies that $\psi_2(t) \sim \psi_1(t) \sim \psi(t)$ when $t \to \infty$. We now are ready to prove a minimal velocity bound.

Proposition 6.10. *For $d_1 > 0$ define $F_-(r/t) := F_-^{d_1/2,d_1}(r/t)$. Then there exists d_1 small enough such that*

$$\left\| F_-(r/t)\psi_2(t) \right\| = \mathcal{O}\big(t^{-\infty}\big)\|\psi\|. \tag{6.38}$$

Proof. The strategy is proving that for d_1 small enough we have

$$\left\| F_-(r/t) \cdot F_+^{\mathrm{dil}}\big(\tilde{A}_C(E)/t\big) \right\| = \mathcal{O}\big(t^{-\infty}\big). \tag{6.39}$$

Since $\sup_{t\geqslant 1} \|\tilde{A}_C(E)/t\| = M < \infty$, we can replace F_+^{dil} with some compactly supported function $F_{+-}^{d/8,d/4,M,M+1}$ such that $F_+^{\mathrm{dil}}(\tilde{A}_C(E)/t) = F_{+-}(\tilde{A}_C(E)/t)$. Using the Helffer–Sjöstrand formula we get

$$\begin{aligned} &F_-(r/t) \cdot F_+^{\mathrm{dil}}\left(\frac{\tilde{A}_C(E)}{t}\right) \\ &\quad = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{F}_{+-}(z) \left\{ F_-(r/t) \cdot \left(\frac{\tilde{A}_C(E)}{t} - z \right)^{-1} \right\} dx\, dy. \end{aligned} \tag{6.40}$$

Define $\tilde{\tilde{A}}_C(E) = F_-^{d_1,2d_1}(r/t)\tilde{A}_C(E)F_-^{d_1,2d_1}(r/t)$ and notice that for sufficiently small d_1 and some T large enough we have

$$\sup_{t\geqslant T} \left\| \tilde{\tilde{A}}_C(E)/t \right\| \leqslant d/9$$

which implies that $F_{+-}^{d/8,d/4,M,M+1}(\tilde{\tilde{A}}_C(E)/t) = 0$ for all $t \geqslant T$. Reasoning as in Lemma 6.1 we may write that for $N \geqslant 1$

$$\left\| F_-^{d_1/2,d_1}(r/t)\big[\big(\tilde{A}_C(E)/t - z\big)^{-1} - \big(\tilde{\tilde{A}}_C(E)/t - z\big)^{-1}\big] \right\| \leqslant \frac{Const(N)}{|\Im(z)|^2} \left(\frac{\langle z \rangle}{|\Im(z)|} \right)^N t^{-N}.$$

Put this back into (6.40) and the proof is finished. □

Define with the d_1 provided by the previous proposition:

$$F_+^{\text{m.v.b.}}(r/t) := F_+^{d_1/3, d_1/2}(r/t), \qquad \psi_3(t) := F_+^{\text{m.v.b.}}(r/t)\psi_2(t). \tag{6.41}$$

Then we have shown $\psi_3(t) \sim \psi_2(t) \sim \psi_1(t) \sim \psi(t)$.

6.5. $\tilde{\boldsymbol{\rho}}$ *is localized above* $\rho_E - \epsilon$

We now formulate Proposition 2.11 in terms of quantum mechanics.

Proposition 6.11. *Fix* $\tilde{\epsilon}_2 > 0$. *Consider* $F_- = F_-^{-2\tilde{\epsilon}_2, -\tilde{\epsilon}_2}$. *Then we have*

$$\left\| F_-\big(\tilde{\boldsymbol{\rho}} - \rho_E + (\partial_E \rho_E)/C\big)\psi_3(t)\right\| = \mathcal{O}\big(t^{-\infty}\big)\|\psi\|. \tag{6.42}$$

Proof. If we look at F_+^{dil} we introduced in (6.37) we can write (we drop the energy dependence)

$$F_+^{\text{dil}}(\tilde{A}_C/t) = F_+^{\frac{d}{16C}, \frac{d}{8C}}\big(\tilde{A}_C/(tC)\big)F_+^{\text{dil}}(\tilde{A}_C/t).$$

Define

$$\Delta\rho := \tilde{\boldsymbol{\rho}} - \rho_E + (\partial_E \rho_E)/C. \tag{6.43}$$

The proposition would then be implied by the estimate

$$\left\| F_-(\Delta\rho)F_+^{\frac{d}{16C}, \frac{d}{8C}}\big(\tilde{A}_C/(tC)\big)F_+^{\text{m.v.b.}}(r/t)F_-^{\text{M.v.b.}}(r/t)f_{1,E}(H)\right\| = \mathcal{O}\big(t^{-\infty}\big). \tag{6.44}$$

The interpretation of the above estimate is that as in the classical situation, $\tilde{\boldsymbol{\rho}} - \rho_E + (\partial_E \rho_E)/C)$ and A_C tend to have the same sign. The strategy of the proof is somewhat similar to the one we have used for (6.39). As a general remark, we will often write $\mathcal{O}(t^{-\infty})$ instead of terms containing commutators of the type we encountered in Lemma 6.2.

If F_{+-} is supported on $\mathbb{R}_+$ and equals 1 on the support of $F_+^{\text{m.v.b.}}F_-^{\text{M.v.b.}}$ then define $\tilde{\tilde{A}}_C$ as

$$f_{1,E}(H)F_{+-}(r/t)F_-^{-\tilde{\epsilon}_2/2, -\tilde{\epsilon}_2/4}(\Delta\rho) \cdot \tilde{A}_C \cdot F_-^{-\tilde{\epsilon}_2/2, -\tilde{\epsilon}_2/4}(\Delta\rho)F_{+-}(r/t)f_{1,E}(H)$$

and let us prove that the spectrum of $\tilde{\tilde{A}}_C/(Ct)$ tends to be negative. Indeed, using the expressions in (6.31), (4.1) and (2.33), we first get that for $t \geqslant 1$

$$\tilde{A}_C/(Ct) = \tilde{F}_-(r/t)\tilde{f}_E(H) \cdot (\rho - \rho_E + \partial_E \rho_E/C)\frac{r}{t} \cdot \tilde{f}_E(H)\tilde{F}_-(r/t) + \mathcal{O}(1/t).$$

Because of the presence of $f_{1,E}(H)$ in the definition of $\tilde{\tilde{A}}_C$ we can use an argument like in Lemma 6.2 by enlarging a bit the support of $f_{1,E}$ and put it near $\tilde{f}_E(H)$, transforming $\tilde{f}_E(H) \cdot (\rho - \rho_E - \partial_E \rho_E/C) \cdot \tilde{f}_E(H)$ into $\Delta\rho + \mathcal{O}(t^{-\infty})$.

Then we do the same thing with the cut-off in $\Delta\rho$ thus making the "leading term" negative in form sense; we finally get

$$\tilde{\tilde{A}}_C/(Ct) \leqslant \mathcal{O}\big(t^{-1}\big), \quad t \geqslant 1.$$

It follows that there exists T large enough such that for every $t \geqslant T = T(C)$:

$$\tilde{\tilde{A}}_C/(Ct) \leqslant \frac{d}{16C}, \qquad F_+^{\frac{d}{16C},\frac{d}{8C}}\big(\tilde{\tilde{A}}_C/(Ct)\big) = 0. \tag{6.45}$$

Then reasoning as in Proposition 6.10, we can insert $F_+^{\frac{d}{16C},\frac{d}{8C}}(\tilde{\tilde{A}}_C/(Ct))$ in (6.44) and up to the use of Helffer–Sjöstrand formula and of various commutator estimates as in Lemma 6.2 we obtain the result. □

A consequence of the above proposition is

Corollary 6.12. *Fix $\tilde{\epsilon}_2 > 0$ as given by the previous proposition. Then choosing $C(E)$ large enough we have*

$$\big\| F_-^{-9\tilde{\epsilon}_2,-8\tilde{\epsilon}_2}(\tilde{\boldsymbol{\rho}} - \rho_E)\psi_3(t)\big\| = \mathcal{O}\big(t^{-\infty}\big)\|\psi\|. \tag{6.46}$$

Proof. The interpretation is again simple: if C is large then $\tilde{\boldsymbol{\rho}} - \rho_E$ cannot be too small, due to the previous proposition. If $\tilde{\rho}$ had commuted with $\partial_E\rho_E$ then this would have been automatic. But even if they do not commute, their commutator becomes small in time on the particular cut-offs which build the state $\psi_3(t)$.

Choose a function F_{+-} supported on $\mathbb{R}_+$ which satisfies

$$F_{+-}(r/t)F_+^{\text{m.v.b.}}(r/t)F_-^{\text{M.v.b.}}(r/t) = F_+^{\text{m.v.b.}}(r/t)F_-^{\text{M.v.b.}}(r/t).$$

Since we have already proven (6.42), the corollary would follow if we can prove (remember the notation $\Delta\rho = \tilde{\boldsymbol{\rho}} - \rho_E + \partial_E\rho_E/C$)

$$\big\| F_-^{-9\tilde{\epsilon}_2,-8\tilde{\epsilon}_2}(\tilde{\boldsymbol{\rho}} - \rho_E)F_+^{-3\tilde{\epsilon}_2,-2\tilde{\epsilon}_2}(\Delta\rho)F_{+-}(r/t)f_{1,E}(H)\big\| = \mathcal{O}\big(t^{-\infty}\big). \tag{6.47}$$

Denote by G_1 the function $F_+^{-5\tilde{\epsilon}_2,-4\tilde{\epsilon}_2}$ and observe that

$$G_1F_+^{-3\tilde{\epsilon}_2,-2\tilde{\epsilon}_2} = F_+^{-3\tilde{\epsilon}_2,-2\tilde{\epsilon}_2}.$$

For every $\phi \in L^2(\mathbb{R}^2)$ define

$$\Psi(t) = F_-^{-9\tilde{\epsilon}_2,-8\tilde{\epsilon}_2}(\tilde{\boldsymbol{\rho}} - \rho_E)F_+^{-3\tilde{\epsilon}_2,-2\tilde{\epsilon}_2}(\Delta\rho)F_{+-}(r/t)f_{1,E}(H)\phi.$$

Then we have

$$\begin{aligned}\tilde{\epsilon}_2\|\Psi(t)\|^2 &\leqslant \langle -7\tilde{\epsilon}_2 - (\tilde{\boldsymbol{\rho}} - \rho_E)\rangle_{\Psi(t)} \\ &= \langle -7\tilde{\epsilon}_2 - (\tilde{\boldsymbol{\rho}} - \rho_E + \partial_E\rho_E/C) + \partial_E\rho_E/C\rangle_{\Psi(t)} \\ &= \langle -7\tilde{\epsilon}_2 - (\Delta\rho)\cdot G_1(\Delta\rho)\rangle_{\Psi(t)} + \langle\partial_E\rho_E/C\rangle_{\Psi(t)} + \mathcal{O}(t^{-\infty})\|\phi\|^2. \end{aligned} \tag{6.48}$$

In the above second line we put G_1 near $\Delta\rho$ at the expense of a usual $\mathcal{O}(t^{-\infty})$ error coming from the commutations. Then on the support of G_1 we have $-7\tilde{\epsilon}_2 - (\Delta\rho)\cdot G_1(\Delta\rho) \leqslant 0$ in the form sense. Finally, if C is large enough we get $\langle\partial_E\rho_E/C\rangle_{\Psi(t)} \leqslant (\tilde{\epsilon}_2/2)\|\Psi(t)\|^2$ and we are done. □

Inspired by the last two results, we define

$$F_+^{\Delta_1}(\Delta\rho) := F_+^{-3\tilde{\epsilon}_2,-2\tilde{\epsilon}_2}(\Delta\rho), \qquad F_+^{\Delta_2}(\tilde{\rho} - \rho_E) := F_+^{-10\tilde{\epsilon}_2,-9\tilde{\epsilon}_2}(\tilde{\rho} - \rho_E) \tag{6.49}$$

and

$$\psi_4(t) = F_+^{\Delta_2}(\tilde{\rho} - \rho_E)F_+^{\Delta_1}(\Delta\rho)\psi_3(t). \tag{6.50}$$

Then we have shown $\psi_4(t) \sim \psi_3(t) \sim \cdots \sim \psi(t)$.

6.6. *$\tilde{\xi}$ is not localized on the negative axis*

We now give the quantum equivalent of Lemma 3.3. Remember that one important ingredient of the proof was (3.13) which said that the radial velocity stayed away from $-\sqrt{2E}$. That is why we start with a preliminary result:

Lemma 6.13. *Assume that $\tilde{\epsilon}_2$ entering the definition of $\psi_4(t)$ is very small. Moreover, assume that $\rho_E + \sqrt{2E} \geqslant 20\tilde{\epsilon}_2$. We then have*

$$\left\| F_-^{3\tilde{\epsilon}_2,4\tilde{\epsilon}_2}(\tilde{\boldsymbol{\rho}} + \sqrt{2E})F_+^{\Delta_2}(\tilde{\boldsymbol{\rho}} - \rho_E)F_{+-}(r/t)f_{1,E}(H)\right\| = \mathcal{O}(t^{-\infty}). \tag{6.51}$$

Proof. The interpretation of this lemma is easy. Since $\psi_4(t)$ is localized on the region, where $\tilde{\rho}$ is essentially larger than ρ_E, and since the periodic solution ρ_E is strictly larger than $-\sqrt{2E}$, then the same must be true for $\tilde{\rho}$.

The strategy we follow is similar to the one we used before. Fix $\phi \in L^2(\mathbb{R}^2)$ and define

$$\Psi(t) := F_-^{3\tilde{\epsilon}_2,4\tilde{\epsilon}_2}(\tilde{\boldsymbol{\rho}} + \sqrt{2E})F_+^{\Delta_2}(\tilde{\boldsymbol{\rho}} - \rho_E)F_{+-}(r/t)f_{1,E}(H)\phi.$$

Define $G_2 := F_+^{-11\tilde{\epsilon}_2,-10\tilde{\epsilon}_2}$ and notice that

$$F_+^{\Delta_2}(\tilde{\rho} - \rho_E)\cdot G_2(\tilde{\rho} - \rho_E) = F_+^{\Delta_2}(\tilde{\rho} - \rho_E).$$

Then we have

$$\begin{aligned}\tilde{\epsilon}_2\|\Psi(t)\|^2 &\leqslant \langle 5\tilde{\epsilon}_2 - (\tilde{\boldsymbol{\rho}} + \sqrt{2E})\rangle_{\Psi(t)} \\ &= \langle 5\tilde{\epsilon}_2 - (\tilde{\boldsymbol{\rho}} - \rho_E) - (\sqrt{2E} + \rho_E)\rangle_{\Psi(t)} \\ &\leqslant \langle 5\tilde{\epsilon}_2 - (\tilde{\boldsymbol{\rho}} - \rho_E)\cdot G_2(\tilde{\boldsymbol{\rho}} - \rho_E) - 20\tilde{\epsilon}_2\rangle_{\Psi(t)} + \mathcal{O}(t^{-\infty})\|\phi\|^2, \end{aligned} \tag{6.52}$$

where we put a G_2 near $\tilde{\boldsymbol{\rho}} - \rho_E$ at the usual expense and replaced $-(\sqrt{2E} + \rho_E)$ by $-20\tilde{\epsilon}_2$. It follows that the first term in the second line is negative and the proof follows. □

Define a new "evolved" state

$$\psi_5(t) = F_+^{\tilde{\epsilon}_2, 2\tilde{\epsilon}_2}(\tilde{\boldsymbol{\rho}} + \sqrt{2E}\,)\psi_4(t). \tag{6.53}$$

Then we have shown $\psi_5(t) \sim \psi_4(t) \sim \cdots \sim \psi(t)$.

We finally give the equivalent of Lemma 3.3 in quantum mechanics.

Lemma 6.14. *There exists $d_2 > 0$ small enough so that for $F_{++}^{-d_2, -d_2/2, d_2/2, d_2}$*

$$\int_1^\infty \frac{1}{t} \left\| \sqrt{F'_{++}}(\tilde{\xi})\psi_5(t) \right\|^2 dt \leqslant Const \cdot \|\psi\|^2. \tag{6.54}$$

Moreover, if $0 < d_3 < d_2/2$ is also sufficiently small then for any F_{+-} with support in $[-d_3, d_3]$ we have

$$\lim_{t\to\infty} F_{+-}(\tilde{\xi})\psi_5(t) = 0. \tag{6.55}$$

Proof. Using all the cut-offs entering in $\psi_5(t)$ we define a bounded and symmetric propagation observable $\Phi(t)$ in such a way that when taking the expectation on a state like in (6.1) we get

$$\langle \Phi(t) \rangle_{\psi(t)} = \langle F_{++}(\tilde{\xi}) \rangle_{\psi_5(t)}.$$

When we differentiate such an expectation with respect to t we are led to the Heisenberg derivative of each cut-off function; in the process we obtain a number of terms which can be regrouped as

$$\partial_t \langle \Phi(t) \rangle_{\psi(t)} = \langle D_{\mathrm{H}}\Phi(t) \rangle_{\psi(t)} = \langle D_{\mathrm{H}}\tilde{\xi} \rangle_{\sqrt{F'_{++}}(\tilde{\xi})\psi_5(t)} + R_1(t). \tag{6.56}$$

As usual, $R_1(t)$ is just a remainder which can be shown (based on the previously obtained propagation estimates) to behave like

$$\int_1^\infty \left| R_1(t) \right| dt \leqslant Const \cdot \|\psi\|^2.$$

Equation (6.56) is the quantum equivalent of (3.12). When we perform the Heisenberg derivative of $\tilde{\xi}$, we obtain two leading terms (i.e. behaving like $1/t$ due to the various cut-offs). Because of the same cut-offs which build $\psi_5(t)$ we can essentially repeat the proof of Lemma 3.3; the non-commutativity is bypassed by putting other cut-offs with larger (or smaller) support near the relevant operators. Let us only mention that here is the place, where Lemma 6.3 comes into play and forces $\tilde{\rho}$ to stay near $\sqrt{2E}$ and hence to be positive. Further details are omitted. □

The next step is proving a quantum equivalent of Lemma 3.4. One can see that the classical computations we did there can easily be translated into the quantum language, as it was the case with the previous lemma. That is why we only formulate the result in terms of adding a new cut-off on our state: if $d_1 > 0$ is the one obtained in Lemma 3.3 then

$$\psi_6(t) = F_+^{d_1/2, d_1}(\tilde{\xi})\psi_5(t). \tag{6.57}$$

We have shown $\psi_6(t) \sim \psi_5(t) \sim \cdots \sim \psi(t)$.

6.7. $\tilde{\rho}$ is localized below $\rho_E + \epsilon$

Remember that $\psi_6(t)$ (and already $\psi_4(t)$ in (6.50)) contains a localization of $\tilde{\rho}$ above $\rho_E - 9\tilde{\epsilon}_2$. We now want to prove the analog of Proposition 3.5 in quantum mechanics, which would provide us with an upper bound for $\tilde{\rho}$ of the form $\rho_E + \epsilon$.

We first start with a propagation estimate of the same type as the one in (3.21); we employ the notations introduced in Lemma 3.6.

Lemma 6.15. *Let $\epsilon := 9\tilde{\epsilon}_2$ and define ϵ_0 as in Lemma* 3.6. *For every $\epsilon_2 > 0$ denote by $F_{+-} = F_{+-}^{-10\epsilon_2, -9\epsilon_2, 9\epsilon_2, 10\epsilon_2}$ and by F_{++} precisely that function as in* (3.4) *for which we have $F'_{++} = F_{+-}$. For every $E' \in [E + \epsilon_0, M]$ define $B_{E'}(t) := \tilde{\rho} - \rho_{E'}$. Then there exists $\epsilon_2 > 0$ small enough such that uniformly in E' we have*

$$\int_1^\infty \frac{1}{t} \left\| \sqrt{F'_{++}}\big(B_{E'}(t)\big) \cdot \psi_6(t) \right\|^2 dt \leqslant Const \cdot \|\psi\|^2 \tag{6.58}$$

and

$$\lim_{t\to\infty} F_{+-}^{-8\epsilon_2, -7\epsilon_2, 7\epsilon_2, 8\epsilon_2}\big(B_{E'}(t)\big)\psi_6(t) = 0. \tag{6.59}$$

Proof. We only prove (6.58). Remember that ϵ_0 is fixed and proportional to $\tilde{\epsilon}_2$ we obtained in the previous subsection. Using all the cut-offs entering $\psi_6(t)$ we define a bounded and symmetric propagation observable $\Phi(t)$ in such a way that when taking the expectation on a state like in (6.1) we get

$$\big\langle \Phi(t) \big\rangle_{\psi(t)} = -\big\langle F_{++}\big(B_{E'}(t)\big)\big\rangle_{\psi_6(t)}.$$

Differentiate with respect to t as in the classical case (see (3.23)) and notice that the "interesting" term is going to be

$$\big\langle \tilde{\xi} \cdot (\xi_{E'} - \tilde{\xi})\big\rangle_{r^{-1/2}\sqrt{F'_{++}}(B_{E'}(t))\cdot\psi_6(t)}.$$

We then perform the same manipulations as we did in order to get (3.24); before that we have to give a proper sense to the inverse of $\tilde{\xi} + \xi_{E'}$. This can be done because ψ_6 contains the localization of $\tilde{\xi}$ on the positive semi-axis (see (6.57)). Indeed, since $\xi_{E'}$ is strictly positive, it is enough to use

$$\big(F_+^{0, d_1/4}(\tilde{\xi}) \cdot \tilde{\xi} + \xi_{E'}\big)^{-1} \geqslant Const > 0.$$

More precisely, we consider the product

$$(\xi_{E'} - \tilde{\xi}) \cdot \big(F_+^{0,d_1/4}(\tilde{\xi}) \cdot \tilde{\xi} + \xi_{E'}\big) = 2(E' - H) + (\tilde{\rho} + \rho_{E'}) \cdot B_{E'}(t) + \mathit{small}(t), \quad (6.60)$$

where $\mathit{small}(t)$ means that after taking the expectation it will converge to zero. Remember that H is localized in a very narrow interval around E (of width $\tilde{\epsilon}$, see (6.1)) Hence reasoning as in the classical case we eventually obtain the desired positivity by making ϵ_2 sufficiently small and we can integrate in the usual way. □

We now give the quantum version of Proposition 3.5.

Proposition 6.16. *Let* $\epsilon = 9\tilde{\epsilon}_2$. *Then*

$$\lim_{t\to\infty} F_+^{10\epsilon,11\epsilon}(\tilde{\rho} - \rho_E)\psi_6(t) = 0. \quad (6.61)$$

Proof. What we need first is to restate (3.26) and (3.27) in terms of operators. Look first at the right-hand side of (3.26). We want to get rid of the θ dependence by introducing a partition of unity in the angular variable. So we can write that there exists J large enough such that

$$F_{+-}^{-2\epsilon_2,-\epsilon_2,\epsilon_2,2\epsilon_2}\big(x - \rho_{E'_k}(\theta)\big) \leqslant \sum_{j=1}^{J} F_{+-}^{-3\epsilon_2,-2\epsilon_2,2\epsilon_2,3\epsilon_2}\big(x - \rho_{E'_k}(2\pi j/J)\big) \cdot \chi_j(\theta), \quad (6.62)$$

where χ_j are functions of the F_{+-} type, $\sum_{j=1}^{J} \chi_j(\theta) = 1$ and the support of each χ_j is sharply localized around $2\pi j/J$. Notice that by fixing the angle to $\theta_j = 2\pi j/J$ we had to enlarge the region, where F_{+-} equals 1. Using (3.26), there exists a large enough N (depending on J) such that for any $1 \leqslant j \leqslant J$ and $x \in \mathbb{R}$ we can write

$$F_+^{6\epsilon,7\epsilon}\big(x - \rho_E(\theta_j)\big) \leqslant \frac{1}{c}\sum_{k=1}^{N} F_{+-}^{-3\epsilon_2,-2\epsilon_2,2\epsilon_2,3\epsilon_2}\big(x - \rho_{E'_k}(\theta_j)\big). \quad (6.63)$$

Let us state a technical result.

Lemma 6.17. *Define* $\Psi(t) := F_+^{10\epsilon,11\epsilon}(\tilde{\rho} - \rho_E)\psi_6(t)$. *Denote by* $\theta_j := 2\pi j/J$. *Then for* J *large enough, there exists* N *and* T *such that if* $t > T$ *we have*

$$\sum_{k=1}^{N}\sum_{j=1}^{J}\big\langle F_{+-}^{-3\epsilon_2,-2\epsilon_2,2\epsilon_2,3\epsilon_2}\big(\tilde{\rho} - \rho_{E'_k}(\theta_j)\big)\big\rangle_{\sqrt{\chi_j}\Psi(t)} \geqslant \frac{c}{2}\big\|\Psi(t)\big\|^2 - \mathcal{O}\big(t^{-\infty}\big)\|\psi\|^2. \quad (6.64)$$

Proof. Using the spectral theorem, we can replace x by $\tilde{\rho}$ in (6.63) and obtain a form inequality. This leads to

$$\begin{aligned} &\sum_{j=1}^{J}\sqrt{\chi_j}(\theta) \cdot F_+^{6\epsilon,7\epsilon}\big(\tilde{\rho} - \rho_E(\theta_j)\big)\sqrt{\chi_j}(\theta) \\ &\quad\leqslant \sum_{k=1}^{N}\sum_{j=1}^{J}\sqrt{\chi_j}(\theta) \cdot \frac{1}{c}F_{+-}^{-3\epsilon_2,-2\epsilon_2,2\epsilon_2,3\epsilon_2}\big(\tilde{\rho} - \rho_{E'_k}(\theta_j)\big) \cdot \sqrt{\chi_j}(\theta). \end{aligned} \quad (6.65)$$

Then the left-hand side of (6.64) will be bounded from below by

$$c\sum_{j=1}^{J}\big\langle \Psi(t), \sqrt{\chi_j}\cdot F_+^{6\epsilon,7\epsilon}\big(\tilde\rho-\rho_E(\theta_j)\big)\cdot\sqrt{\chi_j}\Psi(t)\big\rangle.$$

Remember that $\Psi(t)$ lives in the range of $F_+^{10\epsilon,11\epsilon}(\tilde\rho-\rho_E)$ hence we can rewrite the above sum as

$$\sum_{j=1}^{J}\big\langle \Psi(t), \sqrt{\chi_j}\cdot F_+^{8\epsilon,9\epsilon}(\tilde\rho-\rho_E)\cdot F_+^{6\epsilon,7\epsilon}\big(\tilde\rho-\rho_E(\theta_j)\big)\cdot\sqrt{\chi_j}\Psi(t)\big\rangle + \mathcal{O}\big(t^{-\infty}\big)\|\psi\|^2, \tag{6.66}$$

since the error introduced by the commutation with χ_j and the other F_+ is of order $t^{-\infty}$ due to the cut-offs which build Ψ. The next step is proving that we can get rid of $F_+^{6\epsilon,7\epsilon}(\tilde\rho-\rho_E(\theta_j))$ if the angular partition is fine enough. In fact, we show that

$$\big\|F_+^{8\epsilon,9\epsilon}(\tilde\rho-\rho_E)\cdot F_-^{6\epsilon,7\epsilon}\big(\tilde\rho-\rho_E(\theta_j)\big)\cdot\sqrt{\chi_j}\Psi(t)\big\| = \mathcal{O}\big(t^{-\infty}\big)\|\psi\|. \tag{6.67}$$

Indeed, denote by

$$\Psi_1(t) := F_+^{8\epsilon,9\epsilon}(\tilde\rho-\rho_E)\cdot F_-^{6\epsilon,7\epsilon}\big(\tilde\rho-\rho_E(\theta_j)\big)\cdot\sqrt{\chi_j}\,\Psi(t).$$

We then have

$$\begin{aligned}8\epsilon\big\|\Psi_1(t)\big\|^2 &\leqslant \langle\tilde\rho-\rho_E\rangle_{\Psi_1(t)}\\ &= \big\langle\tilde\rho-\rho_E(\theta_j)\big\rangle_{\Psi_1(t)} + o(1)\cdot\big\|\Psi_1(t)\big\|^2 + \mathcal{O}\big(t^{-\infty}\big)\|\psi\|^2,\end{aligned} \tag{6.68}$$

where we used that $\rho_E(\theta)-\rho_E(\theta_j)$ can be made as small as we want if χ_j has a sharp support; simply put $\tilde\chi_j(\theta)$ near it, where $\tilde\chi_j$ has a slightly larger support than χ_j. We then put near $\tilde\rho-\rho_E(\theta_j)$ a factor of $F_-^{29\epsilon/4,31\epsilon/4}(\tilde\rho-\rho_E(\theta_j))$; the price we pay is again of order $\mathcal{O}(t^{-\infty})\|\psi\|^2$. Hence

$$\begin{aligned}8\epsilon\big\|\Psi_1(t)\big\|^2 &\leqslant \big\langle\big[\tilde\rho-\rho_E(\theta_j)\big]\cdot F_-^{29\epsilon/4,31\epsilon/4}\big(\tilde\rho-\rho_E(\theta_j)\big)\big\rangle_{\Psi_1(t)}\\ &\quad + o(1)\cdot\big\|\Psi_1(t)\big\|^2 + \mathcal{O}\big(t^{-\infty}\big)\big\|\Psi(t)\big\|^2\\ &\leqslant \big(31\epsilon/4 + o(1)\big)\cdot\big\|\Psi_1(t)\big\|^2 + \mathcal{O}\big(t^{-\infty}\big)\|\psi\|^2\end{aligned} \tag{6.69}$$

which ends the proof of (6.67). Now go back to (6.66) and replace $F_+^{6\epsilon,7\epsilon}(\tilde\rho-\rho_E(\theta_j))$ by 1. Finally, replace $F_+^{8\epsilon,9\epsilon}(\tilde\rho-\rho_E)$ by 1 because Ψ is in the range of $F_+^{10\epsilon,11\epsilon}(\tilde\rho-\rho_E)$ and the lemma is proven. □

We now continue the proof of (6.61). Consider separately each term in the left-hand side of (6.64). Denote by (at k and j fixed)

$$\Psi_2(t) := \sqrt{F_{+-}^{-3\epsilon_2,-2\epsilon_2,2\epsilon_2,3\epsilon_2}}\,\big(\tilde{\rho} - \rho_{E'_k}(\theta_j)\big)\sqrt{\chi_j}\,\Psi(t).$$

Notice that if we prove that

$$\big\|\Psi_2(t)\big\|^2 \leqslant \big\langle F_{+-}^{-7\epsilon_2,-6\epsilon_2,6\epsilon_2,7\epsilon_2}(\tilde{\rho} - \rho_{E'_k})\big\rangle_{\Psi_2(t)} + \mathcal{O}\big(t^{-\infty}\big)\|\psi\|^2 \tag{6.70}$$

then we can apply (6.59) (up to another small enlargement of the support). The proof of (6.70) is very similar to what we did in (6.67). Namely, we show that

$$\big\|F_{+}^{6\epsilon_2,7\epsilon_2}(\tilde{\rho} - \rho_{E'_k}) \cdot \Psi_2(t)\big\| = \mathcal{O}\big(t^{-\infty}\big)\|\psi\| \tag{6.71}$$

and

$$\big\|F_{-}^{-7\epsilon_2,-6\epsilon_2}(\tilde{\rho} - \rho_{E'_k}) \cdot \Psi_2(t)\big\| = \mathcal{O}\big(t^{-\infty}\big)\|\psi\|, \tag{6.72}$$

which is true provided the angular partition is fine enough. Hence Proposition 6.16 is proven. □

The above result says that we can put another cut-off on $\psi(t)$ (remember that $\epsilon = 9\tilde{\epsilon}_2$):

$$F_{-}^{12\epsilon,13\epsilon}(\tilde{\rho} - \rho_E)\psi_6(t) \sim \psi(t). \tag{6.73}$$

For further purposes, we are forced to replace it with another cut-off involving $\rho_{E'}$ with $E' = E + 2\epsilon_0$; remember that ϵ_0 in Lemmas 3.6 and 6.15 was small and proportional to ϵ. We see that

$$\tilde{\rho} - \rho_E \leqslant 13\epsilon \quad \text{and} \quad \tilde{\rho} - \rho_{E'} \geqslant 30\epsilon + 2\epsilon_0\Big(\sup_\theta \partial_E \rho_E\Big) := \eta_1 \tag{6.74}$$

are classically incompatible, thus we can rewrite (6.73) as

$$F_{-}^{\eta_1,\eta_1+\epsilon_2}(\tilde{\rho} - \rho_{E'})\psi_6(t) \sim \psi(t), \tag{6.75}$$

up to a $\mathcal{O}(t^{-\infty})\|\psi\|$ error. The reason for doing this replacement will appear clear in the proof of Proposition 8.1.

Since we would still like to have an explicit upper bound for $\tilde{\rho} - \rho_E$, we again notice that

$$\tilde{\rho} - \rho_E \geqslant 2\eta_1 + \epsilon_2 \quad \text{and} \quad \tilde{\rho} - \rho_{E'} \leqslant \eta_1 + \epsilon_2 \tag{6.76}$$

are classically incompatible, thus we can put $F_{-}^{2\eta_1+\epsilon_2,2\eta_1+2\epsilon_2}(\tilde{\rho} - \rho_E)$ on the left-hand side of (6.75) at the expense of a $\mathcal{O}(t^{-\infty})\|\psi\|$ error.

Now define

$$\psi_7(t) := F_{-}^{2\eta_1+\epsilon_2,2\eta_1+2\epsilon_2}(\tilde{\rho} - \rho_E)F_{-}^{\eta_1,\eta_1+\epsilon_2}(\tilde{\rho} - \rho_{E'})\psi_6(t). \tag{6.77}$$

We have shown $\psi_7(t) \sim \psi_6(t) \sim \cdots \sim \psi(t)$.

Then the new state has the property that $\tilde{\rho}$ is effectively localized in a very narrow band around $\rho_E(\theta)$ (see also (6.50)). We are now prepared to show a similar strong localization for $\tilde{\xi}$ around ξ_E.

Corollary 6.18. *Let $\eta_2 := 2\eta_1 + 2\epsilon_2$. Then there exists a constant $M > 0$ large enough so that*

$$\begin{aligned} \left\| F_-^{-(M+1)\sqrt{\eta_2}, -M\sqrt{\eta_2}}(\tilde{\xi} - \xi_E) \cdot \psi_7(t) \right\| &= \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|, \\ \left\| F_+^{M\sqrt{\eta_2}, (M+1)\sqrt{\eta_2}}(\tilde{\xi} - \xi_E) \cdot \psi_7(t) \right\| &= \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|. \end{aligned} \tag{6.78}$$

Proof. We only consider the first norm in (6.78), the other one being analogous. The idea is to use energy conservation together with the already known localization for $\tilde{\rho}$. Define as usual

$$\Psi(t) := F_-^{-(M+1)\sqrt{\eta_2}, -M\sqrt{\eta_2}}(\tilde{\xi} - \xi_E) \cdot \psi_7(t).$$

Then

$$\begin{aligned} M^2\eta_2 \left\| \Psi(t) \right\|^2 &\leqslant \big\langle (\tilde{\xi} - \xi_E)^2 \big\rangle_{\psi(t)} \\ &= \big\langle -\tilde{\xi}^2 - \tilde{\xi} \cdot (\xi_E - \tilde{\xi}) - (\xi_E - \tilde{\xi}) \cdot \tilde{\xi} + \xi_E^2 \big\rangle_{\Psi(t)}. \end{aligned} \tag{6.79}$$

Since we know that on the support of the cut-offs in $\Psi(t)$ we have

$$\tilde{\xi}^2 - \xi_E^2 = 2(H - E) - \tilde{\rho}^2 + \rho_E^2 \sim \tilde{\epsilon} + \eta_2$$

we can rewrite up to the usual errors introduced by commutations (6.79) as

$$\begin{aligned} M^2\eta_2 \left\| \Psi(t) \right\|^2 \leqslant{} & Const \cdot \eta_2 \left\| \Psi(t) \right\|^2 + \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|^2 \\ & - \big\langle \tilde{\xi} \cdot (\xi_E - \tilde{\xi}) + (\xi_E - \tilde{\xi}) \cdot \tilde{\xi} \big\rangle_{\Psi(t)}. \end{aligned} \tag{6.80}$$

Now remember (see (6.57)) that ψ_7 has a localization for $\tilde{\xi}$ above zero; so we can write $\tilde{\xi} = F^2(\tilde{\xi})$, where F is a smooth version of the square root on the positive semi-axis. Hence

$$\begin{aligned} -\big\langle \tilde{\xi} \cdot (\xi_E - \tilde{\xi}) + (\xi_E - \tilde{\xi}) \cdot \tilde{\xi} \big\rangle_{\Psi(t)} ={} & 2\big\langle F(\tilde{\xi}) \cdot (\tilde{\xi} - \xi_E) \cdot F(\tilde{\xi}) \big\rangle_{\Psi(t)} \\ & + \mathcal{O}\big(t^{-1}\big) \left\| \Psi(t) \right\|^2 + \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|^2, \end{aligned} \tag{6.81}$$

where the term $\mathcal{O}(t^{-1})\|\Psi(t)\|^2$ comes from commuting $F(\tilde{\xi})$ with ξ_E. But now the expectation on the right-hand side of (6.81) is essentially negative because of the extra cut-off on $\Psi(t)$. Therefore (6.80) becomes

$$M^2\eta_2 \left\| \Psi(t) \right\|^2 \leqslant \big(Const \cdot \eta_2 + \mathcal{O}\big(t^{-1}\big) \big) \cdot \left\| \Psi(t) \right\|^2 + \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|^2, \tag{6.82}$$

hence choosing M large enough the corollary is proven. □

With the M provided by the above corollary define

$$F^M_{+-} := F_{+-}^{-(M+2)\sqrt{\eta_2},-(M+1)\sqrt{\eta_2},(M+1)\sqrt{\eta_2},(M+2)\sqrt{\eta_2}},$$
$$\psi_8(t) := F^M_{+-}(\tilde{\xi} - \xi_E) \cdot \psi_7(t). \tag{6.83}$$

We have shown $\psi_8(t) \sim \psi_7(t) \sim \cdots \sim \psi(t)$.

6.8. $-\partial_t S(t, r, \theta)$ is close to H

The last quantity which we would like to know that is a priori small on our state is $H + \partial_t S(t, r, \theta)$. We first give a quantum version for Proposition 3.10.

Proposition 6.19. *Take $F^{\text{v.b.}}$ to be of the type* (3.3) *with support on $\mathbb{R}_+$ and equal to* 1 *on the support of the minimal and maximal velocity cut-offs. Abbreviate $-\partial_t S(t, r, \theta)$ by E_t and define*

$$D(t) := 1 - \frac{r}{t}\partial_E \rho_E(\theta) \cdot F^{\text{v.b.}}(r/t) = 1 - \frac{\partial_E \rho_E}{\partial_E \rho_{E_t}} \cdot F^{\text{v.b.}}(r/t).$$

Denote by $c := 1 + \sup_{t\geqslant 1} \|D(t)\| < \infty$. For $0 < \epsilon_3 < 1/3$ denote by F_{+-} either the function $F_{+-}^{2\epsilon_3,3\epsilon_3,c,c+1}$ or $F_{+-}^{-c-1,-c,-3\epsilon_3,-2\epsilon_3}$. Then

$$\int_1^\infty \frac{1}{t} \left\| \sqrt{F_{+-}}\,\big(D(t)\big) \cdot \psi_8(t) \right\|^2 dt \leqslant Const \cdot \|\psi\|^2. \tag{6.84}$$

Proof. The proof is similar in spirit to the one we gave in the classical case. Assume first that we work with $F_{+-} = F_{+-}^{2\epsilon_3,3\epsilon_3,c,c+1}$. Define a bounded and symmetric propagation observable $\Phi(t)$ in such a way that when taking the expectation on a state like in (6.1) we get

$$\big\langle \Phi(t) \big\rangle_{\psi(t)} = -\big\langle F_{++}\big(D(t)\big)\big\rangle_{\psi_8(t)}.$$

Differentiate with respect to t as in the classical case (see (3.46)) and notice that the "interesting" term is going to be

$$\left\langle \frac{1}{t} \cdot \big[D(t) + \mathcal{O}(\epsilon)\big] \right\rangle_{\sqrt{F'_{++}}\,(D(t))\cdot\psi_8(t)} \geqslant \frac{2\epsilon_3 - \mathcal{O}(\epsilon)}{t} \left\| \sqrt{F'_{++}}\,\big(D(t)\big) \cdot \psi_8(t) \right\|^2,$$

where we employed the positivity of D on the support of F'_{++} together with the smallness of $\tilde{\rho} - \rho_E$ and $\tilde{\xi} - \xi_E$. Then we integrate and get the result in (6.84). The case where F_{+-} is supported on the negative axis is similar, only the propagation observable has to be taken with an opposite sign. The proposition is proven. □

We are now ready to add a new cut-off to our state. If ϵ_3 is as given by the above proposition, define

$$F_D := F_{+-}^{-4\epsilon_3,-3\epsilon_3,3\epsilon_3,4\epsilon_3}, \qquad \psi_9(t) := F_D\big(D(t)\big)\psi_8(t). \tag{6.85}$$

We have shown $\psi_9(t) \sim \psi_8(t) \sim \cdots \sim \psi(t)$.

Finally, we can show that on our state H can be well approximated by $-\partial_t S(t,r,\theta)$. We formulate this as a proposition.

Proposition 6.20. *If $M > 0$ consider $F_{+-} = F_{+-}^{-2M\epsilon_3,-M\epsilon_3,M\epsilon_3,2M\epsilon_3}$. Denote again $-\partial_t S(t,r,\theta)$ by E_t and define ($F^{\text{v.b.}}$ is as in the previous proposition)*

$$\tilde{\gamma}_3 := H \cdot \tilde{f}_E(H) + \partial_t S(t,r,\theta) F^{\text{v.b.}}(r/t). \tag{6.86}$$

Then there exists M large enough such that

$$\big\|\big(\mathbf{1} - F_{+-}(\tilde{\gamma}_3)\big)\psi_9(t)\big\| = \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|. \tag{6.87}$$

Proof. The argument relies on the classical analog that can be traced back to (3.48), (3.50) and (3.45). Define

$$\Psi(t) := F_{+}^{M\epsilon_3,2M\epsilon_3}(\tilde{\gamma}_3)\psi_9(t)$$

and compute

$$M\epsilon_3\big\|\Psi(t)\big\|^2 \leqslant \langle\tilde{\gamma}_3\rangle_{\Psi(t)} \leqslant \langle E - E_t + \tilde{\epsilon}\rangle_{\Psi(t)} + \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|^2.$$

But then $|E - E_t|$ can be bounded by $|D|$ times a (big) constant so it can be made smaller than a constant times ϵ_3. Hence if we take M large enough we obtain $\|\Psi(t)\| = \mathcal{O}(t^{-\infty}) \cdot \|\psi\|$. We then follow the same argument "on the other side" and the proposition is proven. □

Now let us introduce another cut-off on our state. If ϵ_3 and M are as given above, define

$$F_E := F_{+-}^{-3M\epsilon_3,-2M\epsilon_3,2M\epsilon_3,3M\epsilon_3}, \qquad \psi_{10}(t) := F_E(\tilde{\gamma}_3)\psi_9(t). \tag{6.88}$$

We have shown $\psi_{10}(t) \sim \psi_9(t) \sim \cdots \sim \psi(t)$.

7. Asymptotic completeness: $\gamma_1^2 + \gamma_2^2$ is integrable

If we look back at (4.19) we see that it would be good to know that when applied on a state like the one we have in (6.1), the "perturbation" $\gamma_1^2 + \gamma_2^2$ decays at least like $t^{-1-\delta}$. This would mean that a Cook-type argument for the existence of Ω_+ could be possible.

We first introduce a regularized version of our gammas. Define (see Propositions 6.19 and 6.20 for various notations)

$$\tilde{\gamma}_1 := \tilde{\rho} - \rho_{E_t}(\theta) \cdot F^{\text{v.b.}}(r/t), \tag{7.1}$$

$$\tilde{\gamma}_2 := \tilde{\xi} - \xi_{E_t}(\theta) \cdot F^{\text{v.b.}}(r/t). \tag{7.2}$$

The main result of this section will be a quantum equivalent of Lemma 3.14. We will try to follow the same steps as we did in the classical case since there is a close analogy with that situation. Of course, here the technique is more involved since we now have to deal with non-commutativity. But still, the main idea is the same: find a Liapunov-type function of the $\tilde{\gamma}$'s whose Heisenberg derivative obeys a certain inequality (see (3.128) for the classical counterpart).

In classical mechanics it was very easy to go from (3.128) to (3.129) because everything commutes. In quantum mechanics we have to be more careful with remainders.

7.1. A quantum version for (3.128)

As we have said before, we closely follow the steps we took in the proof of Lemma 3.14. Remember the "third" gamma we defined in (6.86) which should replace (3.105).

We now introduce the triplet of quantum $\hat{\gamma}$'s (see (3.106)). In order to simplify the writing we adopt the same notation for them as in the classical case. Define

$$\hat{\gamma}_1 := \tilde{\gamma}_1 - \frac{1}{2}\big[(\partial_E \rho_{E_t}) \cdot \tilde{\gamma}_3 + \text{h.c.}\big], \qquad \hat{\gamma}_2 := \tilde{\gamma}_2 - \frac{1}{2}\big[(\partial_E \xi_{E_t}) \cdot \tilde{\gamma}_3 + \text{h.c.}\big], \qquad \hat{\gamma}_3 := \tilde{\gamma}_3, \tag{7.3}$$

where as usual, h.c. means Hermitian conjugate. Notice that the a priori smallness we established for the $\tilde{\gamma}$'s in the previous section can be easily transferred to $\hat{\gamma}$'s at the expense of introducing some other cut-offs. That is, we can prove the existence of M large enough so that for small $\epsilon > 0$ we have

$$\big\|\big[\mathbf{1} - F_{+-}^{-2M\epsilon,-M\epsilon,M\epsilon,2M\epsilon}(\hat{\gamma}_j)\big] \cdot \psi_{10}(t)\big\| = \mathcal{O}\big(t^{-\infty}\big) \cdot \|\psi\|, \quad j \in \{1,2,3\}. \tag{7.4}$$

Let us define the state which also takes into account the smallness on $\hat{\gamma}_1$ and $\hat{\gamma}_2$ (remember that $\hat{\gamma}_3 = \tilde{\gamma}_3$):

$$\hat{F} := F_{+-}^{-4M\epsilon,-3M\epsilon,3M\epsilon,4M\epsilon}, \qquad \psi_{11}(t) := \hat{F}(\hat{\gamma}_1)\hat{F}(\hat{\gamma}_2)\psi_{10}(t). \tag{7.5}$$

We have shown $\psi_{11}(t) \sim \psi_{10}(t) \sim \cdots \sim \psi(t)$.

We now are interested in writing down the Heisenberg derivatives for the $\hat{\gamma}$'s (see (6.36)). As a general rule, performing the Heisenberg derivative generates (up to some commutators) the same result as if the computation was done in classical mechanics by performing the time derivative on a given classical orbit.

Remark. Since at the end we will apply everything on states containing all (eleven by now, at least) cut-offs, we make the convention of denoting by $\mathcal{O}(t^{-n})$ any term which contains a decay of order $1/r^n$, $n \geqslant 1$. Typically such terms will arise from various commutators or conservation laws. This means that we can often neglect the non-commutativity.

Ia. We start with the constraints we have on the $\hat{\gamma}$'s coming from energy conservation. The first one is

$$\begin{aligned} 2H\tilde{f}_E(H) &= \tilde{\rho}^2 + \tilde{\xi}^2 + \mathcal{O}\big(t^{-2}\big) = (\tilde{\gamma}_1 + \rho_{E_t})^2 + (\tilde{\gamma}_2 + \xi_{E_t})^2 + \mathcal{O}\big(t^{-2}\big) \\ &= 2E_t + (\tilde{\gamma}_1 \cdot \rho_{E_t} + \tilde{\gamma}_2 \cdot \xi_{E_t} + \text{h.c.}) + \mathcal{O}\big(\tilde{\gamma}_1^2, \tilde{\gamma}_2^2, t^{-2}\big). \end{aligned} \tag{7.6}$$

Let us comment a bit these equalities which might look strange at first sight. First, as we mentioned in the above remark, the cut-offs entering the $\hat{\gamma}$'s were neglected because they only contribute with $\mathcal{O}(t^{-\infty})$ when faced with the "thinner" cut-offs building our state. Second, the first equality contains $\mathcal{O}(t^{-2})$ which takes into account (see (1.19)) that "$2H - \rho^2 - \xi^2 \sim 1/\bar{r}^2$."

If A and B are bounded operators, by $\mathcal{O}(A, B)$ we denote any finite linear combination of products of bounded operators containing either A or B as factors. The operators entering these products will have "good" commutation properties.

An equivalent expression is

$$2\tilde{\gamma}_3 = (\tilde{\gamma}_1 \cdot \rho_{E_t} + \tilde{\gamma}_2 \cdot \xi_{E_t} + \text{h.c.}) + \mathcal{O}\big(\tilde{\gamma}_1^2, \tilde{\gamma}_2^2, t^{-2}\big). \tag{7.7}$$

Rewriting (7.7) with $\hat{\gamma}$'s we again get an "almost" linear dependence between $\hat{\gamma}_1$ and $\hat{\gamma}_2$; one of the ingredients is the identity $1 - \rho_E \partial_E \rho_E = \xi_E \partial_E \xi_E$:

$$2\hat{\gamma}_1 \cdot \rho_{E_t} + 2\hat{\gamma}_2 \cdot \xi_{E_t} = \mathcal{O}\big(\hat{\gamma}_j^2, t^{-1}\big), \quad j \in \{1, 2, 3\}, \tag{7.8}$$

where we chose to drop the symmetric form at the expense of an extra $\mathcal{O}(t^{-1})$ error.

Ib. Before starting to compute various Heisenberg derivatives we have to spend some time studying the commutation properties of our γ's. Technically the following formal identity will be important:

$$i\left[-i\partial_r - \partial_r S, -\frac{i}{r}\partial_\theta - \frac{\partial_\theta S}{r}\right] = -\frac{1}{r}\left(-\frac{i}{r}\partial_\theta - \frac{\partial_\theta S}{r}\right).$$

It implies

$$i[\tilde{\gamma}_1, \tilde{\gamma}_2] \simeq -r^{-1}\tilde{\gamma}_2 + \mathcal{O}\big(t^{-2}\big) = \mathcal{O}\big(\tilde{\gamma}/t, t^{-2}\big). \tag{7.9}$$

As a consequence of (7.9), by commuting $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$ with $\tilde{\gamma}_3$ expressed as in (7.7) we either gain an extra $1/t$ and keep the number of $\tilde{\gamma}_i$'s as before or we gain a $1/t^2$ and lower the number of $\tilde{\gamma}_i$'s by one. Hence we can write

$$i[\tilde{\gamma}_j, \tilde{\gamma}_3] = \mathcal{O}\big(\tilde{\gamma}/t, \tilde{\gamma}^2/t, t^{-3}\big), \quad j \in \{1, 2\}. \tag{7.10}$$

And thus

$$i[\tilde{\gamma}_j, \tilde{\gamma}_k] = \mathcal{O}\big(\tilde{\gamma}/t, t^{-2}\big), \qquad i[\hat{\gamma}_j, \hat{\gamma}_k] = \mathcal{O}\big(\hat{\gamma}/t, t^{-2}\big), \quad j, k \in \{1, 2, 3\}. \tag{7.11}$$

II. We continue with the Heisenberg derivative of $\hat{\gamma}_3$. Write the equation which defines E_t:

$$t/r = \partial_E \rho_{E_t}(\theta) \tag{7.12}$$

and compute the Heisenberg derivative on both sides (notice that $D_{\mathrm{H}}\hat{\gamma}_3 = -D_{\mathrm{H}} E_t + \mathcal{O}(t^{-\infty})$):

$$1/r - \big(t/r^2\big)\tilde{\rho} + \mathcal{O}\big(t^{-2}\big) = -\big(\partial_E^2 \rho_{E_t}\big) \cdot (D_{\mathrm{H}}\hat{\gamma}_3) + (\partial_E \xi_{E_t}/r) \cdot \tilde{\xi} + \mathcal{O}\big(t^{-2}\big). \tag{7.13}$$

Employing $1 - \rho_E \partial_E \rho_E = \xi_E \partial_E \xi_E$ again we are led to

$$D_{\mathrm{H}}\hat{\gamma}_3 = D_{\mathrm{H}}\tilde{\gamma}_3 = \frac{1}{r\partial_E^2 \rho_{E_t}}\big\{(\partial_E \rho_{E_t}) \cdot \tilde{\gamma}_1 + (\partial_E \xi_{E_t}) \cdot \tilde{\gamma}_2\big\} + \mathcal{O}\big(t^{-2}\big), \tag{7.14}$$

and finally, using (7.12) in order to get rid of r on the right-hand side:

$$D_{\mathrm{H}}\hat{\gamma}_3 = \frac{\partial_E \rho_{E_t}}{t\partial_E^2 \rho_{E_t}}\big\{(\partial_E \rho_{E_t}) \cdot \hat{\gamma}_1 + (\partial_E \xi_{E_t}) \cdot \hat{\gamma}_2\big\} + \frac{\partial_E \rho_{E_t}}{t\partial_E^2 \rho_{E_t}}\big[(\partial_E \rho_{E_t})^2 + (\partial_E \xi_{E_t})^2\big] \cdot \hat{\gamma}_3 + \mathcal{O}\big(t^{-2}\big). \tag{7.15}$$

III. Next comes the Heisenberg derivative of $\hat{\gamma}_1$:

$$\begin{aligned} D_{\mathrm{H}}\hat{\gamma}_1 &= D_{\mathrm{H}}\tilde{\gamma}_1 - (\partial_E \xi_{E_t}/r) \cdot \tilde{\xi} \cdot \hat{\gamma}_3 - (\partial_E \rho_{E_t}) \cdot D_{\mathrm{H}}\tilde{\gamma}_3 + \mathcal{O}\big(\hat{\gamma}^2/t, t^{-2}\big) \\ &= (\xi_{E_t}/r) \cdot \hat{\gamma}_2 + \mathcal{O}\big(\hat{\gamma}^2/t, t^{-2}\big) \\ &= -\big[\rho_{E_t} \cdot (\partial_E \rho_{E_t})/t\big] \cdot \hat{\gamma}_1 + \mathcal{O}\big(\hat{\gamma}^2/t, t^{-2}\big), \end{aligned} \tag{7.16}$$

where the last equality came from (7.8) and (7.12). As in the classical case, define

$$f_1(t, r, \theta) := \frac{1}{\partial_E \rho_{E_t}} > 0. \tag{7.17}$$

We see that its Heisenberg derivative gives (use (7.12) and (7.14)):

$$\begin{aligned} D_{\mathrm{H}} f_1 &= \big(\partial_E^2 \rho_{E_t}\big)/(\partial_E \rho_{E_t})^2 \cdot (D_{\mathrm{H}}\hat{\gamma}_3) - (f_1/t) \cdot (\partial_E \xi_{E_t}) \cdot \tilde{\xi} + \mathcal{O}\big(t^{-2}\big) \\ &= -f_1 \cdot (1 - \rho_{E_t}\partial_E \rho_{E_t})/t + \mathcal{O}\big(\hat{\gamma}/t, t^{-2}\big). \end{aligned} \tag{7.18}$$

Combining this with (7.16) we obtain

$$D_{\mathrm{H}}(f_1\hat{\gamma}_1) = -(1/t)(f_1\hat{\gamma}_1) + \mathcal{O}\big(\hat{\gamma}^2/t, t^{-2}\big). \tag{7.19}$$

IV. Now we compute the Heisenberg derivative of $\hat{\gamma}_2$:

$$D_{\mathrm{H}}\hat{\gamma}_2 = -\frac{b + \xi_{E_t}}{r}\hat{\gamma}_1 + \frac{b\rho_{E_t}}{r\xi_{E_t}}\hat{\gamma}_2 + \mathcal{O}\big(\hat{\gamma}^2/t, t^{-2}\big). \tag{7.20}$$

We remark that the right-hand side of the above equation contains $\hat{\gamma}_3$ only in the quadratic remainder. Then

$$D_{\mathrm{H}}[\xi_{E_t}] = -\frac{b + \xi_{E_t}}{r}\rho_{E_t} + \mathcal{O}\big(\hat{\gamma}/t, t^{-2}\big)$$

and with the integrating factor

$$f_2(t, r, \theta) := \xi_{E_t}/(\partial_E \rho_{E_t}) \tag{7.21}$$

we obtain

$$D_{\mathrm{H}}(f_2\hat{\gamma}_2) = -\frac{b + \xi_{E_t}}{t}(\partial_E \rho_{E_t})\xi_{E_t} \cdot (f_1\hat{\gamma}_1) - (1/t)(f_2\hat{\gamma}_2) + \mathcal{O}\big(\hat{\gamma}^2/t, t^{-2}\big). \tag{7.22}$$

Now we are ready to rewrite (7.15) in a more convenient form. Define

$$f_3(t,r,\theta) = -\left(\partial_E^2 \rho_{E_t} / \partial_E \rho_{E_t}\right). \tag{7.23}$$

Using the identity $(\partial_E \xi_{E_t})^2 + (\partial_E \rho_{E_t})^2 = -\rho_{E_t} \partial_E^2 \rho_{E_t} - \xi_{E_t} \partial_E^2 \xi_{E_t}$ together with the "linear" dependence (7.8) we obtain

$$D_{\mathrm{H}}(f_3 \hat{\gamma}_3) = \left[a_{31}(t)/t\right] \cdot (f_1 \hat{\gamma}_1) - (1/t)(f_3 \hat{\gamma}_3) + \mathcal{O}\left(\hat{\gamma}^2/t, t^{-2}\right), \tag{7.24}$$

where a_{31} is an operator uniformly bounded in time.

V. Let us now give a differential inequality involving all three $\hat{\gamma}$'s. First, rewrite (7.19), (7.22) and (7.24) as:

$$D_{\mathrm{H}}(f_j \hat{\gamma}_j) = \sum_{k=1}^{3} \left[a_{jk}(t)/t\right] \cdot (f_k \hat{\gamma}_k) + \mathcal{O}\left(\hat{\gamma}^2/t, t^{-2}\right), \quad j \in \{1,2,3\}, \tag{7.25}$$

where $a_{jj} = -1$ for all j, and $a_{12} = a_{13} = a_{23} = a_{32} = 0$. As in the classical case, the matrix $\{a\}$ is lower triangular. Notice again that when the energy is localized around $E > E_d$, we have upper and lower bounds for f_j's uniform in t; there exist uniform in time upper bounds for a_{jk}'s, too.

Define the Liapunov-type function of $\hat{\gamma}$'s

$$L_C := C \cdot \left(\hat{\gamma}_1 f_1^2 \hat{\gamma}_1\right) + \hat{\gamma}_2 f_2^2 \hat{\gamma}_2 + \hat{\gamma}_3 f_3^2 \hat{\gamma}_3, \tag{7.26}$$

where $C > 0$ is a very large positive constant only depending on the energy localization. Now let us see how we choose C. Compute

$$\begin{aligned} D_{\mathrm{H}} L_C = {} & -(2/t) L_C + (2/t) \cdot \left[a_{21} \cdot (f_2 \hat{\gamma}_2)(f_1 \hat{\gamma}_1)\right] \\ & + (2/t) \cdot \left[a_{31} \cdot (f_3 \hat{\gamma}_3)(f_1 \hat{\gamma}_1)\right] + \mathcal{O}\left(\hat{\gamma}^3/t, \hat{\gamma}/t^2\right). \end{aligned} \tag{7.27}$$

We see that the cross terms can be bounded in the form sense by ($j \in \{2,3\}$):

$$2\left|\left\langle (f_j \hat{\gamma}_j)(f_1 \hat{\gamma}_1)\right\rangle_\varphi\right| \leqslant \frac{1}{\sqrt{C}} \left[\left\langle (f_j \hat{\gamma}_j)^*(f_j \hat{\gamma}_j) + C (f_1 \hat{\gamma}_1)^*(f_1 \hat{\gamma}_1)\right\rangle_\varphi\right] \leqslant \frac{1}{\sqrt{C}} \langle L_C \rangle_\varphi.$$

Moreover, the a priori smallness of the $\hat{\gamma}$'s stated in (7.5) enables us to bound in form sense any product of three gammas with (up to a constant) ϵL_C. We conclude that for every $\delta > 0$, we can choose $C(\delta)$ sufficiently large such that in the form sense we have L_{C_δ}:

$$D_{\mathrm{H}} L_{C_\delta} \leqslant -\frac{2 - \delta/2}{t} L_{C_\delta} + \mathcal{O}\left(\hat{\gamma}/t^2\right). \tag{7.28}$$

As in the classical case, we abuse notation and write L_δ instead of L_{C_δ}.

7.2. A propagation estimate for the $\hat{\gamma}$'s

The next step is proving that when restricted to states like $\psi_{11}(t)$ (see (7.5)), our gammas decay better than $t^{-1/2}$ as is the case in the classical situation.

Proposition 7.1. *Fix a small* $0 < \delta \ll 2$ *in* (7.28). *Then there exists* (*a sufficiently small*) $\epsilon_4 > 0$ *so that the following estimate holds*:

$$\lim_{t\to\infty} \left\| \mathbf{1}_{(1,\infty)}\left(t^{1+\epsilon_4} L_\delta\right) \psi_{11}(t) \right\| = 0.$$

The proof is long and complicated, so we split it into several parts.

7.2.1. Starting the proof of Proposition 7.1

Let Γ be a cut-off function of the type F_+ with $a = 1/2$ and $b = 1$. Then Proposition 7.1 follows if we can show that

$$\lim_{t\to\infty} \left\| \sqrt{\Gamma}\left(t^{1+\epsilon_4} L_\delta\right) \psi_{11}(t) \right\|^2 = 0. \tag{7.29}$$

Equivalently, introducing

$$F(t) := \left\langle \psi_{11}(t), \Gamma\left(t^{1+\epsilon_4} L_\delta\right) \psi_{11}(t) \right\rangle,$$

we need to prove that $F(t) \to 0$ when $t \to \infty$. We define for $t, \mu > 1$

$$F(t,\mu) := \left\langle \psi_{11}(t), \Gamma\left(\mu^{-1} t^{1+2\epsilon_4} L_\delta\right) \psi_{11}(t) \right\rangle; \qquad B(t,\mu) := \mu^{-1} t^{1+2\epsilon_4} L_\delta. \tag{7.30}$$

Assume for the moment two properties (for a similar procedure see [4, Section 6.13]):

(A) For every $a > 0$, there exists t_a independent of $\mu > 1$ such that for all $t \geqslant t_a$

$$F(t,\mu) \leqslant F(t_a,\mu) + a/2; \quad \text{and}$$

(B) There exists $\mu_a > 1$ so that whenever $\mu \geqslant \mu_a$

$$F(t_a,\mu) < a/2.$$

Given (A) and (B), Proposition 7.1 readily follows by observing that $F(t) = F(t, t^{\epsilon_4})$. Indeed for any given $a > 0$ we have for all $t \geqslant \max\{t_a, \mu_a^{1/\epsilon_4}\}$

$$0 \leqslant F(t) \leqslant F\left(t_a, t^{\epsilon_4}\right) + a/2 < a.$$

Hence what remains to be proved is (A) and (B). Notice that (B) immediately follows from (A) and the fact that Γ is supported away from zero, yielding

$$\operatorname*{s-lim}_{\mu\to\infty} \Gamma\left(\mu^{-1} t_a^{1+2\epsilon_4} L_\delta\right) = 0.$$

As for (A) we write

$$F(t,\mu)=F(t_0,\mu)+\int_{t_0}^{t}\partial_\tau F(\tau,\mu)\,d\tau.$$

The idea is to find a function $\eta\in L^1((1,\infty))$ such that

$$\sup_{\mu>1}\partial_\tau F(\tau,\mu)\leqslant\eta(\tau). \tag{7.31}$$

By taking the partial derivative with respect to τ in $F(\tau,\mu)$, we obtain various terms containing the Heisenberg derivative of the cut-off factors in $\psi_{11}(\tau)$ (their total sum is denoted by $R(\tau,\mu)$) and one term with the Heisenberg derivative of $\Gamma(B(t,\mu))$ (see (7.30)). Remember that the Heisenberg derivative of a time dependent family of operators $A(t)$ is denoted by $D_{\mathrm{H}}A(t)$ and it means $\partial_t A(t)+i[H,A(t)]$.

7.2.2. Estimating $R(\tau,\mu)$

Let us first deal with $R(\tau,\mu)$, i.e. the terms coming up by performing the H-derivative of the various cut-offs building $\psi_{11}(t)$. Take first the term generated by the maximal velocity cut-off $F_-(r/\tau)=F_-^{\mathrm{M.v.b.}}(r/\tau)$; it may be written as

$$\frac{2}{\tau}\Re\big\langle\psi_{11}(\tau),\Gamma(B_{\tau,\mu})\cdot(\text{other cut-offs})\\ \times\sqrt{-F_-'}(r/\tau)(r/\tau-\boldsymbol{\rho})\sqrt{-F_-'}(r/\tau)\psi(\tau)\big\rangle;\quad B_{\tau,\mu}:=B(\tau,\mu)=\frac{\tau^{1+2\epsilon_4}}{\mu}L_\delta. \tag{7.32}$$

In order to be able to use Lemma 6.4, we need to put the left factor $\sqrt{-F_-'}$ next to the $\psi(t)$ in the first entry through repeated commutations. When commuting with the "old" cut-offs one gains a decay of $1/\tau$, so the remainders are integrable.

Hence the only problematic terms could arise from the commutation with Γ. Since we use the first equality in (6.5), we are motivated to study various commutators of $B(\tau,\mu)$.

Lemma 7.2. *For every $G\in C_0^\infty(\mathbb{R}_+)$ we have*

$$\big\|\big[G(r/t),(B_{t,\mu}-z)^{-1}\big]\big\|\leqslant Const\frac{\langle z\rangle^{1/2}}{|\Im(z)|^2}\cdot\frac{t^{-(1-2\epsilon_4)/2}}{\mu^{1/2}}.$$

Proof. We rely on the identity

$$\big[G(r/t),(B_{t,\mu}-z)^{-1}\big]=(B_{t,\mu}-z)^{-1}\big[B_{t,\mu},G(r/t)\big](B_{t,\mu}-z)^{-1},$$

where

$$[B_{t,\mu}, G] = \frac{t^{1+2\epsilon_4}}{\mu}\Biggl\{C_\delta \hat{\gamma}_1\bigl[f_1^2\hat{\gamma}_1, G\bigr] + C_\delta[\hat{\gamma}_1, G]f_1^2\hat{\gamma}_1 + \sum_{j=2}^{3}\bigl([\hat{\gamma}_j, G]f_j^2\hat{\gamma}_j + \hat{\gamma}_j\bigl[f_j^2\hat{\gamma}_j, G\bigr]\bigr)\Biggr\}. \tag{7.33}$$

Since the $\hat{\gamma}_j$'s are essentially first-order derivatives, when we commute them with $G(r/t)$ we gain a factor of $1/t$. Hence the lemma would follow from the estimate

$$\frac{t^{\frac{1+2\epsilon_4}{2}}}{\mu^{1/2}}\bigl\|\hat{\gamma}_j(B_{t,\mu} - z)^{-1}\bigr\| \leqslant Const \cdot \frac{\langle z\rangle^{1/2}}{|\Im(z)|}; \quad j = 1, 2. \tag{7.34}$$

Clearly (7.34) follows from the quadratic estimate (notice that $\hat{\gamma}_j^2 \leqslant Const \cdot L_\delta$ in form sense)

$$\begin{aligned}\bigl\|\hat{\gamma}_j(B_{t,\mu} - z)^{-1}\phi\bigr\|^2 &\leqslant Const \cdot \frac{\mu}{t^{1+2\epsilon_4}}\bigl\langle (B_{t,\mu} - z)^{-1}\phi, B_{t,\mu}(B_{t,\mu} - z)^{-1}\phi\bigr\rangle \\ &\leqslant Const \cdot \frac{\mu}{t^{1+2\epsilon_4}}\left\{\frac{1}{|\Im(z)|} + \frac{|z|}{|\Im(z)|^2}\right\}\|\phi\|^2. \qquad \Box\end{aligned} \tag{7.35}$$

Corollary 7.3. *Under the conditions of Lemma* 7.2

$$\bigl\|\bigl[B_{t,\mu}, G(r/t)\bigr](B_{t,\mu} - z)^{-1}\bigr\| \leqslant Const \cdot \frac{\langle z\rangle^{1/2}}{|\Im(z)|} \cdot \frac{t^{-(1-2\epsilon_4)/2}}{\mu^{1/2}}.$$

Proof. We commute the "free" $\hat{\gamma}$'s in (7.33) to the right. These commutations introduce an extra decay of t^{-1}, hence the corresponding terms are bounded by $C\frac{t^{-(1-2\epsilon_4)}}{\mu|\Im(z)|}$. In addition we use (7.34). $\Box$

Lemma 7.4. *Suppose $F, G \in C_0^\infty(\mathbb{R}_+)$ and $F = 1$ on a neighborhood of the support of G. Then for every integer $N \geqslant 1$, there exists a constant $C > 0$ independent of $\mu > 1$ such that*

$$\bigl\|\bigl(1 - F(r/t)\bigr)(B_{t,\mu} - z)^{-1}G(r/t)\bigr\| \leqslant \frac{C}{|\Im(z)|}\left(\frac{\langle z\rangle}{|\Im(z)|}\right)^N t^{-N(1-2\epsilon_4)/2}.$$

Proof. One can find a function G_1 such that $G_1G = G$ and $FG_1 = G_1$. Then for any $N \in \mathbb{N}$ we may write (abbreviating below $F = F(r/t)$, $G = G(r/t)$, $G_1 = G_1(r/t)$ and $B = B_{t,\mu}$)

$$(1 - F)(B - z)^{-1}G = (1 - F)(B - z)^{-1}G_1^N G.$$

Due to the support conditions we have

$$(1 - F)(B - z)^{-1}G = (1 - F)\mathrm{ad}_{G_1}^N\bigl((B - z)^{-1}\bigr)G, \tag{7.36}$$

where $\mathrm{ad}_{G_1}^0(B) = B$ and $\mathrm{ad}_{G_1}^k(B) = [\mathrm{ad}_{G_1}^{k-1}(B), G_1]$ for $k \geqslant 1$.

As before, one may argue that by each commutation with $G_1(\cdot/t)$ we gain an extra decay of t^{-1}. We may bound

$$\left\|\mathrm{ad}_{G_1}^k(B)(B-z)^{-1}\right\| \leqslant C_k \frac{t^{-k+1+2\epsilon_4}}{\mu|\Im(z)|} \leqslant C_k \cdot \frac{\langle z\rangle^{1/2}}{|\Im(z)|} \cdot \frac{t^{-k+1+2\epsilon_4}}{\mu^{1/2}}, \quad k \geqslant 2. \tag{7.37}$$

We now investigate the Nth order commutator in (7.36). We simplify notation by abbreviating $\mathrm{ad}_{G_1}^k(B)$ as ad^k. Then

$$\begin{aligned} &\mathrm{ad}_{G_1}^N\big((B-z)^{-1}\big) \\ &\quad= \sum_{k_1+\cdots+k_n=N} C_{k_1,\ldots,k_n}(B_{t,\mu}-z)^{-1}\mathrm{ad}^{k_1}\cdots(B_{t,\mu}-z)^{-1}\mathrm{ad}^{k_n}(B_{t,\mu}-z)^{-1}. \end{aligned} \tag{7.38}$$

When we estimate the norm of each term in the above sum, we make a distinction between the factors with $k=1$ and those with $k>1$. Choose a term with the total number of factors to be $n \leqslant N$ and assume that we have n_1 factors of ad^1 and n_2 factors with $k \geqslant 2$; clearly $n_1+n_2=n$ and $n_2 \leqslant (N-n_1)/2$. We use Corollary 7.3 for the n_1 factors and the second inequality in (7.37) for the remaining factors obtaining a bound of the form (uniformly in $\mu>1$):

$$\frac{C}{|\Im(z)|}\frac{\langle z\rangle^{n/2}}{|\Im(z)|^n} \cdot t^{-l_1+1+2\epsilon_4}\cdots t^{-l_{n_2}+1+2\epsilon_4} \cdot t^{-n_1(1-2\epsilon_4)/2}, \tag{7.39}$$

where $l_1+\cdots+l_{n_2}=N-n_1$ and each $l \geqslant 2$.

Since there at most $(N-n_1)/2$ factors of the form $t^{-l+1+2\epsilon_4}$ we may bound the time dependence in (7.39) by

$$t^{-N+n_1} \cdot t^{(N-n_1)(1+2\epsilon_4)/2} \cdot t^{-n_1(1-2\epsilon_4)/2} = t^{-N(1-2\epsilon_4)/2}.$$

As for the z dependence of the bound we notice that

$$\frac{\langle z\rangle^{n/2}}{|\Im(z)|^n} \leqslant \left(\frac{\langle z\rangle}{|\Im(z)|}\right)^N, \quad n \leqslant N,$$

and the lemma follows. □

Now let us go back to (7.32) and see what happens when we commute $\sqrt{-F_-'}$ with $\Gamma(B_{\tau,\mu})$. We use the formula (6.5): by introducing the estimate from Lemma 7.2 with $G=\sqrt{-F_-'}$ in that formula we get that the commutator brings an extra decay (to the already existing $1/\tau$ in front of the scalar product) of order $t^{-1/2+\epsilon_4}$, uniformly in $\mu>1$; notice that the integral with respect to z is also absolutely convergent (put $k=2$ in (6.4)). Finally, apply Lemma 6.4 and we are done with all the contributions coming from the Heisenberg derivative of the maximal velocity cut-off.

But there are some other cut-offs which have to be differentiated. Take for instance the contribution coming from the Heisenberg derivative of $F_+^{\Delta_2}(\tilde{\boldsymbol{\rho}}-\rho_E)$ (see (6.43) and (6.50)). Denote for simplicity $F_+^{\Delta_2}$ with F_+. Then $D_{\mathrm{H}}F_+$ will have only one "dangerous" term with a decay of just $1/r$ but this one will also contain F_+' which is supported on the classical forbidden region (see (6.46)) hence integrable. The good thing here is that we do not have to commute anything with $\Gamma(B_{\tau,\mu})$.

There is a third type of terms in $R(\tau,\mu)$, coming for instance from the H-derivative of $F_+(\tilde{\xi})$ in (6.57). In this case we again have to commute $\sqrt{F'_+}(\tilde{\xi})$ with $\Gamma(B_{\tau,\mu})$ in order to apply Lemma 6.14. If $F^{\mathrm{v.b.}}$ is like in Proposition 6.19, then it suffices to show the bound

$$\left\| \int\limits_{\mathbb{C}} \bar{\partial}\tilde{\Gamma}(z)\big[\sqrt{F'_+}(\tilde{\xi}), (B_{\tau,\mu}-z)^{-1}\big]F^{\mathrm{v.b.}}(r/\tau)\,dx\,dy\right\| \leqslant C\tau^{-1/2+\epsilon_4}, \tag{7.40}$$

uniformly in $\mu > 1$. To prove (7.40) we expand the commutator

$$\big[\sqrt{F'_+}(\tilde{\xi}), (B_{\tau,\mu}-z)^{-1}\big] = -(B_{\tau,\mu}-z)^{-1}\big[\sqrt{F'_+}(\tilde{\xi}), B_{\tau,\mu}\big](B_{\tau,\mu}-z)^{-1},$$

and substitute into the integral in (7.40). As before we verify the absolute integrability of the integral by providing a bound for the integrand that exhibits appropriate z- and τ-decay.

The argument closely follows the one we used in the proof Lemma 7.2, with just one notable difference: when commuting $\sqrt{F'_+}(\tilde{\xi})$ with $B_{\tau,\mu}$ we do not automatically get a $1/\tau$ but rather a $1/r$ decay; however the presence of $F^{\mathrm{v.b.}}$ in the integrand will transform $1/r$ into $1/\tau$.

Now let us give details. We pick a smooth function G_{+-} of the type F_{+-} supported on $\mathbb{R}_+$ and equal to 1 on the support of the function $F^{\mathrm{v.b.}}$. Using Lemmas 7.4 and 6.1 we may put a factor $G_{+-}(r/\tau)$ next to the commutator $[\sqrt{F'_+}(\tilde{\xi}), B_{\tau,\mu}]$ since the commutators with $(B_{\tau,\mu}-z)^{-1}$ induced by this operation produce integrable terms in agreement with (7.40).

Computing $[\sqrt{F'_+}(\tilde{\xi}), B_{\tau,\mu}]$ as in (7.33), we have to deal with

$$\frac{\tau^{1+2\epsilon_4}}{\mu}(B_{\tau,\mu}-z)^{-1}\tilde{\gamma}_j\big[\tilde{\gamma}_j, \sqrt{F'_+}(\tilde{\xi})\big]G_{+-}(r/\tau)(B_{\tau,\mu}-z)^{-1}, \quad j\in\{1,2,3\}. \tag{7.41}$$

(and a similar expression with $\tilde{\gamma}_j$ to the right). By coupling one resolvent with $\tilde{\gamma}_j$ and estimating the norm as in (7.34), we see that (7.40) follows from

$$\big\|\big[\tilde{\gamma}_j, \sqrt{F'_+}(\tilde{\xi})\big]r\big\| \leqslant Const, \tag{7.42}$$

(this is just a consequence of the fact that $i[\tilde{\gamma}_j, \tilde{\xi}]$ brings an extra $1/r$ factor). Finally, $1/r$ is transformed into $1/\tau$ by the factor $G_{+-}(r/\tau)$ and we are done.

We therefore conclude that $|R(\tau,\mu)|$ is integrable in τ uniformly in $\mu > 1$, cf. (7.31).

7.2.3. The Heisenberg derivative of $\Gamma(B_{\tau,\mu})$

We continue the verification of (7.31) (for some $\eta \in L^1$) by considering the remaining contribution from the Heisenberg derivative of $\Gamma(B_{\tau,\mu})$. Formally using (6.7) with A given by $\partial_\tau + iH$, we get

$$\begin{aligned} \big\langle D_{\mathrm{H}}\Gamma(B_{\tau,\mu})\big\rangle_{\psi_{11}} &= \mu^{-1}\tau^{1+2\epsilon_4}\langle C_\tau\rangle_{\Psi_\mu} + \langle R_1\rangle_{\psi_{11}} + \langle R_2\rangle_{\psi_{11}}; \\ C_\tau &= (1+2\epsilon_4)\tau^{-1}L_\delta + D_{\mathrm{H}}L_\delta, \quad \Psi_\mu = \sqrt{\Gamma'(B_{\tau,\mu})}\psi_{11}. \end{aligned} \tag{7.43}$$

We now concentrate on the term involving the “first commutator” C_τ. If we could use (7.28) then by choosing ϵ_4 small enough, the dangerous term which only decays like $1/\tau$ becomes negative so we can discard it. Remember that (7.28) was derived having in mind that by slightly enlarging the supports of the various cut-offs building ψ_{11} we may put them anywhere we want at the expense of $\mathcal{O}(\tau^{-\infty})\|\psi\|$ errors. Hence if we prove that we can commute the old cut-offs over $\sqrt{\Gamma'(B_{\tau,\mu})}$ in the same way, then we are done. But this is essentially contained in Lemmas 7.2 and 7.4 and then in the proof following (7.40). We give no other details.

Now let us go back to the investigation of C_τ in (7.43). Namely, we treat “the quantum errors” introduced by $\mathcal{O}(\hat{\gamma}/\tau^2)$ in the right-hand side of (7.28). In this case we can use the bound $\|\hat{\gamma}_j\Psi_\mu(\tau)\| \leqslant Const\,\mu^{1/2}\tau^{-1/2-\epsilon_4}$ which put back into (7.43) leads to a contribution of order

$$\mu^{-1}\tau^{1+2\epsilon_4}\cdot\mu^{1/2}\tau^{-1/2-\epsilon_4}\cdot\tau^{-2} \leqslant \mu^{-1/2}\tau^{-3/2+\epsilon_4},$$

which clearly implies uniform integrability.

We now treat the last two terms on the right-hand side of (7.43). Since they involve a commutator between L_δ and $D_{\mathrm{H}}L_\delta$, we are motivated to write it as

$$[L_\delta, D_{\mathrm{H}}L_\delta] \simeq \sum_{j,k,l=1}^{3} A_{jkl}\tilde{\gamma}_j\tilde{\gamma}_k\tilde{\gamma}_l + \sum_{j,k=1}^{3} A_{jk}\tilde{\gamma}_j\tilde{\gamma}_k + \sum_{j=1}^{3} A_j\tilde{\gamma}_j + A;$$
$$A_{jkl} = \mathcal{O}(\tau^{-2}),\quad A_{jk} = \mathcal{O}(\tau^{-3}),\quad A_j = \mathcal{O}(\tau^{-4}),\quad A = \mathcal{O}(\tau^{-5}), \tag{7.44}$$

which is obtained by repeatedly applying (7.11). With this formula we can now prove

Lemma 7.5. *The remainder R_1 in* (7.43) *obeys*

$$\sup_{\mu>1}\bigl|\langle R_1\rangle_{\psi_1}\bigr| \leqslant Const\,\tau^{-3/2+3\epsilon_4}. \tag{7.45}$$

Proof. Looking at (6.6), we see that the relevant quantity to bound is

$$\frac{\tau^{2+4\epsilon_4}}{\mu^2}\int_{\mathbb{C}} |\bar{\partial}\tilde{\Gamma}|(z)\bigl\|(B_{\tau,\mu}-z)^{-1}[L_\delta, D_{\mathrm{H}}L_\delta](B_{\tau,\mu}-z)^{-2}\psi_{11}(\tau)\bigr\|\,dx\,dy; \tag{7.46}$$

and a similar expression with the powers of resolvents interchanged (which may be treated similarly).

We now insert each term from (7.44) into (7.46) and check the decay in τ. Let us start with

$$\frac{\tau^{2+4\epsilon_4}}{\mu^2}\bigl\|(B_{\tau,\mu}-z)^{-1}A_{jkl}\hat{\gamma}_j\hat{\gamma}_k\hat{\gamma}_l(B_{\tau,\mu}-z)^{-2}\psi_{11}(\tau)\bigr\|;$$

we claim there is a uniform upper bound of the form

$$C\frac{\langle z\rangle^{1/2+m}}{|\Im(z)|^{2+m}}\tau^{-3/2+3\epsilon_4}\mu^{-1/2},\quad m \geqslant 1, \tag{7.47}$$

which together with (6.4) yields absolutely integrability in z and agreement with (7.45). The other terms from (7.44) will obey the same bound.

First, notice that we may rewrite the middle term as $\hat{\gamma}_j A_{jkl}\hat{\gamma}_k\hat{\gamma}_l$ since the commutator between A_{jkl} and $\hat{\gamma}_j$ behaves like τ^{-3} and therefore may be treated along with the terms $A_{jk}\hat{\gamma}_j\hat{\gamma}_k$ from (7.44). Bounding the factor $\hat{\gamma}_j$ by use of (7.34) yields the upper bounds

$$C\frac{\tau^{4\epsilon_4}}{\mu^2}\left\|(B_{\tau,\mu}-z)^{-1}\hat{\gamma}_j\right\|\cdot\left\|\hat{\gamma}_k\hat{\gamma}_l\hat{\Psi}(\tau)\right\|\leqslant Const\cdot\frac{\tau^{3\epsilon_4-1/2}}{\mu^{3/2}}\frac{\langle z\rangle^{1/2}}{|\Im(z)|}\left\|\hat{\gamma}_k\hat{\gamma}_l\hat{\Psi}(\tau)\right\|,$$
$$\hat{\Psi}(\tau)=(B_{\tau,\mu}-z)^{-2}\psi_{11}(\tau).$$

Next, write

$$\hat{\gamma}_k\hat{\gamma}_l\hat{\Psi}(\tau)=\hat{\gamma}_k(B_{\tau,\mu}-z)^{-1}\hat{\gamma}_l(B_{\tau,\mu}-z)^{-1}\psi_{11}(\tau)+\hat{\gamma}_k(B_{\tau,\mu}-z)^{-1}[B_{\tau,\mu},\hat{\gamma}_l]\hat{\Psi}(\tau).$$

Substituting we have to estimate

$$\frac{\tau^{3\epsilon_4-1/2}}{\mu^{3/2}}\frac{\langle z\rangle^{1/2}}{|\Im(z)|}\left\|\hat{\gamma}_k(B_{\tau,\mu}-z)^{-1}\right\|\cdot\left\|\hat{\gamma}_l(B_{\tau,\mu}-z)^{-1}\right\| \tag{7.48}$$

and

$$\frac{\tau^{3\epsilon_4-1/2}}{\mu^{3/2}}\frac{\langle z\rangle^{1/2}}{|\Im(z)|}\left\|\hat{\gamma}_k(B_{\tau,\mu}-z)^{-1}\right\|\cdot\left\|[B_{\tau,\mu},\hat{\gamma}_l]\hat{\Psi}(\tau)\right\|. \tag{7.49}$$

Introducing (7.34) in (7.48) we get the bound

$$\frac{2}{\mu^{1/2}}\tau^{-3/2+\epsilon_4}\frac{\langle z\rangle^{3/2}}{|\Im(z)|^3}$$

which clearly is of the form (7.47).

Let us focus on (7.49). Introducing again (7.34) we get

$$\frac{\sqrt{2}}{\mu}\tau^{-1+2\epsilon_4}\frac{\langle z\rangle}{|\Im(z)|^2}\left\|[B_{\tau,\mu},\hat{\gamma}_l]\hat{\Psi}(\tau)\right\|.$$

Computing the commutator as in (7.33) yields

$$[B_{\tau,\mu},\hat{\gamma}_l]=\frac{\tau^{1+2\epsilon_4}}{\mu}\left\{\mathcal{O}\left(r^{-1}\right)\hat{\gamma}_i+\mathcal{O}\left(r^{-2}\right)\right\},\quad i\neq l.$$

We substitute and use the minimal velocity cut-off from ψ_{11} to transform $1/r$ into $1/\tau$, cf. Lemma 7.4. For the first term we then use (7.34) again. In conclusion, both terms contribute with a bound of the form (7.47) and we may deduce that the contribution coming from the first term on the right-hand side of (7.44) to $\langle R_1\rangle_{\psi_1}$ behaves as in (7.45).

As for the contribution to $\langle R_1\rangle_{\psi_1}$ coming from the second term on the right-hand side of (7.44) the situation is now better since we trade one $\hat{\gamma}_j$ with an extra $1/\tau$ decay in A_{jk}; similarly for the remaining terms. Details are omitted. □

There is a completely similar bound for R_2 as for R_1 in (7.45). We skip the proof which is similar. We conclude that (7.31) holds for an integrable η. Proposition 7.1 is proven.

7.2.4. A propagation estimate for L_δ

Lemma 7.6. *Consider a function of the type F_- with $a = 1$ and $b = 2$ and denote it by Γ_1. There exists $\epsilon_4 > 0$ small enough so that with $B_t := t^{1+\epsilon_4} L_\delta$ and $\psi_{11}(t)$ given by* (7.5) *we have*

$$\int_1^\infty \frac{1}{t} \left\| \sqrt{-\Gamma_1'}(B_t)\psi_{11}(t) \right\|^2 dt \leqslant Const \cdot \|\psi\|^2.$$

Proof. We proceed as in the previous section by constructing a bounded propagation observable. To simplify notation, we only give its expectation on $\psi(t)$ which equals:

$$V(t) := \langle \psi_{11}(t), \Gamma_1(B_t)\psi_{11}(t) \rangle. \tag{7.50}$$

Differentiating $V(t)$ we get

$$\partial_t V(t) = -\left\langle \sqrt{-\Gamma_1'}(B_t)\psi_{11}(t), (D_{\mathrm{H}} B_t)\sqrt{-\Gamma_1'}(B_t)\psi_{11}(t) \right\rangle + R(t), \tag{7.51}$$

where R contains remainders of the type R_1 and R_2 as in (7.43) together with Heisenberg derivatives of the other cut-off functions which build $\psi_{11}(t)$. Using various previous estimates, cf. the proof of Proposition 7.1, we may prove that

$$\int_1^\infty \left| R(t) \right| dt \leqslant Const \cdot \|\psi\|^2,$$

hence we only have to deal with the first term. Firstly, rewrite it as

$$-t^{1+\epsilon_4} \left\langle (1+\epsilon_4)\frac{L_\delta}{t} + D_{\mathrm{H}} L_\delta \right\rangle_\Psi, \quad \Psi := \sqrt{-\Gamma_1'}(B_t)\psi_{11}(t). \tag{7.52}$$

Up to an integrable remainder this term is "positive," in fact (see the proof of Proposition 7.1):

$$\begin{aligned} &-t^{1+\epsilon_4} \left\langle (1+\epsilon_4)\frac{L_\delta}{t} + D_{\mathrm{H}} L_\delta \right\rangle_\Psi \\ &\quad \geqslant \frac{1-\delta-\epsilon_4}{t} \langle B_t \rangle_\Psi + \mathcal{O}\big(t^{-3/2}\big)\|\psi\|^2 \\ &\quad \geqslant \frac{1-\delta-\epsilon_4}{t} \left\| \sqrt{-\Gamma_1'}(B_t)\psi_{11} \right\|^2 + \mathcal{O}\big(t^{-3/2}\big)\|\psi\|^2, \end{aligned} \tag{7.53}$$

where the last inequality comes from the fact that Γ_1' is supported in $[1, 2]$. □

8. Asymptotic completeness: existence of Ω_+

We will now prove statement II of Theorem 4.2. By a covering argument it suffices to show the following.

Proposition 8.1. *Assume we have $E > E_d$, $E \notin \sigma_{\mathrm{pp}}(H)$. Suppose that $\psi(t)$ is as in* (6.1) *with $\tilde{\epsilon} > 0$ very small. Then there exists a vector $\phi \in L^2((E_d, \infty) \times \mathbb{T})$ such that*

$$\lim_{t\to\infty} \big\| U_0(t)\phi - \psi(t) \big\| = 0.$$

Proof. The proposition is an easy consequence of the unitarity of $U_0(t)$ (since this operator is essentially a change of variables) and of the existence of the following limit:

$$\phi := \lim_{t\to\infty} U_0^*(t)\psi(t). \tag{8.1}$$

The proof of (8.1) is complicated and uses the propagation estimates we have obtained so far. Because of the various localization properties we have already proven, we can replace $\psi(t)$ by

$$\psi_{12}(t) := \Gamma_1\big(t^{1+\epsilon_4} L_\delta\big)\psi_{11}(t) \tag{8.2}$$

since the difference between them tends to zero in time. There are fifteen cut-offs which build $\psi_{12}(t)$ and we would like to keep track of their Heisenberg derivatives in a more efficient way. Moreover, they split into two categories: the first one contains those cut-offs introduced through a weak propagation estimate (proven by constructing a bounded propagation observable whose Heisenberg derivative has a sign) and the second one contains the cut-offs whose complementary localizations are in classically forbidden regions.

Let us look back for the cut-offs belonging to the first category. In order to keep track of them more easily, we introduce unified notations for them as follows: $F_1 := F_D$, $A_1 = D(t)$ (see (6.85)), $F_2 := F_+^{\mathrm{dil}}$, $A_2 := \tilde{A}_C(E)/t$ (see (6.37)), $F_3 := F_+^{d_1/2, d_1}$, $A_3 := \tilde{\xi}$ (see (6.57)), $F_4 := F_-^{\eta_1, \eta_1+\epsilon_2}$, $A_4 = \tilde{\rho} - \rho_{E'}$ (see (6.77)), $F_5 := F_-^{\mathrm{M.v.b.}}$, $A_5 := r/t$ (see (6.30)), and $F_6 := \Gamma_1$, $A_6 := t^{1+\epsilon_4} L_\delta$ (see (8.2)).

We do the same thing with the second category: $F_7 := F_+^{\mathrm{m.v.b.}}$, $A_7 := r/t$ (see (6.41)), $F_8 := F_+^{\Delta_2}$, $A_8 := \tilde{\rho} - \rho_E$ (see (6.50)), $F_9 := F_+^{\Delta_1}$, $A_9 := \tilde{\rho} - \rho_E + \partial_E \rho_E / C$ (see again (6.50)), $F_{10} := F_+^{\epsilon_2, 2\epsilon_2}$, $A_{10} := \tilde{\rho} + \sqrt{2E}$ (see (6.53)), $F_{11} := F_-^{2\eta_1+\epsilon_2, \eta_1+2\epsilon_2}$, $A_{11} = \tilde{\rho} - \rho_E$, $F_{12} := F_{+-}^M$, $A_{12} := \tilde{\xi} - \xi_E$ (see (6.83)), $F_{13} := F_E$, $A_{13} := \tilde{\gamma}_3 = \hat{\gamma}_3$ (see (6.88)), $F_{14} := \hat{F}$, $A_{14} := \hat{\gamma}_1$ (see (7.5)) and finally $F_{15} := \hat{F}$, $A_{15} := \hat{\gamma}_2$ (see again (7.5)).

We remark that the order of the above cut-offs is not important when applied on $\psi(t)$ in (6.1), since every commutation will at least be of order $\mathcal{O}(t^{-1/2+\epsilon_4/2})$. We rewrite then

$$\psi_f(t) := F_1\big(A_1(t)\big) \cdot F_2(A_2) \cdots F_{15}\big(A_{15}(t)\big)\psi(t). \tag{8.3}$$

Define $Q(t, \varphi) = \langle U_0(t)\varphi, \psi_f(t)\rangle$, where $\varphi \in C_0^\infty((E_d, \infty) \times \mathbb{T})$. Since $\psi_f(t) - \psi(t) \to 0$, the existence of the limit in (8.1) is equivalent to the existence of the limit $\lim_{t\to\infty} U_0^*(t)\psi_f(t)$. Moreover, combining the Cauchy criterion for the existence of a limit with the Cook argument and the Riesz representation theorem for linear functionals on L^2, we see that this limit exists if for every $\varepsilon > 0$, there exists $T_\varepsilon > 1$ such that for all $t_2 \geqslant t_1 > T_\varepsilon$ and all $\varphi \in C_0^\infty((E_d, \infty) \times \mathbb{T})$, we have

$$\int_{t_1}^{t_2} \left|\partial_t Q(t,\varphi)\right| dt \leqslant \varepsilon \|\varphi\|_{L^2((E_d,\infty)\times\mathbb{T})}. \tag{8.4}$$

Proving (8.4) will be the task in the remaining of this subsection. We start by expanding the derivative of Q with respect to t. Performing the derivative with respect to t and using (4.19) we get (by $d_{\mathrm{H}}\psi_f$ we denote the Heisenberg derivative acting on the cut-offs in ψ_f)

$$\partial_t Q(t,\varphi) = \frac{i}{8}\left\langle U_0(t)\varphi, \frac{1}{r^2}\psi_f(t)\right\rangle - \frac{i}{2}\left\langle \gamma^2 U_0(t)\varphi, \psi_f(t)\right\rangle + \left\langle U_0(t)\varphi, d_{\mathrm{H}}\psi_f(t)\right\rangle. \tag{8.5}$$

The first term on the right-hand side of (8.5) is clearly bounded up to a constant by $\|\varphi\|/t^2$, because F_1 implies in particular that r/t is bounded and away from zero at the same time. Hence after integration we get an estimate as in (8.4). The second term is technically more complicated: by various commutations involving the cut-offs in $\psi_f(t)$ (we skip the details) one can prove that

$$\gamma^2\psi_f(t) = \tilde{\gamma}^2\psi_f(t) + \mathcal{O}\left(t^{-\infty}\right). \tag{8.6}$$

Invoking (8.6), the second term in (8.5) becomes $-i/2\langle U_0(t)\varphi, \tilde{\gamma}^2\psi_f(t)\rangle$. Because of the presence of $F_6(A_6)$ in $\psi_f(t)$, we choose two functions Γ_2 and Γ_3 with a slightly wider support than Γ_1 and decompose

$$\tilde{\gamma}_1^2\psi_f(t) = \tilde{\gamma}_1\Gamma_3\left(t^{1+\epsilon_4}L_\delta\right)\tilde{\gamma}_1\Gamma_2\left(t^{1+\epsilon_4}L_\delta\right)\psi_f(t) + \mathcal{O}\left(t^{-\infty}\right).$$

Using (7.3) we can easily show that there is a constant independent of time so that in the form sense $\tilde{\gamma}_1^2 \leqslant \mathit{Const}\cdot L_\delta$ which means that $\|\tilde{\gamma}_1\Gamma_3(t^{1+\epsilon_4}L_\delta)\| \leqslant \mathit{Const}\cdot t^{-1/2-\epsilon_4/2}$ and similarly for $\tilde{\gamma}_1\Gamma_2(t^{1+\epsilon_4}L_\delta)$; we conclude that the first term on the right-hand side is $\mathcal{O}(t^{-1-\epsilon_4})$. After integration we get a bound in agreement with (8.4).

We now look at the third term on the right-hand side of (8.5), the one containing the Heisenberg derivatives of all cut-off functions. Acting with d_{H} on (8.3) we get

$$d_{\mathrm{H}}\psi_f(t) = \sum_{j=1}^{15} F_1\left(A_1(t)\right)\cdots\left\{D_{\mathrm{H}}F_j\left(A_j(t)\right)\right\}\cdots F_{15}\left(A_{15}(t)\right)\psi(t). \tag{8.7}$$

Here D_{H} denotes as usual the Heisenberg derivative. We keep in mind that all of the F_j''s of the type F_+' or F_-' have definite signs. The functions of type F_{+-}' do not have this property but can be rewritten as $F_{+-}' = g_+ - g_-$, where the terms are non-negative and have non-overlapping supports. To fix a uniform notation let us write in general $F_j' = g_{j+} - g_{j-}$. Then writing

$$\begin{aligned} S_j(t) := {} & \sqrt{g_{j+}\left(A_j(t)\right)}\left\{D_{\mathrm{H}}A_j(t)\right\}\sqrt{g_{j+}\left(A_j(t)\right)} \\ & + \sqrt{g_{j-}\left(A_j(t)\right)}\left\{-D_{\mathrm{H}}A_j(t)\right\}\sqrt{g_{j-}\left(A_j(t)\right)} \end{aligned} \tag{8.8}$$

one obtains (we can put $f_{1,E}$ on the right-hand side since $f_{1,E}f_E = f_E$ and $f_{1,E}\tilde{f}_E = f_{1,E}$; see also (6.1))

$$d_{\mathrm{H}}\psi_f(t)=f_{1,E}(H)\sum_{j=1}^{15}F_1\big(A_1(t)\big)\cdots S_j(t)\cdots F_{15}\big(A_{15}(t)\big)\psi(t)+\mathcal{O}\big(t^{-1-\epsilon_4}\big). \tag{8.9}$$

Also notice that for every j we have

$$\big\|f_{1,E}(H)F_1\big(A_1(t)\big)\cdots S_j(t)\cdots F_{15}\big(A_{15}(t)\big)\big\|\leqslant Const/t. \tag{8.10}$$

We then have

$$\begin{aligned}&\big\langle U_0(t)\varphi, d_{\mathrm{H}}\psi_f(t)\big\rangle\\&\quad=\sum_{j=1}^{15}\big\langle\sqrt{F_1}(A_1)\cdots\sqrt{F_{j-1}}(A_{j-1})\sqrt{F_{j+1}}(A_{j+1})\cdots\sqrt{F_{15}}(A_{15})f_{1,E}(H)U_0(t)\varphi,\\&\qquad S_j(t)\sqrt{F_1}(A_1)\cdots\sqrt{F_{j-1}}(A_{j-1})\sqrt{F_{j+1}}(A_{j+1})\cdots\sqrt{F_{15}}(A_{15})\psi(t)\big\rangle\\&\qquad+\mathcal{O}\big(t^{-1-\epsilon_4}\big)\|\varphi\|,\end{aligned} \tag{8.11}$$

where we used the fact that by commuting any two cut-off functions we get an integrable contribution.

The next step is to see that from S_8 up to S_{15} we have a $\mathcal{O}(t^{-\infty})\cdot\|\phi\|$ contribution because the supports of $g_{j\pm}$ entering them are localized in the classically forbidden regions. Hence we can rewrite (8.11) as

$$\begin{aligned}&\big\langle U_0(t)\varphi, d_{\mathrm{H}}\psi_f(t)\big\rangle\\&\quad=\sum_{j=1}^{6}\big\langle\sqrt{F_1}(A_1)\cdots\sqrt{F_{j-1}}(A_{j-1})\sqrt{F_{j+1}}(A_{j+1})\cdots\sqrt{F_{15}}(A_{15})f_{1,E}(H)U_0(t)\varphi,\\&\qquad S_j(t)\sqrt{F_1}(A_1)\cdots\sqrt{F_{j-1}}(A_{j-1})\sqrt{F_{j+1}}(A_{j+1})\cdots\sqrt{F_{15}}(A_{15})\psi(t)\big\rangle\\&\qquad+\mathcal{O}\big(t^{-1-\epsilon_4}\big)\|\varphi\|,\end{aligned} \tag{8.12}$$

where now the sum only runs over the first six cut-offs.

The next lemma plays a crucial role in what follows. For $\psi\in L^2(\mathbb{R}_+\times\mathbb{T})$ (as above) and $j\in\{1,\dots,15\}$ we define

$$\psi_{t,j}:=\cdots\sqrt{F_{j-1}}\,(A_{j-1})\sqrt{g_{j+}+g_{j-}}\sqrt{F_{j+1}}\,(A_{j+1})\cdots\sqrt{F_{15}}\,(A_{15})f_{1,E}(H)\psi. \tag{8.13}$$

Lemma 8.2. *There exist two constants $0<c<C<\infty$ such that for all $\psi,\phi\in L^2(\mathbb{R}_+\times\mathbb{T})$ and $j\in\{1,\dots,6\}$ one has (for $\epsilon_4>0$ small)*

$$\begin{aligned}&\frac{c}{t}\|\psi_{t,j}\|^2-\mathcal{O}\big(t^{-1-\epsilon_4}\big)\|\psi\|^2\\&\quad\leqslant\big\langle\sqrt{F_1}\,(A_1)\cdots\sqrt{F_{j-1}}\,(A_{j-1})\sqrt{F_{j+1}}\,(A_{j+1})\cdots\sqrt{F_{15}}\,(A_{15})f_{1,E}(H)\psi,\\&\qquad S_j(t)\sqrt{F_1}\,(A_1)\cdots\sqrt{F_{j-1}}\,(A_{j-1})\sqrt{F_{j+1}}\,(A_{j+1})\cdots\sqrt{F_{15}}\,(A_{15})f_{1,E}(H)\psi\big\rangle\end{aligned} \tag{8.14}$$

and

$$\begin{aligned} &\big|\big\langle \sqrt{F_1}\,(A_1)\cdots\sqrt{F_{j-1}}\,(A_{j-1})\sqrt{F_{j+1}}\,(A_{j+1})\cdots\sqrt{F_{15}}\,(A_{15}) f_E^1(H)\phi, \\ &\quad S_j(t)\sqrt{F_1}\,(A_1)\cdots\sqrt{F_{j-1}}\,(A_{j-1})\sqrt{F_{j+1}}\,(A_{j+1})\cdots\sqrt{F_{15}}\,(A_{15}) f_E^1(H)\psi\big\rangle\big| \\ &\quad \leqslant \frac{C}{t}\|\psi_{t,j}\|\|\phi_{t,j}\| + \mathcal{O}\big(t^{-1-\epsilon_4}\big)\|\psi\|\|\phi\|. \end{aligned} \tag{8.15}$$

Proof. The first estimate says that each S_j, $j \in \{1,\ldots,6\}$ is "positive." That is, upon restricting the Heisenberg derivatives of every observable to states containing all other cut-offs we always get the plus sign. We used this sort of "definite sign" property whenever we had to obtain a weak propagation estimate; the only truly important fact here is that all five S_j's are simultaneously positive. Before verifying the property for each term, we would like to give a simple explanation to this apparently striking coincidence. Consider the function $x:(1,\infty)\mapsto \mathbb{R}$, $x(t) = \pm 10t^{-1}$. We see that $(dx/dt) = -x/t$ thus both $F_+^{-2,-1}(x(t))$ and $F_-^{1,2}(x(t))$ are increasing with t. This is the phenomenon behind the "positivity" in the case when $j = 1$ (see below). As a final remark, let us notice that these six cut-offs are chosen in such a way that the growth of their approximate characteristic functions indicates the tendency of a trajectory to be drawn to the spiraling attractor.

In fact if we interpret the product of cut-off functions as an approximate characteristic function of the attractor, the approximate positivity of its Heisenberg derivative indicates the increasing probability that $\psi(t)$ is "in the attractor."

$j = 1$. Since F_1 is a F_{+-} function, we write the derivative $F_1' = g_+ - g_-$. We look at the operators $\sqrt{g_+}(r/t)\cdot D_{\mathrm{H}}D(t)\cdot\sqrt{g_+}(r/t)$ and $-\sqrt{g_-}(r/t)\cdot D_{\mathrm{H}}D(t)\cdot\sqrt{g_-}(r/t)$. From the proof of Proposition 6.19 it follows that these are essentially positive when the other cut-offs are taken into account.

$j = 2$. F_2 is of F_+ type and

$$S_2(t) = \sqrt{F_+'}\,\big(\tilde{A}_C(E)/t\big)\cdot D_{\mathrm{H}}\tilde{A}_C(E)/t\cdot\sqrt{F_+'}\,\big(\tilde{A}_C(E)/t\big).$$

For its "positivity" go back to Lemma 6.9.

$j = 3$. F_3 is again of F_+ type and

$$S_3(t) := \sqrt{F_+'}\,(\tilde{\xi})\cdot D_{\mathrm{H}}\tilde{\xi}\cdot\sqrt{F_+'}\,(\tilde{\xi}).$$

See for details Lemma 6.14.

$j = 4$. F_4 is of F_- type and

$$S_4(t) = -\sqrt{-F_-'}\,(\tilde{\rho}-\rho_{E'})\cdot D_{\mathrm{H}}(\tilde{\rho}-\rho_{E'})\sqrt{-F_-'}\,(\tilde{\rho}-\rho_{E'}).$$

We treated such terms in Lemma 6.15; we see that $D_{\mathrm{H}}(\tilde{\rho}-\rho_{E'})$ is "almost" $E - E' = -2\epsilon_0 < 0$ if ϵ_2 is small enough.

$j=5$. F_5 is of F_- type (the maximal velocity cut-off), so

$$S_5(t)=-\sqrt{-F'_-}\,(r/t)i[H,r/t]\sqrt{-F'_-}\,(r/t).$$

Its "positivity" comes from the considerations we made in the proof of Lemma 6.4.

$j=6$. F_6 is again of F_- type and

$$S_6(t):=-\sqrt{-F'_-}\,\bigl(t^{1+\epsilon_4}L_\delta\bigr)\cdot D_{\mathrm{H}}\bigl(t^{1+\epsilon_4}L_\delta\bigr)\cdot\sqrt{-F'_-}\,\bigl(t^{1+\epsilon_4}L_\delta\bigr).$$

See for details Lemma 7.6.

About (8.15): these estimates are boundedness properties which easily may be deduced from the above considerations (see also (8.10)). □

Completion of the proof of Proposition 8.1. Introducing (8.15) in (8.12) and applying the Cauchy–Schwarz inequality we obtain

$$\bigl|\bigl\langle U_0(t)\varphi, d_{\mathrm{H}}\psi_f(t)\bigr\rangle\bigr|\leqslant C\left(\frac{1}{t}\sum_{j=1}^{6}\bigl\|\bigl(U_0(t)\varphi\bigr)_{t,j}\bigr\|^2\right)^{1/2}\left(\frac{1}{t}\sum_{j=1}^{6}\bigl\|\bigl(e^{-itH}\psi\bigr)_{t,j}\bigr\|^2\right)^{1/2} + Ct^{-1-\epsilon_4}\|\varphi\|. \tag{8.16}$$

Recall that we had to look at an integral as in (8.4). The proposition would be concluded if we could prove that

$$\int_{t_1}^{t_2}\left(\frac{1}{t}\sum_{j=1}^{6}\bigl\|\bigl(U_0(t)\varphi\bigr)_{t,j}\bigr\|^2\right)^{1/2}\left(\frac{1}{t}\sum_{j=1}^{6}\bigl\|\bigl(e^{-itH}\psi\bigr)_{t,j}\bigr\|^2\right)^{1/2}dt\leqslant\varepsilon\|\varphi\|. \tag{8.17}$$

This will be achieved as soon as we obtain the next two estimates

$$\int_1^\infty\frac{1}{t}\sum_{j=1}^{6}\bigl\|\bigl(U_0(t)\varphi\bigr)_{t,j}\bigr\|^2\,dt\leqslant Const\cdot\|\varphi\|^2 \tag{8.18}$$

and

$$\int_1^\infty\frac{1}{t}\sum_{j=1}^{6}\bigl\|\bigl(e^{-itH}\psi\bigr)_{t,j}\bigr\|^2\,dt\leqslant Const\cdot\|\psi\|^2. \tag{8.19}$$

Since (8.19) follows from the propagation estimates we have obtained so far, we are only left with proving (8.18). This is the place, where the simultaneous "positivity" of S_j, $j\in\{1,\dots,6\}$ from (8.14) plays a central role.

Now let us prove (8.18). We introduce the function (remember that the order of cut-offs does not count)

$$V(t,\varphi)=\big\|f_{1,E}(H)\sqrt{F_{15}}\big(A_{15}(t)\big)\cdots\sqrt{F_6}\big(A_6(t)\big)\cdots\sqrt{F_1}\big(A_1(t)\big)U_0(t)\varphi\big\|^2.$$

Differentiate $V(t,\varphi)$ with respect to t. When the derivative acts on $U_0(t)\varphi$ we get $\cdots\sqrt{F_1}(A_1(t))\cdot(H+\frac{1}{8r^2}-\frac{\gamma^2}{2})U_0(t)\varphi$; see (4.19). Using that

$$f_{1,E}(H)\sqrt{F_1}\,(r/t)\big\{\tilde{\gamma}^2-\gamma^2\big\}=\mathcal{O}\big(t^{-\infty}\big)$$

and

$$\tilde{\gamma}^2\sqrt{F_6}\,\big(A_6(t)\big)=\mathcal{O}\big(t^{-1-\epsilon_4}\big),$$

we get

$$\partial_t V(t,\varphi)=2\sum_{j=1}^{15}\Re\big\langle\cdots\big\{D_{\mathrm{H}}\sqrt{F_j}\,\big(A_j(t)\big)\big\}\cdots f_{1,E}(H)U_0(t)\varphi,\\ \cdots\sqrt{F_j}\,\big(A_j(t)\big)\cdots f_{1,E}(H)U_0(t)\varphi\big\rangle+\mathcal{O}\big(t^{-1-\epsilon_4}\big)\|\varphi\|^2. \tag{8.20}$$

Rearranging the above scalar products, employing (8.14) and noticing that the terms with $j\geqslant 7$ give $\mathcal{O}(t^{-\infty})\|\varphi\|^2$ contributions, we get

$$\partial_t V(t,\varphi)\geqslant\frac{c}{t}\sum_{j=1}^{6}\big\|\big(U_0(t)\varphi\big)_{t,j}\big\|^2-\mathcal{O}\big(t^{-1-\epsilon_4}\big)\|\varphi\|^2. \tag{8.21}$$

We can now integrate and obtain (8.18); hence the proof of the proposition is complete. □

We have therefore proven both I and II of Theorem 4.2, or equivalently, that the limits defined in (1.23) exist and define unitary operators which are mutually inverse. As for III we refer to the discussion at the end of Section 5. □

9. Approximate dynamics for negative times

The first issue we want to explain in this section is the behavior of our system in the distant past. The approximate dynamics $U_0(t)$ in (4.11) and (4.12) only makes sense for positive times and shows that in the distant future every scattering state will spiral away from the origin.

For negative times, the picture is reversed. Our task is to find an approximate dynamics with the usual spiraling feature in the distant past, and which shows how the particle is drawn to the origin. In other words, instead of looking for attracting periodic solutions at positive times, we now search for attracting periodic solutions at negative times.

We are therefore interested in obtaining C^1 and periodic solutions to the system of equations

$$\begin{cases} \partial_\theta \rho = b + \xi, \\ \partial_\theta \xi = -\dfrac{b+\xi}{\xi}\rho, \quad \theta \geqslant 0,\ \rho(0) \in (-\sqrt{2E}, \sqrt{2E}\,) \text{ and } \xi(0) \in (0, \sqrt{2E}\,), \\ \xi^2 + \rho^2 = 2E, \end{cases} \tag{9.1}$$

with the supplementary condition

$$\int_0^{2\pi} \frac{\rho}{\xi}(\theta)\, d\theta < 0. \tag{9.2}$$

Let us explain the meaning of (9.2). Assume that we have such solutions and denote them by ρ_e and ξ_e. Consider the initial value problem ($t \geqslant -1$):

$$\frac{d\tilde{r}}{dt} = \rho_e(\tilde{\theta}), \qquad \frac{d\tilde{\theta}}{dt} = \frac{\xi_e(\tilde{\theta})}{\tilde{r}}, \qquad \big(\tilde{r}(-1), \tilde{\theta}(-1)\big) = (1, 0). \tag{9.3}$$

It is easy to check that at least for t close to -1 the above system admits a solution $(\tilde{r}, \tilde{\theta})$ which also solves the Hamilton equations, thus it corresponds to a real orbit at energy E. We notice that the above system gives

$$\tilde{r}(t) = \tilde{r}\big(\tilde{\theta}(t)\big) = \exp\left\{ \int_0^{\tilde{\theta}(t)} (\rho_e/\xi_e)(\varphi)\, d\varphi \right\}, \tag{9.4}$$

thus $\mathrm{Ran}(\tilde{\theta}) = [0, \infty)$ and $\tilde{r}(t)$ decreases "in mean" after each complete revolution around the origin and collapses to it in a finite amount of time. Nevertheless, going backwards in t we see the spiraling behavior again.

Now let us investigate the existence of such solutions. Define $b_1(\theta) := b(-\theta)$. Then b_1 is negative, periodic and if $E > E_d(b_1)$ then (see Corollary 2.7) we have a unique periodic solution to

$$\begin{cases} \partial_\theta \rho = b_1 + \xi, \\ \partial_\theta \xi = -\dfrac{b_1+\xi}{\xi}\rho, \quad \theta \geqslant 0,\ \rho(0) \in (-\sqrt{2E}, \sqrt{2E}\,) \text{ and } \xi(0) \in (0, \sqrt{2E}\,), \\ \xi^2 + \rho^2 = 2E, \end{cases} \tag{9.5}$$

with the supplementary condition

$$\int_0^{2\pi} \frac{\rho}{\xi}(\theta)\, d\theta > 0. \tag{9.6}$$

The intimate connection between attractive solutions at positive times and attractive solutions at negative times is given by the following proposition.

Proposition 9.1. *Assume that* (ρ_E, ξ_E) *is a periodic solution which solves* (9.5) *and obeys* (9.6). *Denote by* $\rho_e(\theta) := -\rho_E(-\theta)$ *and* $\xi_e(\theta) = \xi_E(-\theta)$. *Then* (ρ_e, ξ_e) *solves* (9.1) *and obeys* (9.2). *Reciprocally, assume that* (ρ_e, ξ_e) *is a periodic solution which solves* (9.1) *and obeys* (9.2). *Denote by* $\rho_E(\theta) := -\rho_e(-\theta)$ *and* $\xi_E(\theta) = \xi_e(-\theta)$. *Then* (ρ_E, ξ_E) *solves* (9.5) *and obeys* (9.6).

Proof. Simple computation. □

The next important thing is knowing the energy range for which we can construct attracting solutions for negative times. Denote by $E'_d(b)$ the infimum of all energies for which (9.1) has a solution obeying (9.2). It is obvious (using the above proposition) that $E'_d(b) = E_d(b_1)$. But is it true that $E'_d(b) = E_d(b)$? In other words, can we prove that "E_d" corresponding to $b(\theta)$ equals the "E_d" associated to $b(-\theta)$? The answer is affirmative.

Proposition 9.2. *The critical energies are equal*: $E_d(b_1) = E'_d(b) = E_d(b)$.

Proof. Assume $E_d(b) < E_d(b_1)$. Then consider the critical solution (ρ_1, ξ_1) to (9.5) corresponding to $E_d(b_1)$ and satisfying $\int_0^{2\pi} \frac{\rho_1}{\xi_1}(\theta)\, d\theta = 0$. Then define $\rho(\theta) := -\rho_1(-\theta)$ and $\xi(\theta) := \xi_1(-\theta)$, and notice that they solve (9.1) but with a zero integral condition. If we look at its associated "real orbit" $(r(t), \theta(t))$ we see that it exists for all $t \geqslant 0$ since $r(t) = \exp\{\int_0^{\theta(t)} (\rho/\xi)(\varphi)\, d\varphi\}$ is bounded from below and above. Then consider

$$A(t) = \big[\rho\big(\theta(t)\big) - \rho_d\big(\theta(t)\big)\big] \cdot r(t)$$

and see that $A'(t) \geqslant E_d(b_1) - E_d(b) > 0$ for $t \geqslant 0$, which contradicts its boundedness. Then $E_d(b_1) < E_d(b)$ is contradicted by a similar argument and we are done. □

Remark. For every $E > E_d$ there are exactly two branches of C^1 periodic orbits which solve (9.1), both having $\xi > 0$ but their integral condition is with opposite signs. While the "propagating" ρ_E is increasing and concave in energy, the "collapsing" ρ_e is decreasing and convex in energy. Then notice that $r \cdot \rho_e(E, \theta)$ solves the eikonal equation and provides a Hamilton–Jacobi function for negative times $t < 0$. We then can define a direct and inverse flow and finally construct the approximate dynamics for negative times using the same ideas as in the case of U_0. We give no further details.

10. Open problems

We mention two related problems concerning dynamics and spectral theory for magnetic fields considered in this paper.

(a) Dynamics below E_d: we write $\rho = \omega \cos(\varphi)$, $\xi = \omega \sin(\varphi)$ with $\omega = \sqrt{2E}$, and introduce a new time τ with $d\tau/dt = 1/r(t)$. The variables (φ, θ) move on a torus $\mathbb{T}$ according to the differential equation

$$\frac{d}{d\tau}(\varphi, \theta) = \big(-b(\theta) - \omega \sin(\varphi), \omega \sin(\varphi)\big).$$

For $0 < E < E_d$ it turns out that $\varphi(\tau) \to \infty$ and $\theta(\tau) \to \infty$ as $t \to \infty$. We can write

$$r(\tau) = r(0)e^{\int_0^\tau \omega \cos(\varphi(\tau'))\, d\tau'}.$$

Are there any orbits with energy $E < E_d$ for which $r(\tau) \to \infty$ as $\tau \to \infty$? There are none if $b(\theta)$ is constant and nonzero.

(b) We know that the spectrum of H is $[0, \infty)$. What is the nature of the spectrum in $[0, E_d]$? For the constant b case, reasoning as in the Miller–Simon model (see [2, Theorem 6.2]) we can show it is pure point.

Acknowledgments

This work was completed during a stay at Institut Mittag-Leffler. The authors want to express their gratitude for invitations and financial support. H.C. and E.S. also acknowledge support from the Danish F.N.U. grant Mathematical Physics and Partial Differential Equations.

References

[1] H. Cornean, I. Herbst, E. Skibsted, Classical and quantum dynamics for 2D-electromagnetic potentials asymptotically homogeneous of degree zero, http://arxiv.org/abs/math-ph/0703089.
[2] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, Schrödinger Operators with Application to Quantum Mechanics and Global Geometry, Springer-Verlag, Berlin, 1987.
[3] J. Dereziński, C. Gérard, Long-range scattering in the position representation, J. Math. Phys. 38 (8) (1997) 3925–3942.
[4] J. Dereziński, C. Gérard, Scattering Theory of Classical and Quantum N-Particle Systems, Springer-Verlag, Berlin, 1997.
[5] V. Enss, Quantum scattering with long-range magnetic fields, in: Operator Calculus and Spectral Theory, Lambrecht, 1991, in: Oper. Theory Adv. Appl., vol. 57, Birkhäuser, Basel, 1992, pp. 61–70.
[6] A. Hassell, R. Melrose, A. Vasy, Spectral and scattering theory for symbolic potentials of order zero, Adv. Math. 181 (1) (2004) 1–87.
[7] I. Herbst, Spectral and scattering theory for Schrödinger operators with potentials independent of $|\mathbf{x}|$, Amer. J. Math. 113 (3) (1991) 509–565.
[8] I. Herbst, E. Skibsted, Quantum scattering for potentials independent of $|\mathbf{x}|$: asymptotic completeness for high and low energies, Comm. Partial Differential Equations 29 (3–4) (2004) 547–610.
[9] L. Hörmander, The existence of wave operators in scattering theory, Math. Z. 146 (1) (1976) 69–91.
[10] L. Hörmander, The Analysis of Linear Partial Differential Operators IV, Springer-Verlag, Berlin, 1985.
[11] H. Isozaki, H. Kitada, Modified wave operators with time-independent modifiers, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 32 (1) (1985) 77–104.
[12] M. Loss, B. Thaller, Scattering of particles by long-range magnetic fields, Ann. Phys. 176 (1) (1987) 159–180.
[13] M. Loss, B. Thaller, Short-range scattering in long-range magnetic fields: the relativistic case, J. Differential Equations 73 (2) (1988) 225–236.
[14] J.S. Møller, An abstract radiation condition and applications to N-body systems, Rev. Math. Phys. 12 (5) (2000) 767–803.
[15] E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. 78 (3) (1981) 391–408.
[16] F. Nicoleau, D. Robert, Thèorie de la diffusion quantique pour des perturbations à longue et courte portèe du champ magnétique, Ann. Fac. Sci. Toulouse Math. (5) 12 (2) (1991) 185–194.
[17] M. Reed, B. Simon, Methods of Modern Mathematical Physics III. Scattering Theory, Academic Press, New York, 1979.
[18] D. Robert, On scattering theory for long range perturbations of Laplace operators, J. Anal. Math. 59 (1992) 189–203.
[19] P. Roux, D. Yafaev, The scattering matrix for the Schrödinger operator with a long-range electromagnetic potential, J. Math. Phys. 44 (7) (2003) 2762–2786.
[20] D. Yafaev, Wave operators for the Schrödinger operators, Teor. Mat. Fiz. 45 (1980) 224–234.