

# STATIONARY SCATTERING THEORY ON MANIFOLDS, I

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ABSTRACT. We study spectral theory for the Schrödinger operator on manifolds possessing an escape function. A particular class of examples are manifolds with Euclidean and/or hyperbolic ends. Certain exterior domains for possibly unbounded obstacles are included. We prove a number of results that are considered as “classical“ for the Euclidean space: Rellich’s theorem, the limiting absorption principle and radiation condition bounds. The Sommerfeld uniqueness result is presented as an application. The proofs of these results are given by an extensive use of commutator arguments. These arguments have a classical spirit (essentially) not involving energy cutoffs or microlocal analysis and require, presumably, minimum regularity and decay properties of perturbations. This paper has interest of its own right, but it also serves as a basis for the stationary scattering theory developed fully in the sequel [IS3].

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## 1. INTRODUCTION

Let  $(M, g)$  be a connected Riemannian manifold of dimension  $d \geq 1$ . In this paper we study the spectral theory for the geometric Schrödinger operator

$$H = H_0 + V; \quad H_0 = -\frac{1}{2}\Delta = \frac{1}{2}p_i^* g^{ij} p_j, \quad p_i = -i\partial_i,$$

on the Hilbert space  $\mathcal{H} = L^2(M)$ . The potential  $V$  is real-valued and bounded, and the self-adjointness of  $H$  is realized by the Dirichlet boundary condition. Assuming an *end* structure on  $M$ , we prove in this paper Rellich's theorem, the limiting absorption principle, radiation condition bounds and the Sommerfeld uniqueness result. Our assumptions are general enough to cover for example manifolds with finitely many ends of (mixed) Euclidean and hyperbolic types studied recently by Kumura [Ku4]. Another example is a "scattering manifold" as introduced by Melrose [Me]. Our theory also covers certain exterior domains for possibly unbounded and non-smooth obstacles in a manifold. For the Euclidean model certain unbounded regular exterior domains were studied previously by Constantin [Co], Minskii [Min] and Il'in [Il1, Il2, Il3]. To prove the above results we invent a commutator argument with some *weight inside*. This commutator argument has a classical spirit, to some extent resembling [Min, Sa1, Ku4] (see [Ku4] for a more extensive list of references). In particular we are not going to use Mourre theory [Mo, JP, Do]. Rather our "conjugate operator"  $A$  is the generator of a semigroup of (a kind of) radial translations and not a group of radial dilations which for a limited class of metrics fits into standard Mourre theory [Do], and the commutator includes an appropriate *weight inside* depending on the context (see Lemmas 3.3, 3.4, 4.3 and 5.4). This paper extensively employs explicit commutator computations of differential operators and to a limited degree tools from functional analysis (primarily semigroup theory). Similarly microlocal analysis is virtually absent in this paper. As an advantage our assumptions on regularity and decay properties of perturbations appear minimal.

Based on the results of this paper, in [IS3] we develop fully the stationary scattering theory in a similar but slightly more restrictive framework, and in particular we provide a complete characterization of asymptotics for appropriate generalized eigenfunctions at infinity.

## 1.1. Setting and results.

1.1.1. *Basic setting.* We shall study manifolds for which there exist ends in a somewhat disguised form.

**Condition 1.1.** Let  $(M, g)$  be a connected Riemannian manifold. There exist a function  $r \in C^\infty(M)$  with image  $r(M) = [1, \infty)$  and constants  $c > 0$  and  $r_0 \geq 2$  such that:

- (1) The gradient vector field  $\omega = \text{grad } r \in \mathfrak{X}(M)$  is forward complete in the sense that the integral curve of  $\omega$  is defined for any initial point  $x \in M$  and any non-negative time parameter  $t \geq 0$ .
- (2) The bound  $|dr| = |\omega| \geq c$  holds on  $\{x \in M \mid r(x) > r_0/2\}$ .

We call each component of the open subset  $E = \{x \in M \mid r(x) > r_0\}$  an *end* of  $M$ , and the function  $r$  may model a distance function there. The last interpretation is supported by a part of (1.6b) below too. The set  $E$  is obviously the continuous

disjoint union of  $r$ -spheres

$$S_R = \{x \in M \mid r(x) = R\}; \quad R > r_0,$$

which are submanifolds of  $M$  due to (2) of Condition 1.1 and the implicit function theorem. Then we can canonically construct the *spherical coordinates* on  $E$  along the vector field  $\omega$ , however, since these coordinates are not used in this paper, we omit the construction here. Note that spherical coordinates will be important in our sequel paper [IS3].

Let us impose more assumptions on the geometry of  $E$  in terms of the radius function  $r$ . Choose  $\chi \in C^\infty(\mathbb{R})$  such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2, \end{cases} \quad \chi \geq 0, \quad \chi' \leq 0, \quad (1.1)$$

and set

$$\eta = 1 - \chi(2r/r_0), \quad \tilde{\eta} = \eta|dr|^{-2} = (1 - \chi(2r/r_0))|dr|^{-2}.$$

We introduce the ‘‘radial differential operator’’:

$$A = \operatorname{Re} p^r = \frac{1}{2}(p^r + (p^r)^*); \quad p^r = -i\nabla^r, \quad \nabla^r = \nabla_\omega = g^{ij}(\nabla_i r)\nabla_j, \quad (1.2)$$

and also the tensor  $\ell$  and the associated differential operator  $L$ :

$$\ell = g - \tilde{\eta} dr \otimes dr, \quad L = p_i^* \ell^{ij} p_j. \quad (1.3)$$

As we can see with ease, the tensor  $\ell$  may be identified with the pull-back of  $g$  to the  $r$ -spheres, and  $L$  with the spherical part of  $-\Delta$ . We remark that the tensor  $\ell$  clearly satisfies

$$0 \leq \ell \leq g, \quad \ell^{\bullet i}(\nabla r)_i = (1 - \eta)dr, \quad (1.4)$$

where the first bounds of (1.4) are understood as quadratic form estimates on fibers of the tangent bundle of  $M$ . The quantities of (1.3) will play a major role in this paper.

Recall a local expression of the Levi-Civita connection  $\nabla$ : If we denote the Christoffel symbol by  $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$ , then for any smooth function  $f$  on  $M$

$$(\nabla f)_i = (\nabla_i f) = (df)_i = \partial_i f, \quad (\nabla^2 f)_{ij} = \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f. \quad (1.5)$$

Note that  $\nabla^2 f$  is the geometric Hessian of  $f$ .

**Condition 1.2.** There exist constants  $\sigma, \tau, C > 0$  such that globally on  $M$

$$r \left( \nabla^2 r - \frac{1}{2}\eta|dr|^{-4}(\nabla^r|dr|^2)dr \otimes dr \right) \geq \frac{1}{2}\sigma|dr|^2\ell - Cr^{-\tau}g \quad (1.6a)$$

as quadratic forms on fibers of the tangent bundle of  $M$ , and

$$|dr|^2 \leq C, \quad |\nabla^r|dr|^2| \leq Cr^{-1-\tau/2}, \quad \Delta r \leq C, \quad |\ell^{\bullet i}\nabla_i \Delta r| \leq Cr^{-1-\tau/2}. \quad (1.6b)$$

Condition 1.2 says that the ends are geometrically growing. For any  $R > r_0$  we let  $\iota_R: S_R \hookrightarrow M$  be the inclusion. In case where  $r$  is an exact distance function, i.e.  $|dr| = 1$  on  $E$ , the Hessian  $\nabla^2 r$  has no radial components in the spherical coordinates, and  $(\nabla^2 r)|_{S_R}$  can be identified with the pull-back  $\iota_R^*(\nabla^2 r)$ , which is exactly the second fundamental form of  $S_R$ , and  $(\Delta r)|_{S_R}$  with the mean curvature

$\text{tr}[\iota_R^*(\nabla^2 r)]$  of  $S_R$ . In general under Condition 1.2 the radial components of  $\nabla^2 r$  do not necessarily vanish, but we may still somehow regard the quantity

$$\nabla^2 r - \frac{1}{2}\eta|dr|^{-4}(\nabla^r|dr|^2)dr \otimes dr$$

as the second fundamental form with negligible error, cf. Lemma 2.1, and hence the bound (1.6a) implies that the ends are growing, bounding the minimal curvature of  $S_R$  below: For any  $\sigma' \in (0, \sigma)$  there exists  $R_{\sigma'} \geq r_0$  such that for all  $R \geq R_{\sigma'}$

$$R\iota_R^*(\nabla^2 r) = R\iota_R^*\left(\nabla^2 r - \frac{1}{2}\eta|dr|^{-4}(\nabla^r|dr|^2)dr \otimes dr\right) \geq \frac{1}{2}\sigma'|dr|^2\iota_R^*g.$$

The bounds in (1.6b) together with (1.6a), see also Lemma 2.1, are connected to the regularity properties of the mean curvature of  $S_R$ , and, in particular, we can bound the maximal curvature above, since  $\iota_R^*(\nabla^2 r)$  is strictly positive for  $R > r_0$  large enough. We also remark that in agreement with conventions one could reasonably call the radius function  $r$  an *escape function*, since as a consequence of the convexity property (1.6a) the complete geodesics with  $r$  globally large enough are non-trapped. One benefit of our indirect description of the geometry of  $(M, g)$  is that it is obviously stable under small perturbations. When it is difficult to compute an exact distance function, we may choose a more useful distance-like function to verify the conditions.

Finally we impose a long-range type condition on the potential  $V$ . More precisely, taking into account a metric quantity related to the volume growth of the ends, we formulate it in terms of an *effective potential*  $q$  defined by

$$q = V + \frac{1}{8}\tilde{\eta}\left[(\Delta r)^2 + 2\nabla^r \Delta r\right]. \quad (1.7)$$

This quantity naturally shows up. In fact, using (1.3) and the expressions

$$A = p^r - \frac{1}{2}(\Delta r) = (p^r)^* + \frac{1}{2}(\Delta r), \quad (1.8)$$

we can rewrite the Schrödinger operator  $H$  in the form

$$H = \frac{1}{2}A\tilde{\eta}A + \frac{1}{2}L + q + \frac{1}{4}(\nabla^r \tilde{\eta})(\Delta r). \quad (1.9)$$

**Condition 1.3.** The potential  $V$  is a real-valued function belonging to  $L^\infty(M)$ . Moreover, there exists a splitting by real-valued functions:

$$q = q_1 + q_2; \quad q_1 \in C^1(M) \cap L^\infty(M), \quad q_2 \in L^\infty(M),$$

such that for some  $\rho', C > 0$  the following bounds hold globally on  $M$ :

$$\nabla^r q_1 \leq Cr^{-1-\rho'}, \quad |q_2| \leq Cr^{-1-\rho'}. \quad (1.10)$$

A setting similar to Conditions 1.1–1.3 is used in [IS2]. See also [Ku2, Ku3, IS1]. In Subsection 1.2 we shall discuss concrete models of manifolds satisfying Conditions 1.1–1.3 along with additional Conditions 1.6 and 1.9 below. The models include manifolds with asymptotically Euclidean and/or hyperbolic ends, and their conic regions. Some more general regions with unbounded obstacles are included too. We remark that in this paper only derivatives of  $r$  of order at most four are used quantitatively. Throughout our presentation we use the convention that  $c$  is used for a “small” positive constant while  $C$  is used for a “big” positive constant, however their particular values not being important. On the other hand the parameters  $\sigma$ ,  $\tau$  and  $\rho$  appearing in Condition 1.9 are intimately related to scattering properties of quantum particles on the model manifold. This will be demonstrated in [IS3].

Now let us mention the self-adjoint realizations of  $H$  and  $H_0$ . Since  $(M, g)$  can be incomplete, the operators  $H$  and  $H_0$  are not necessarily essentially self-adjoint

on  $C_c^\infty(M)$ . We realize  $H_0$  as a self-adjoint operator by imposing the Dirichlet boundary condition, i.e.  $H_0$  is the unique self-adjoint operator associated with the closure of the quadratic form

$$\langle H_0 \rangle_\psi = \langle \psi, -\frac{1}{2}\Delta\psi \rangle, \quad \psi \in C_c^\infty(M).$$

We denote the form closure and the self-adjoint realization by the same symbol  $H_0$ . Define the associated Sobolev spaces  $\mathcal{H}^s$  by

$$\mathcal{H}^s = (H_0 + 1)^{-s/2}\mathcal{H}, \quad s \in \mathbb{R}. \quad (1.11)$$

Then  $H_0$  may be understood as a closed quadratic form on  $Q(H_0) = \mathcal{H}^1$ . Equivalently,  $H_0$  makes sense also as a bounded operator  $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ , whose action coincides with that for distributions. By the definition of the Friedrichs extension the self-adjoint realization of  $H_0$  is the restriction of such distributional  $H_0: \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$  to the domain:

$$\mathcal{D}(H_0) = \{\psi \in \mathcal{H}^1 \mid H_0\psi \in \mathcal{H}\} \subseteq \mathcal{H}.$$

Since  $V$  is real-valued and bounded by Condition 1.3, we can realize the self-adjoint operator  $H = H_0 + V$  simply as

$$H = H_0 + V, \quad \mathcal{D}(H) = \mathcal{D}(H_0).$$

In contrast to (1.11) we introduce the weighted Hilbert space  $\mathcal{H}_s$  for  $s \in \mathbb{R}$  by

$$\mathcal{H}_s = r^{-s}\mathcal{H}.$$

We also denote the locally  $L^2$ -space by

$$\mathcal{H}_{\text{loc}} = L_{\text{loc}}^2(M).$$

We consider the  $r$ -balls  $B_R = \{r(x) < R\}$  and the characteristic functions

$$F_\nu = F(B_{R_{\nu+1}} \setminus B_{R_\nu}), \quad R_\nu = 2^\nu, \quad \nu \geq 0, \quad (1.12)$$

where  $F(\Omega)$  is used for sharp characteristic function of a subset  $\Omega \subseteq M$ . Define the associated Besov spaces  $B$  and  $B^*$  by

$$\begin{aligned} B &= \{\psi \in \mathcal{H}_{\text{loc}} \mid \|\psi\|_B < \infty\}, \quad \|\psi\|_B = \sum_{\nu=0}^{\infty} R_\nu^{1/2} \|F_\nu\psi\|_{\mathcal{H}}, \\ B^* &= \{\psi \in \mathcal{H}_{\text{loc}} \mid \|\psi\|_{B^*} < \infty\}, \quad \|\psi\|_{B^*} = \sup_{\nu \geq 0} R_\nu^{-1/2} \|F_\nu\psi\|_{\mathcal{H}}, \end{aligned} \quad (1.13)$$

respectively. We also define  $B_0^*$  to be the closure of  $C_c^\infty(M)$  in  $B^*$ . Recall the nesting holding for any  $s > 1/2$ :

$$\mathcal{H}_s \subsetneq B \subsetneq \mathcal{H}_{1/2} \subsetneq \mathcal{H} \subsetneq \mathcal{H}_{-1/2} \subsetneq B_0^* \subsetneq B^* \subsetneq \mathcal{H}_{-s}.$$

Using the function  $\chi \in C^\infty(\mathbb{R})$  of (1.1), define  $\chi_n, \bar{\chi}_n, \chi_{m,n} \in C^\infty(M)$  for  $n > m \geq 0$  by

$$\chi_n = \chi(r/R_n), \quad \bar{\chi}_n = 1 - \chi_n, \quad \chi_{m,n} = \bar{\chi}_m \chi_n. \quad (1.14)$$

Let us introduce an auxiliary space:

$$\mathcal{N} = \{\psi \in \mathcal{H}_{\text{loc}} \mid \chi_n\psi \in \mathcal{H}^1 \text{ for all } n \geq 0\}.$$

This is the space of functions that satisfy the Dirichlet boundary condition, possibly with infinite  $\mathcal{H}^1$ -norm on  $M$ . Note that under Conditions 1.1–1.3 the manifold  $M$

may be, e.g. a half-space in the Euclidean space, and there could be a “boundary” even for large  $r$ , which is “invisible” from inside  $M$ .

1.1.2. *Rellich’s theorem.* Our first theorem is Rellich’s theorem, the absence of  $B_0^*$ -eigenfunctions with eigenvalues above a certain “critical energy”  $\lambda_0 \in \mathbb{R}$  given by

$$\lambda_0 = \limsup_{r \rightarrow \infty} q_1 = \lim_{R \rightarrow \infty} \left( \sup \{ q_1(x) \mid r(x) \geq R \} \right) < \infty. \quad (1.15)$$

For the Euclidean and the hyperbolic spaces and many other examples the critical energy  $\lambda_0$  can be computed explicitly, see Subsection 1.2, and the essential spectrum  $\sigma_{\text{ess}}(H) = [\lambda_0, \infty)$ . The latter is usually seen in terms of Weyl sequences, see [Ku1].

**Theorem 1.4.** *Suppose Conditions 1.1–1.3, and let  $\lambda > \lambda_0$ . If a function  $\phi \in \mathcal{H}_{\text{loc}}$  satisfies that*

- (1)  $(H - \lambda)\phi = 0$  in the distributional sense,
- (2)  $\bar{\chi}_m \phi \in \mathcal{N} \cap B_0^*$  for all  $m \geq 0$  large enough,

then  $\phi = 0$  in  $M$ .

**Corollary 1.5.** *The operator  $H$  has no eigenvalues above  $\lambda_0$ :  $\sigma_{\text{pp}}(H) \cap (\lambda_0, \infty) = \emptyset$ .*

Note in order to verify (2) it suffices to have  $\bar{\chi}_m \phi \in \mathcal{N} \cap B_0^*$  for a single value of  $m$ . For any function  $\phi$  obeying the conditions of Theorem 1.4 we have  $\chi_{m,n} \phi \in \mathcal{D}(H)$  for all  $m$  large enough and  $n > m$ . See the discussion on our self-adjoint realization of  $H$  above. We can drop the space  $\mathcal{N}$  if the  $r$ -annuli  $B_{R_{\nu+1}} \setminus B_{R_{\nu}}$  are relatively compact in  $M$  for all large  $\nu \geq 0$ . Note that an  $r$ -ball  $B_R$ ,  $R \in \mathbb{R}$ , may be unbounded under Conditions 1.1–1.3. If on the other hand  $M$  is complete and  $B_R$  is bounded it follows from the Hopf–Rinow theorem [Jo, Theorem 1.4.8] that  $B_R$  is relatively compact. Corollary 1.5, a direct consequence of Theorem 1.4, was proven in a somewhat similar setting in [IS2]. For a little more precise comparison with [IS2], see Remark 2.3 (3).

1.1.3. *Limiting absorption principle and radiation condition.* Next we discuss the limiting absorption principle and the radiation condition related to the resolvent

$$R(z) = (H - z)^{-1}.$$

We first establish a locally uniform bound for the resolvent  $R(z)$  as a map:  $B \rightarrow B^*$ . Let us impose a compactness condition.

**Condition 1.6.** In addition to Conditions 1.1–1.3, there exists an open subset  $\mathcal{I} \subseteq (\lambda_0, \infty)$  such that for any  $n \geq 0$  and relatively compact open subset  $I \subseteq \mathcal{I}$  the mapping

$$\chi_n P_H(I): \mathcal{H} \rightarrow \mathcal{H}$$

is compact, where  $P_H(I)$  denotes the spectral projection onto  $I$  for  $H$ .

Due to Rellich’s compact embedding theorem [RS, Theorem XIII.65], “boundedness” of  $r$ -balls provides a criterion for Condition 1.6: If  $M$  is complete and each  $r$ -ball  $B_R$ ,  $R \geq 1$ , is bounded there, then Condition 1.6 is satisfied for  $\mathcal{I} = (\lambda_0, \infty)$ . More generally, even if  $M$  is incomplete, it suffices that each  $r$ -ball  $B_R$ ,  $R \geq 1$ , is isometric to a bounded subset of a complete manifold. Condition 1.6 in fact includes even more general situations where  $M$  has several ends possibly with different critical energies and where  $r$ -balls are unbounded, cf. [Ku4]. We shall discuss an example in Subsection 1.2.

For notational simplicity we set for a large  $C > 0$

$$\begin{aligned} h &= \nabla^2 r - \frac{1}{2}\eta|\mathrm{d}r|^{-4}(\nabla^r|\mathrm{d}r|^2)\mathrm{d}r \otimes \mathrm{d}r + 2Cr^{-1-\tau}g \\ &\geq \frac{1}{2}\sigma r^{-1}|\mathrm{d}r|^2\ell + Cr^{-1-\tau}g \geq 0, \end{aligned} \quad (1.16)$$

cf. (1.6a). We may consider  $h$ , as well as  $\nabla^2 r$ , as the second fundamental form with negligible error. For any open subset  $I \subseteq \mathcal{I}$  let us denote

$$I_{\pm} = \{z = \lambda \pm i\Gamma \in \mathbb{C} \mid \lambda \in I, \Gamma \in (0, 1)\},$$

respectively. We also use the notation  $\langle T \rangle_{\psi} = \langle \psi, T\psi \rangle$ .

**Theorem 1.7.** *Suppose Condition 1.6 and let  $I \subseteq \mathcal{I}$  be any relatively compact open subset. Then there exists  $C > 0$  such that for any  $\phi = R(z)\psi$  with  $z \in I_{\pm}$  and  $\psi \in B$*

$$\|\phi\|_{B^*} + \|p^r\phi\|_{B^*} + \langle p_i^* h^{ij} p_j \rangle_{\phi}^{1/2} + \|H_0\phi\|_{B^*} \leq C\|\psi\|_B. \quad (1.17)$$

**Corollary 1.8.** *The operator  $H$  has no singular continuous spectrum on  $\mathcal{I}$ :  $\sigma_{\mathrm{sc}}(H) \cap \mathcal{I} = \emptyset$ .*

Absence of singular continuous spectrum is a standard application of the uniform boundedness of  $R(z)$  in an appropriate operator space. Corollary 1.8 applies in particular to the Dirichlet Laplacian on  $\mathbb{R}^d \setminus K$  for any compact set  $K$ , which to our knowledge was first proved in [DaSi].

The Besov boundedness (1.17) does not immediately imply the limiting absorption principle. Before showing it we establish radiation condition bounds under an additional (minor) regularity condition. These bounds will be crucial for our application [IS3].

**Condition 1.9.** In addition to Condition 1.6 with the same  $\tau > 0$  appearing there, there exist splittings  $q_1 = q_{11} + q_{12}$  and  $q_2 = q_{21} + q_{22}$  by real-valued functions

$$q_{11} \in C^2(M) \cap L^{\infty}(M), \quad q_{12}, q_{21} \in C^1(M) \cap L^{\infty}(M), \quad q_{22} \in L^{\infty}(M)$$

and constants  $\rho, C > 0$  such that for  $\alpha = 0, 1$

$$\begin{aligned} |\ell^{\bullet i} \nabla_i |\mathrm{d}r|^2| &\leq Cr^{-1-\tau/2} & |\nabla^r q_{11}| &\leq Cr^{-(1+\rho/2)/2}, & |\ell^{\bullet i} \nabla_i q_{11}| &\leq Cr^{-1-\rho/2}, \\ |\mathrm{d}\nabla^r q_{11}| &\leq Cr^{-1-\rho/2}, & |\mathrm{d}q_{12}| &\leq Cr^{-1-\rho/2}, & |(\nabla^r)^{\alpha} q_{21}| &\leq Cr^{-\alpha-\rho}, \\ q_{21} \nabla^r q_{11} &\leq Cr^{-1-\rho}, & |q_{22}| &\leq Cr^{-1-\rho/2}. \end{aligned}$$

Our radiation condition bounds are stated in terms of the radial derivative  $A$  defined in (1.2) and an asymptotic complex phase  $a$  given below. Pick a smooth decreasing function  $r_{\lambda} \geq r_0$  of  $\lambda > \lambda_0$  such that

$$\lambda + \lambda_0 - 2q_1 \geq 0 \text{ for } r \geq r_{\lambda}/2, \quad (1.18)$$

and that  $r_{\lambda} = r_0$  for all  $\lambda$  large enough. Then we set for  $z = \lambda \pm i\Gamma \in \mathcal{I} \cup \mathcal{I}_{\pm}$

$$a = a_z = \eta_{\lambda} \left[ |\mathrm{d}r| \sqrt{2(z - q_1)} \pm \frac{1}{4}(p^r q_{11}) / (z - q_1) \right]; \quad \eta_{\lambda} = 1 - \chi(2r/r_{\lambda}), \quad (1.19)$$

respectively, where the branch of square root is chosen such that  $\mathrm{Re} \sqrt{w} > 0$  for  $w \in \mathbb{C} \setminus (-\infty, 0]$ . Note that the phase  $a = a_{\pm}$  of (1.19) is an approximate solution to the radial Riccati equation

$$\pm p^r a + a^2 - 2|\mathrm{d}r|^2(z - q_1) = 0 \quad (1.20)$$

in the sense that it makes the quantity on the left-hand side of (1.20) small for large  $r \geq 1$ . The first term in the brackets of (1.19) alone already gives an approximate

solution to the same equation, however with the second term a better approximation is obtained, cf. Lemma 5.1 and Remark 5.3. Let

$$\beta_c = \frac{1}{2} \min\{\sigma, \tau, \rho\} > 0. \quad (1.21)$$

**Theorem 1.10.** *Suppose Condition 1.9, and let  $I \subseteq \mathcal{I}$  be any relatively compact open subset. Then for all  $\beta \in [0, \beta_c)$  there exists  $C > 0$  such that for any  $\phi = R(z)\psi$  with  $\psi \in r^{-\beta}B$  and  $z \in I_{\pm}$*

$$\|r^{\beta}(A \mp a)\phi\|_{B^*} + \langle p_i^* r^{2\beta} h^{ij} p_j \rangle_{\phi}^{1/2} \leq C \|r^{\beta}\psi\|_B, \quad (1.22)$$

respectively.

As an application we obtain the limiting absorption principle.

**Corollary 1.11.** *Suppose Condition 1.9, and let  $I \subseteq \mathcal{I}$  be any relatively compact open subset. For any  $s > 1/2$  and  $\epsilon \in (0, \min\{(2s-1)/(2s+1), \beta_c/(\beta_c+1)\})$  there exists  $C > 0$  such that for  $\alpha = 0, 1$  and any  $z, z' \in I_+$  or  $z, z' \in I_-$*

$$\|p^{\alpha}R(z) - p^{\alpha}R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C|z - z'|^{\epsilon}. \quad (1.23)$$

In particular, the operators  $p^{\alpha}R(z)$ ,  $\alpha = 0, 1$ , attain uniform limits as  $I_{\pm} \ni z \rightarrow \lambda \in I$  in the norm topology of  $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$ , say denoted by

$$p^{\alpha}R(\lambda \pm i0) = \lim_{I_{\pm} \ni z \rightarrow \lambda} p^{\alpha}R(z), \quad \lambda \in I, \quad (1.24)$$

respectively. These limits  $p^{\alpha}R(\lambda \pm i0) \in \mathcal{B}(B, B^*)$ , and  $R(\lambda \pm i0): B \rightarrow \mathcal{N} \cap B^*$ .

Now we have the limiting resolvents  $R(\lambda \pm i0)$ . The radiation condition bounds for real spectral parameters follow directly from Theorem 1.10.

**Corollary 1.12.** *Suppose Condition 1.9, and let  $I \subseteq \mathcal{I}$  be any relatively compact open subset. Then for all  $\beta \in [0, \beta_c)$  there exists  $C > 0$  such that for any  $\phi = R(\lambda \pm i0)\psi$  with  $\psi \in r^{-\beta}B$  and  $\lambda \in I$*

$$\|r^{\beta}(A \mp a)\phi\|_{B^*} + \langle p_i^* r^{2\beta} h^{ij} p_j \rangle_{\phi}^{1/2} \leq C \|r^{\beta}\psi\|_B, \quad (1.25)$$

respectively.

We shall see in Subsection 1.2 that for the Euclidean and the hyperbolic spaces without potential  $V$  we have  $\beta_c \geq 1$ . Hence in these cases the bound (1.25) hold for any  $\beta \in [0, 1)$ . We remark that for the Euclidean space and a sufficiently regular potential the bound (1.25) is well-known for  $\beta \in [0, 1)$ , cf. [Is, Sa1, HS]. However in this case one can actually allow  $\beta \in [1, 2)$ , cf. [HS].

As another application of the radiation condition bounds we can characterize the limiting resolvents  $R(\lambda \pm i0)$ . For the Euclidean space such characterization is usually referred to as the *Sommerfeld uniqueness result*, see for example [Is].

**Corollary 1.13.** *Suppose Condition 1.9, and let  $\lambda \in \mathcal{I}$ ,  $\phi \in \mathcal{H}_{\text{loc}}$  and  $\psi \in r^{-\beta}B$  with  $\beta \in [0, \beta_c)$ . Then  $\phi = R(\lambda \pm i0)\psi$  holds if and only if both of the following conditions hold:*

- (i)  $(H - \lambda)\phi = \psi$  in the distributional sense.
- (ii)  $\phi \in \mathcal{N} \cap r^{\beta}B^*$  and  $(A \mp a)\phi \in r^{-\beta}B_0^*$ .

**1.2. Discussion of simple models.** Let us provide several examples here.



1.2.1. *Ends of warped-product type.* Let  $(M, g)$  be a complete Riemannian manifold. Suppose that there exist a relatively compact open subset  $B \subseteq M$  and a  $(d - 1)$ -dimensional closed Riemannian manifold  $(S, h)$  such that isometrically

$$M \setminus B \cong [2, \infty) \times S, \quad \partial B \cong \{2\} \times S,$$

and that in the coordinates  $(r, \sigma) \in [2, \infty) \times S$  the metric  $g$  is of warped-product type:

$$g(r, \sigma) = dr \otimes dr + f(r)h(\sigma); \quad h(\sigma) = h_{\alpha\beta}(\sigma) d\sigma^\alpha \otimes d\sigma^\beta. \quad (1.26)$$

Here the Greek indices run over  $2, \dots, d$ . To make contact to Condition 1.1 we identify  $\{r_0\} \times S = S \subseteq M$  for any fixed  $r_0 \geq 4$  and modify the coordinate  $r$  suitably to become a globally defined smooth function. Such a modified  $r$  obviously conforms with the bounds in (1.6b) of Condition 1.1 on a compact subset. Below we examine in more detail in this particular setting the content of a number of the bounds of Subsection 1.1 by specifying  $f$  explicitly. Whence we consider  $E \cong (r_0, \infty) \times S$ , and more generally, the spherical coordinates are well-defined on  $M \setminus B$  and the Christoffel symbols are computed there as follows:

$$\begin{aligned} \Gamma_{rr}^r &= 0, & \Gamma_{r\alpha}^r &= \Gamma_{\alpha r}^r = 0, & \Gamma_{rr}^\alpha &= 0, \\ \Gamma_{\alpha\beta}^r &= -\frac{1}{2}f'h_{\alpha\beta}, & \Gamma_{r\beta}^\alpha &= \Gamma_{\beta r}^\alpha = \frac{1}{2}(f'/f)\delta^{\alpha\beta}, & \Gamma_{\beta\gamma}^\alpha &= (\Gamma_S)_{\beta\gamma}^\alpha, \end{aligned}$$

where  $\delta^{\alpha\beta}$  denotes Kronecker's  $\delta$  and  $\Gamma_S$  the Christoffel symbol for  $h$  on  $S$ . Hence

$$|dr|^2 = 1, \quad \nabla^2 r = \frac{1}{2}f'h, \quad \Delta r = \frac{d-1}{2}f'/f, \quad \iota_R^* \nabla^3 r = 0. \quad (1.27)$$

For the last calculation  $\iota_R^* \nabla^3 r = 0$  (to be relevant only to [IS3]) we used the compatibility condition (2.2). Now we can verify the conditions of Subsection 1.1 (with  $V \equiv 0$ ) for the following examples.

**Examples 1.14.** (1) Let

$$f(r) = r^\theta; \quad \theta > 0.$$

Condition 1.9 is satisfied for  $\sigma = \theta$ , any  $\tau > 0$ ,  $\rho' = 2$  and  $\rho = 6$ , and the critical energy is  $\lambda_0 = 0$ . The Euclidean space corresponds to  $f(r) = r^2$  and  $S$  being the standard unit sphere.

(2) Let

$$f(r) = \exp(\delta r^\theta); \quad 0 < \delta, \quad 0 < \theta < 1.$$

Condition 1.9 is satisfied for any  $\sigma > 0$ , any  $\tau > 0$ ,  $\rho' = 2 - 2\theta$  and  $\rho = 6 - 4\theta$ , and the critical energy is  $\lambda_0 = 0$ .

(3) Let

$$f(r) = C \exp(\kappa r + \delta_\theta(r)); \quad C, \kappa > 0, \quad \theta < 1.$$

Here  $\theta$  is an order parameter in the sense that the derivatives

$$\delta_\theta^{(k)}(r) = O(r^{\theta-k}); \quad k = 0, 1, 2, \dots$$

Condition 1.9 is satisfied for any  $\sigma > 0$ , any  $\tau > 0$ ,  $\rho' = 1 - \theta$  and  $\rho = 4 - 2\theta$ , and the critical energy is  $\lambda_0 = (d - 1)^2 \kappa^2 / 32$ . The hyperbolic space corresponds to  $f(r) = (\sinh r)^2$  and  $S$  being the standard unit sphere, for which  $\theta < 1$  may be arbitrary.

Note that, if we can choose  $2\beta_c = \min\{\sigma, \tau, \rho\} > 1$ , the above models also fulfill Condition 1.16 (2) of [IS3]. In particular all of the results of [IS3] apply to these examples.

Furthermore we can perturb the models of Examples 1.14. For example, we can add to (1.26) some lower order terms, whether warped-product type or not. We can also put any compact obstacle or attach handles topologically. Obstacles can be non-compact if the gradient vector field  $\omega$  is inward pointing as follows.

**Example 1.15.** Let  $(M, g)$  be any of the Riemannian manifolds discussed above. Let  $\Omega \subseteq M$  be a domain such that its  $r$ -sections  $U_R = \Omega \cap (\{R\} \times S)$  are increasing: For some  $R_0 \geq 3$

$$U_R \subseteq U_{R'} \quad \text{for all } R' \geq R \geq R_0 - 1.$$

We can modify the function  $r$  on  $\Omega \setminus ([R, \infty) \times S)$  so that the gradient vector field  $\omega$  is forward complete on  $\Omega$ . Then  $\Omega$  satisfies Condition 1.9 with the same  $\sigma, \tau, \rho', \rho$  as those of  $M$ , and Condition 1.16 (2) [IS3] is fulfilled as well. This construction includes solid cones in the Euclidean and the hyperbolic spaces (for example half-spaces) for which one has  $U_R = U_{R'}$ .

1.2.2. *Unbounded obstacles in  $\mathbb{R}^2$ .* We give examples of manifolds  $M \subseteq \mathbb{R}^2$  equipped with the Euclidean metric and possessing non-compact obstacles. Again  $V \equiv 0$  and we shall refer to conditions of our sequel [IS3].

**Examples 1.16.** 1) Let  $[x]$  denote the integer part of  $x > 0$  and let  $[x]_- = [x]$  for  $x \notin \mathbb{N}$  and  $[n]_- = n - 1$  for  $n \in \mathbb{N}$ . Consider the ‘‘saw-tooth region’’ defined in terms of a parameter  $K > 0$  as

$$M = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > K(x - [x]_-)/(1 + [x]_-)\}.$$

Define then  $r \geq 1$  by the formula

$$r^2 = 1 + x^2 + (y + K)^2.$$

In this case  $\omega$  is forward complete and the other conditions of this paper, as well as Condition 1.16 (2) of [IS3], are fulfilled too.

2) Consider

$$M = \{(x, y) \in \mathbb{R}^2 \mid xy < 1\}.$$

Define then in terms of a parameter  $K > 2$  a function  $r \geq 1$  by the formula

$$r^2 = x^2 + y^2 + \frac{K}{2} \ln((y - x)^2 + 2).$$

We compute at the boundary  $\partial M \subseteq \mathbb{R}^2$

$$\frac{1}{2} \nabla r^2 \cdot \nabla(xy) = 2 - K + 2K(x^2 + y^2)^{-1}.$$

This expression is negative for  $r$  big, more precisely for  $x^2 + y^2 > 2K/(K - 2)$ , and forward completeness is fulfilled at infinity. In  $M$  (as well as at  $\partial M \subseteq \mathbb{R}^2$ )

$$\frac{1}{2} \nabla r^2 = (x + K(x - y)((y - x)^2 + 2)^{-1}, y + K(y - x)((y - x)^2 + 2)^{-1}).$$

Using the identity  $dr = 2^{-1}r^{-1}\nabla r^2$  we then obtain

$$|rdr|^2 = x^2 + y^2 + 2K^2 \frac{(y - x)^2}{((y - x)^2 + 2)^2} + 2K \frac{(y - x)^2}{(y - x)^2 + 2},$$

and hence for any non-negative integer  $\alpha$

$$\nabla^\alpha(|dr|^2 - 1) = \ln((y-x)^2 + 2)O(r^{-2-|\alpha|}).$$

Similarly for the convexity we compute

$$r\nabla^2 r = \ell + \ln((y-x)^2 + 2)O(r^{-2}).$$

We can easily show that the conditions of this paper are fulfilled for any  $\sigma, \tau, \rho < 2$  (in particular for some  $\sigma, \tau, \rho > 1$ ). More generally we can again verify Condition 1.16 (2) of [IS3] and hence obtain the conclusions of [IS3].

- 3) Fix  $\kappa \in (0, 1)$ , let  $\theta := xy^{-\kappa}$  for  $y > 0$  and let  $r^2 := \kappa x^2 + y^2$ . Consider  $M \subset \mathbb{R}^2$  with an end described as

$$E = \{(x, y) \in \mathbb{R} \times \mathbb{R}_+ \mid r > r_0, \quad -1 < \theta < 1\},$$

which is a cylinder in the variables  $r$  and  $\theta$ . The conditions of this paper are indeed fulfilled, cf. [Min]. However they are not met with  $2\beta_c > 1$  as required in [IS3]. If on the other hand  $\kappa \geq 1$  we can let  $r^2 := x^2 + y^2$  for this model and indeed the conditions of this paper are fulfilled for any  $\sigma, \tau, \rho < 2$ . Whence [IS3] is applicable for  $\kappa \geq 1$ . This agrees with Example 1.15 as well as with [Co].

1.2.3. *Multi-ends with different critical energies.* Here we discuss Condition 1.6. Let us consider the simplest situation: Let  $M$  be the 1-dimensional Euclidean space  $\mathbb{R}$ , which has exactly two ends, and the Schrödinger operator  $H$  be given by

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V \quad \text{on } \mathcal{H} = L^2(\mathbb{R}),$$

where  $V \in C^\infty(\mathbb{R})$  is equal to different constants  $\lambda_0 < \lambda_1$  on the two ends:

$$V(x) = \begin{cases} \lambda_0 & \text{for } x \geq 1, \\ \lambda_1 & \text{for } x \leq -1. \end{cases}$$

If we choose  $r \in C^\infty(\mathbb{R})$  such that

$$r = \begin{cases} x & \text{for } x \geq 2, \\ 1 & \text{for } x \leq 1, \end{cases}$$

then clearly Conditions 1.1–1.3 are satisfied with critical energy  $\lambda_0$ . In this case, although the  $r$ -balls are unbounded, Condition 1.6 is certainly satisfied:

**Lemma 1.17.** *Under the above setting Condition 1.6 holds with  $\mathcal{I} = (\lambda_0, \lambda_1)$ .*

*Proof.* Fix any  $n \geq 0$  and any relatively compact open subset  $I \subseteq \mathcal{I}$ . We let  $\{\psi_k\}_{k \geq 0} \subseteq \mathcal{H}$  be a bounded sequence, and set  $\phi_k = \chi_n P_H(I)\psi_k$ . It is clear that the sequence  $\{\phi_k\}_{k \geq 0}$  is bounded in the (usual) Sobolev space  $H^1(\mathbb{R})$ . Hence by Rellich's compact embedding theorem and the diagonal argument it suffices to show that

$$\limsup_{\nu \rightarrow \infty} \sup_k \|\check{\chi}_\nu \phi_k\| = 0; \quad \check{\chi}_\nu(x) = 1 - \chi(-x/R_\nu),$$

cf. (1.1). We choose  $f \in C_0^\infty(\mathcal{I})$  with  $f = 1$  on a neighborhood of  $I$ , and decompose

$$\check{\chi}_\nu \phi_k = f(H)\check{\chi}_\nu \phi_k + (1 - f(H))\check{\chi}_\nu \phi_k.$$

The terms on the right-hand side above can be estimated by a commutator method. We omit detailed computations, but it is typical to estimate them by using the Helffer–Sjöstrand formula as follows: Uniformly in  $k, \nu \geq 0$

$$\|(1 - f(H))\check{\chi}_\nu \phi_k\| = \|[\check{\chi}_\nu, f(H)]P_H(I)\psi_k\| \leq C_1 R_\nu^{-1};$$

similarly, since we have uniformly in  $k \geq 0$  and  $\nu \geq 1$

$$\begin{aligned} (\sup \text{supp } f) \|f(H)\check{\chi}_\nu \phi_k\|^2 &\geq \langle f(H)\check{\chi}_\nu \phi_k, Hf(H)\check{\chi}_\nu \phi_k \rangle \\ &= \langle f(H)\check{\chi}_\nu \phi_k, \check{\chi}_{\nu-1} H \check{\chi}_{\nu-1} f(H)\check{\chi}_\nu \phi_k \rangle \\ &\quad + \langle [f(H), \check{\chi}_{\nu-1}] \check{\chi}_\nu \phi_k, H \check{\chi}_{\nu-1} f(H)\check{\chi}_\nu \phi_k \rangle \\ &\quad + \langle f(H)\check{\chi}_\nu \phi_k, H[f(H), \check{\chi}_{\nu-1}] \check{\chi}_\nu \phi_k \rangle \\ &\geq \lambda_1 \|\check{\chi}_{\nu-1} f(H)\check{\chi}_\nu \phi_k\|^2 - C_2 R_\nu^{-1} \\ &\geq \lambda_1 \|f(H)\check{\chi}_\nu \phi_k\|^2 - C_3 R_\nu^{-1}, \end{aligned}$$

it follows that

$$\|f(H)\check{\chi}_\nu \phi_k\| \leq C_4 R_\nu^{-1/2}.$$

Hence we are done.  $\square$

By Lemma 1.17 the results of Subsection 1.1 holds true for  $\mathcal{I} = (\lambda_0, \lambda_1)$ . However, here we note that we may retake  $r \in C^\infty(\mathbb{R})$  such that

$$r = |x| \quad \text{for } |x| \geq 2.$$

Then we have Conditions 1.1–1.3 with critical energy  $\lambda_1$ , and also Condition 1.6 for  $\mathcal{I} = (\lambda_1, \infty)$ , since now the  $r$ -balls are bounded and Rellich’s compact embedding theorem applies. Hence we actually have the results of Subsection 1.1 for  $\mathcal{I} = (\lambda_0, \infty) \setminus \{\lambda_1\}$ .

The above arguments easily generalize to a manifold with several ends possibly with different critical energies. For such a model the results of Subsection 1.1 hold true above the minimal critical energy except possibly for the other critical energies, or thresholds. (Of course Theorem 1.4 holds true also at these thresholds.)

In a multi-end setting the limiting absorption principle above the minimal critical energy is obtained in [Ku4]. There Kumura deals with an exact distance function for which a strong (short-range type) condition on asymptotics for  $\Delta r$  holds, cf. (2.21). On the other hand Kumura does not require bounds on the first derivative of  $\Delta r$  as done for the escape function of this paper. Whence [Ku4] is not directly comparable with ours. However it would be possible to modify our arguments to cover situations with less regularity as in [Ku4, IS2]. For simplicity of presentation we are not going to do this, however we have devoted Remarks 2.3 to some elaboration.

**1.2.4. Comparison with our previous model** [IS1]. Below we extract and reformulate the essential parts of the conditions of [IS1] in a form similar to the setting of the present paper. These conditions are more restrictive than those of the present paper.

**Condition 1.18.** Let  $(M, g)$  be a connected and complete Riemannian manifold, and let  $V \in L^\infty(M)$  be real-valued. There exist an unbounded function  $r \in C^\infty(M)$  and constants  $\delta, \kappa, \eta, C > 0$  and  $r_0 \geq 2$  such that:

- (1) The  $r$ -balls  $B_R = \{x \in M \mid r(x) < R\}$ ,  $R > 0$ , are relatively compact in  $M$ .

(2) The following relations hold for  $r(x) = R > r_0/2$ :

$$|dr| = 1, \quad R\iota_R^*(\nabla^2 r) \geq \frac{1}{2}(1 + \delta)\iota_R^*g.$$

(3) The following estimates hold globally on  $M$  for  $\alpha = 0, 1$ :

$$r \geq 1, \quad |\nabla^\alpha \Delta r| \leq Cr^{-1/2-\alpha-\kappa}, \quad |V| \leq Cr^{-1-\eta}. \quad (1.28)$$

**Example 1.19.** Obviously Condition 1.18 follows from Conditions 1.1–1.4 of [IS1]. On the other hand Condition 1.18 constitutes what is used in the proofs of [IS1] and consequently the results of [IS1] remain valid under Condition 1.18. Clearly the second bound of (1.28) implies the *metric short-range condition*

$$(\Delta r)^2 = O(r^{-1-2\kappa}),$$

and, in order to simplify  $a$  of (1.19), let us here propose to set  $q_1 = 0$ . Then clearly

$$q_2 = O(r^{-1-\min\{2\kappa, 1/2+\kappa, \eta\}}).$$

Hence Condition 1.18 suffices for applying this paper. Note that we are not claiming that [IS3] is applicable without additional conditions (not to be examined here).

## 2. GENERATORS OF RADIAL TRANSLATIONS

**2.1. Elementary tensor analysis.** Here we fix our convention for the covariant derivatives. We formulate and use them always in terms of local expressions, but for a coordinate-independent representation, see [Ch, p. 34].

We shall denote two tensors by the same symbol if they are related with each other through the canonical identification  $TM \cong T^*M$ , and distinguish them by super- and subscripts. We denote  $TM \cong T^*M$  by  $T$  for short, and set  $T^p = T^{\otimes p}$ . The covariant derivative  $\nabla$  acts as a linear operator  $\Gamma(T^p) \rightarrow \Gamma(T^{p+1})$  and is defined for  $t \in \Gamma(T^p)$  by

$$(\nabla t)_{ji_1 \dots i_p} = \nabla_j t_{i_1 \dots i_p} = \partial_j t_{i_1 \dots i_p} - \sum_{s=1}^p \Gamma_{j i_s}^k t_{i_1 \dots k \dots i_p}. \quad (2.1)$$

Here  $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$  is the Christoffel symbol, and we adopt the convention that a new subscript is always added to the left as in (2.1). By the identification  $TM \cong T^*M$  it suffices to discuss an expression only for the subscripts. In fact, we have the compatibility condition

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl} = 0, \quad (2.2)$$

and then by (2.1) and (2.2) the covariant derivative can be computed for the tensors of any type. For example, for  $t \in \Gamma(T) = \Gamma(T^1)$

$$(\nabla t)_j^i = g^{ik}(\nabla t)_{jk} = g^{ik}(\partial_j t_k - \Gamma_{jk}^l t_l) = g^{ik}(\partial_j g_{kl} t^l - \Gamma_{jk}^l g_{lm} t^m) = \partial_j t^i + \Gamma_{jk}^i t^k,$$

and this extends to the general case with ease. The covariant derivative acts as a derivation with respect to tensor product, i.e. for  $t \in \Gamma(T^p)$  and  $u \in \Gamma(T^q)$

$$(\nabla(t \otimes u))_{ji_1 \dots i_{p+q}} = (\nabla t)_{ji_1 \dots i_p} u_{i_{p+1} \dots i_{p+q}} + t_{i_1 \dots i_p} (\nabla u)_{ji_{p+1} \dots i_{p+q}}. \quad (2.3)$$

The formal adjoint  $\nabla^* : \Gamma(T^{p+1}) \rightarrow \Gamma(T^p)$  is defined to satisfy

$$\int \overline{u_{ji_1 \dots i_p}} (\nabla t)^{ji_1 \dots i_p} (\det g)^{1/2} dx = \int \overline{(\nabla^* u)_{i_1 \dots i_p}} t^{i_1 \dots i_p} (\det g)^{1/2} dx$$

for  $u \in \Gamma(T^{p+1})$  and  $t \in \Gamma(T^p)$  compactly supported in a coordinate neighborhood. Actually we can write it in a divergence form: For  $u \in \Gamma(T^{p+1})$

$$(\nabla^* u)_{i_1 \dots i_p} = -(\operatorname{div} u)_{i_1 \dots i_p} = -(\nabla u)_j^j{}_{i_1 \dots i_p} = -g^{jk}(\nabla u)_{jki_1 \dots i_p}.$$

Finally let us give several remarks. It is clear that for any function  $f \in \Gamma(T^0) = C^\infty(M)$  the second covariant derivative  $\nabla^2 f = \nabla \nabla f$  is symmetric, i.e.

$$(\nabla^2 f)_{ij} = (\nabla^2 f)_{ji} = \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f, \quad (2.4)$$

and we have expressions for the Laplace–Beltrami operator  $\Delta$ :

$$\Delta f = (\nabla^2 f)_i^i = g^{ij}(\nabla^2 f)_{ij} = \operatorname{tr} \nabla^2 f = \operatorname{div} \nabla f.$$

We note that covariant differentiation and contraction are commuting operations. Whence we have, for example, for  $t \in \Gamma(T)$  and  $u \in \Gamma(T^{p+1})$

$$\begin{aligned} \nabla_k t^j u_{ji_1 \dots i_p} &= (\nabla t)_k^j u_{ji_1 \dots i_p} + t^j (\nabla u)_{kji_1 \dots i_p}, \\ \nabla_j (\nabla t)_i^i &= (\nabla^2 t)_{ji}^i = g^{ik} (\nabla^2 t)_{jik}. \end{aligned} \quad (2.5)$$

Let us verify various estimates that can be deduced from Conditions 1.1 and 1.2.

**Lemma 2.1.** *Suppose Conditions 1.1 and 1.2. Then one has*

$$(\nabla |dr|^2)^i = 2(\nabla^2 r)^{ij} (\nabla r)_j, \quad (2.6a)$$

and there exists  $C > 0$  such that

$$\nabla^2 r \leq Cg, \quad \nabla^2 r \geq -Cr^{-1-\tau/2}g, \quad \Delta r \geq -Cr^{-1-\tau/2}. \quad (2.6b)$$

*Proof.* The formula (2.6a) is a consequence of the above tensor analysis. Since the tensor  $h$  of (1.16) is non-negative, we have

$$h \leq (\operatorname{tr} h)g = (\Delta r - \frac{1}{2}\eta |dr|^{-2} (\nabla^r |dr|^2) + C_1 dr^{-1-\tau})g \leq C_2 g,$$

and then it follows that

$$\nabla^2 r \leq C_2 g + \frac{1}{2}\eta |dr|^{-4} (\nabla^r |dr|^2) dr \otimes dr - C_1 r^{-1-\tau} g \leq C_3 g.$$

This verifies the first bound of (2.6b). The second bound of (2.6b) follows from (1.6a), and the third bound follows by taking the trace of the second bound.  $\square$

**Remark.** Clearly a combination of bounds of Conditions 1.2 and 1.9 amounts to requiring  $|\nabla |dr|^2| \leq Cr^{-1-\tau/2}$ . Suppose in addition that for some  $\sigma' > \sigma$

$$R \iota_R^* (\nabla^2 r) \geq \frac{1}{2} \sigma' |dr|^2 \iota_R^* g; \quad R \geq r_0. \quad (2.7)$$

Then by the Cauchy-Schwarz inequality and (2.6a) in fact (1.6a) holds. Whence (1.6a) and (2.7) can be considered as being essentially equivalent.

**2.2. Semigroups of radial translations.** In this subsection we define and discuss semigroups of *unnormalized* radial translations (in [IS3] we are going to use spherical coordinates defined by normalized radial translations). Let

$$y: \mathcal{M} \rightarrow M, \quad (t, x) \mapsto y(t, x) = \exp(t\omega)(x); \quad \mathcal{M} \subseteq \mathbb{R} \times M,$$

be the maximal flow generated by the vector field  $\omega$ . By (1) of Condition 1.1 the set  $\mathcal{M}$  contains a neighborhood of  $[0, \infty) \times M$  in  $\mathbb{R} \times M$ . Note that by definition it satisfies, in local coordinates,

$$\partial_t y^i(t, x) = \omega^i(y(t, x)) = (\nabla r)^i(y(t, x)), \quad y(0, x) = x. \quad (2.8)$$

We define the ‘‘radial translations’’  $T(t): \mathcal{H} \rightarrow \mathcal{H}$ ,  $t \in \mathbb{R}$ , by

$$\begin{aligned} (T(t)\psi)(x) &= J(t, x)^{1/2} (\det g(y(t, x))/\det g(x))^{1/4} \psi(y(t, x)) \\ &= \exp\left(\int_0^t \frac{1}{2}(\Delta r)(y(s, x)) ds\right) \psi(y(t, x)) \end{aligned} \quad (2.9)$$

if  $(t, x) \in \mathcal{M}$ , and  $(T(t)\psi)(x) = 0$  otherwise, where  $J(t, \cdot)$  is the Jacobian of the mapping  $y(t, \cdot): M \rightarrow M$ . Here let us verify the well-definedness and the equivalence of the two expressions in (2.9):

*Verification of (2.9).* If we set the pull-back

$$(g^*)_{ij}(t, x) = g_{\alpha\beta}(y(t, x))[\partial_i y^\alpha(t, x)][\partial_j y^\beta(t, x)], \quad (2.10)$$

then we can write

$$J(t, x)^2 \det g(y(t, x)) = \det g^*(t, x). \quad (2.11)$$

Since  $g$  and  $g^*$  are subject to the same transformation rule under change of coordinates, so are  $\det g$  and  $\det g^*$ , and whence the first expression of (2.9) is coordinate-invariant and well-defined.

Next we prove the second equality of (2.9). Note that by the coordinate-invariance noted above we can choose specific coordinates to prove it. Let us consistently use the Roman and the Greek indices to denote quantities concerning  $x$  and  $y = y(t, x)$ , respectively. Differentiating the expression (2.10) and using the compatibility condition (2.2) and the equation (2.8), we can compute

$$\begin{aligned} \frac{\partial}{\partial t}(g^*)_{ij} &= [\Gamma_{\gamma\alpha}^\delta g_{\delta\beta} + \Gamma_{\gamma\beta}^\delta g_{\alpha\delta}](\nabla r)^\gamma (\partial_i y^\alpha)(\partial_j y^\beta) \\ &\quad + g_{\alpha\beta}(\partial_\gamma(\nabla r)^\alpha)(\partial_i y^\gamma)(\partial_j y^\beta) + g_{\alpha\beta}(\partial_\gamma(\nabla r)^\beta)(\partial_i y^\alpha)(\partial_j y^\gamma) \\ &= 2(\nabla^2 r)_{\alpha\beta}(\partial_i y^\alpha)(\partial_j y^\beta). \end{aligned} \quad (2.12)$$

Let us choose local coordinates around  $x$  such that the matrix  $((g^*)_{ij})_{i,j}$  is diagonal, and then introduce the orthonormal basis of tangents at  $y$

$$t_i = (g^*)_{ii}^{-1/2}(\partial_i y^\alpha) \frac{\partial}{\partial y^\alpha}; \quad i = 1, \dots, d.$$

Then by (2.12)

$$\sum_{i=1}^d (g^*)_{ii}^{-1} \frac{\partial}{\partial t}(g^*)_{ii} = 2 \sum_{i=1}^d (\nabla^2 r)(t_i, t_i) = 2 \operatorname{tr}((\nabla^2 r)_{\alpha\beta})_{\alpha,\beta} = 2\Delta r. \quad (2.13)$$

On the other hand, we have by (2.11)

$$\begin{aligned} \frac{\partial}{\partial t} \ln (J(t, x)^2 \det g(y(t, x))/\det g(x)) &= \frac{\partial}{\partial t} \ln \left( \prod_{i=1}^d (g^*)_{ii} \right) \\ &= \sum_i (g^*)_{ii}^{-1} \frac{\partial}{\partial t}(g^*)_{ii}. \end{aligned} \quad (2.14)$$

Hence the second equality of (2.9) follows by (2.13) and (2.14).  $\square$

Now it follows by the former expression of (2.9) that for any  $\psi \in \mathcal{H}$

$$\|T(t)\psi\| = \left( \int_{M(t)} |\psi(x)|^2 (\det g(x))^{1/2} dx \right)^{1/2}; \quad M(t) = y(\max\{t, 0\}, M),$$

and hence  $T(t)$ ,  $t \geq 0$ , form a strongly continuous one-parameter semigroup of surjective partial isometries, and  $T(-t)$ ,  $t \geq 0$ , form that of isometries, being the adjoints of each other:  $T(t)^* = T(-t)$ . We remark that in general  $T(t)$ ,  $t \in \mathbb{R}$ , do not form a group, but, if the gradient vector field  $\omega$  is both forward and backward complete, then they do and hence are unitary.

Next we investigate the generators  $A_{\pm}$  of semigroups  $T(\pm t)$ ,  $t \geq 0$ . We let

$$\begin{aligned} \mathcal{D}(A_{\pm}) &= \{\psi \in \mathcal{H} \mid \lim_{t \rightarrow 0^+} (\pm it)^{-1}(T(\pm t)\psi - \psi) \text{ exists in } \mathcal{H}\}, \\ A_{\pm}\psi &= \lim_{t \rightarrow 0^+} (\pm it)^{-1}(T(\pm t)\psi - \psi) \quad \text{for } \psi \in \mathcal{D}(A_{\pm}). \end{aligned}$$

respectively. By the Hille–Yosida theorem [RS, Theorem X.47a] the operators  $A_{\pm}$  are densely defined closed operators on  $\mathcal{H}$ . We note that  $T(-t)$  preserves  $C_c^{\infty}(M)$  (see the proof of Lemma 2.6 below) and whence by [RS, Theorem X.49] this space is a core for  $A_{-}$ . In particular  $A_{-}$  is symmetric. In addition we easily verify that  $A_{-} \subseteq A_{+}^*$  and therefore, cf. [RS, Theorem X.47a],

$$A_{\pm} = A_{\mp}^*, \tag{2.15}$$

respectively. Moreover we have inclusions

$$C_c^{\infty}(M) \subseteq \mathcal{D}(H) \subseteq \mathcal{H}^1 \subseteq \mathcal{D}(A_{\pm}), \tag{2.16}$$

and  $A_{\pm}$  coincide with the (maximal) distributional differential operator  $A$  on  $\mathcal{D}(A_{\pm})$ :

$$A_{\pm} = A = \operatorname{Re} p^r = \frac{1}{2}(p^r + (p^r)^*) \quad \text{on } \mathcal{D}(A_{\pm}), \tag{2.17}$$

respectively, cf. (1.2). In fact  $\mathcal{D}(A_{+})$  is exactly the domain of the maximal distributional differential operator  $A = \frac{1}{2}(p^r + (p^r)^*)$ . We may call  $A = A_{+}$  the *conjugate operator associated with  $r$*  although  $A_{-}$  would deserve the same name due to the inclusion relations (2.16). We are going to demonstrate that the corresponding commutator with  $H$  tends to be positive although this will be in a different sense from that of [Mo].

**2.3. Commutators with weight inside.** Using (2.4) and (2.5) we could compute the simple commutator  $[H, iA]$ . However, in the later sections we shall actually use more general commutators with a weight  $\Theta$  inside:

$$[H, iA]_{\Theta} := i(H\Theta A - A\Theta H). \tag{2.18}$$

Hence, in this subsection, we explicitly compute the weighted commutator (2.18), and precisely formulate how we should realize it as an operator.

Let  $\Theta = \Theta(r)$  be a non-negative smooth function only of  $r$  with bounded derivatives. More explicitly, if we denote its derivatives in  $r$  by primes such as  $\Theta'$ , then

$$\Theta \geq 0, \quad |\Theta^{(k)}| \leq C_k, \quad k = 0, 1, 2, \dots \tag{2.19}$$

We first define the weighted commutator (2.18) as a quadratic form on  $C_c^{\infty}(M)$ , and then extend it onto  $\mathcal{H}^1$  by the following lemma. Throughout the paper we shall always use the notation  $[H, iA]_{\Theta}$  in this extended sense.



**Lemma 2.2.** *Suppose Conditions 1.1–1.3, and let  $\Theta$  be a non-negative smooth function of  $r$  with bounded derivatives (2.19). Then, as quadratic forms on  $C_c^\infty(M)$ ,*

$$\begin{aligned} [H, iA]_\Theta &= A\Theta'A + p_i^*\Theta(\nabla^2 r)^{ik}p_k - \text{Im}(\Theta'(dr)_i(\nabla^2 r)^{ij}p_j) - \frac{1}{4}|dr|^4\Theta''' + q_\Theta \\ &\quad - 2\text{Im}(q_2\Theta p^r) - \frac{1}{2}\text{Im}(\Theta(\nabla_i\Delta r)\ell^{ij}p_j) - \text{Re}(|dr|^2\Theta'H); \\ q_\Theta &= -(\nabla^r q_1)\Theta + q_2(\Delta r)\Theta + \frac{1}{8}(\nabla^r\tilde{\eta})(\Delta r)^2\Theta \\ &\quad + \frac{1}{4}(1-\eta)(\nabla^r\Delta r)\Theta' + |dr|^2q_2\Theta' - \frac{1}{4}(\nabla^r|dr|^2)\Theta''. \end{aligned} \quad (2.20)$$

*In particular by formally absorbing  $q_1$  into  $q_2$  (undoing commutator on  $q_1$ ) and using the Cauchy-Schwarz inequality  $[H, iA]_\Theta$  restricted to  $C_c^\infty(M)$  extends to a bounded form on  $\mathcal{H}^1$ , and whence  $[H, iA]_\Theta$  can be regarded as a bounded operator  $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ .*

**Remarks 2.3.** Of course, there are several variations of the expression for  $[H, A]_\Theta$ .

- (1) To verify the latter part of the assertion the expression (2.28) below would be more natural and convenient. However, in our application in the later sections, we will always use the expression (2.20).
- (2) In [Ku4] Kumura obtains the limiting absorption principle with  $V \equiv 0$  but without any regularity on  $\Delta r$  (for an exact distance function  $r$ ). This involves strong asymptotics hypotheses, in particular it is required in hyperbolic type ends that

$$\Delta r = \beta + O(r^{-\delta}); \quad \beta > 0, \delta > 1. \quad (2.21)$$

However Kumura does not impose bounds on derivatives. In such a low regularity setting it would be more useful to utilize an expression avoiding the derivatives of  $\Delta r$ , rather than (2.20). Here is such an alternative:

$$\begin{aligned} [H_0, iA]_\Theta &= A\Theta'A + p_i^*\Theta(\nabla^2 r)^{ik}p_k - \text{Im}(\Theta'(dr)_i(\nabla^2 r)^{ij}p_j) - \frac{1}{4}|dr|^4\Theta''' \\ &\quad - \frac{1}{4}(\nabla^r|dr|^2)\Theta'' - \frac{1}{4}|dr|^2(\Delta r)\Theta'' + \frac{1}{2}\text{Im}((\Delta r)\Theta'A) \\ &\quad + \text{Re}((\Delta r)\Theta H_0) - \frac{1}{2}\text{Re}(\Theta p_i^*(\Delta r)g^{ij}p_j) - \text{Re}(|dr|^2\Theta'H_0). \end{aligned}$$

For example under the condition (2.21), due to cancellations

$$\begin{aligned} [H_0, iA]_\Theta &= A\Theta'A + p_i^*\Theta(\nabla^2 r)^{ik}p_k - \text{Im}(\Theta'(dr)_i(\nabla^2 r)^{ij}p_j) - \frac{1}{4}|dr|^4\Theta''' \\ &\quad - \frac{1}{4}(\nabla^r|dr|^2)\Theta'' - \frac{1}{4}O(r^{-\delta})\Theta'' + \frac{1}{2}\text{Im}(O(r^{-\delta})\Theta'A) \\ &\quad + \text{Re}(O(r^{-\delta})\Theta H_0) - \frac{1}{2}p_i^*O(r^{-\delta})\Theta g^{ij}p_j - \text{Re}(|dr|^2\Theta'H_0). \end{aligned}$$

The last expression has the same leading terms as those of (2.20), and hence the proofs of the following sections should be applicable in particular to the setting of [Ku4]. There would also appear derivatives of  $\Delta r$  in these proofs, but after “undoing of the commutator” they would contribute to a remainder term only (denoted there by “ $Q$ ”). We shall not give details. We note that (2.21) reasonably may be called “short-range”. On the other hand Example 1.14 (3) includes “long-range” hyperbolic type ends.

- (3) More generally than (2.21), if we assumed short- and long-range splitting conditions for  $\Delta r$  as in [IS2], it should be possible to obtain and slightly generalize the results there. Nevertheless, for simplicity of presentation, we shall not elaborate on details.

For the proof of Lemma 2.2 we shall use the following commutator computation:

**Lemma 2.4.** *Let  $\tilde{g} \in \Gamma(T^{0,2})$  be symmetric, and set*

$$\tilde{H}_0 = -\frac{1}{2}\tilde{\Delta} = \frac{1}{2}p_i^* \tilde{g}^{ij} p_j.$$

*Then, as a quadratic form on  $C_c^\infty(M)$ ,*

$$[\tilde{H}_0, iA] = \frac{1}{2}p_i^* \{ \tilde{g}^i_j (\nabla^2 r)^{jk} + (\nabla^2 r)^i_j \tilde{g}^{jk} - (\nabla^r \tilde{g})^{ik} \} p_k - \frac{1}{4}(\tilde{\Delta} \Delta r).$$

*Proof.* Noting the expression (1.8) and the general identity holding for any  $f \in C^\infty(M)$ :

$$p_i^* f \tilde{g}^{ij} p_j = 2 \operatorname{Re}(f \tilde{H}_0) + \frac{1}{2}(\tilde{\Delta} f), \quad (2.22)$$

we have

$$\begin{aligned} [\tilde{H}_0, iA] &= 2 \operatorname{Im}((p^r)^* \tilde{H}_0) + \operatorname{Re}((\Delta r) \tilde{H}_0) \\ &= 2 \operatorname{Re}(\tilde{H}_0 \nabla^r) + \frac{1}{2}p_i^* (\Delta r) \tilde{g}^{ij} p_j - \frac{1}{4}(\tilde{\Delta} \Delta r), \end{aligned} \quad (2.23)$$

Let us compute the first term on the right-hand side of (2.23) in the form of expectation. This actually prevents unnecessary complication otherwise coming from covariant derivatives on higher-order tensors. For any state  $\psi \in C_c^\infty(M)$

$$\begin{aligned} &\langle 2 \operatorname{Re}(\tilde{H}_0 \nabla^r) \rangle_\psi \\ &= \operatorname{Re} \langle \nabla^* \tilde{g}^{\bullet i} (\nabla \psi)_i, \nabla^r \psi \rangle \\ &= \operatorname{Re} \langle \tilde{g}^{ji} (\nabla \psi)_i, (\nabla^2 r)_j^k (\nabla \psi)_k \rangle + \operatorname{Re} \langle \tilde{g}^{ji} (\nabla \psi)_i, (\nabla r)^k (\nabla^2 \psi)_{jk} \rangle \\ &= \operatorname{Re} \langle p_i^* \tilde{g}^{ij} (\nabla^2 r)_j^k p_k \rangle_\psi \\ &\quad + \frac{1}{2} \left[ \langle (\nabla \psi)_i, \tilde{g}^{ij} (\nabla r)^k (\nabla^2 \psi)_{kj} \rangle + \langle (\nabla^2 \psi)_{kj}, \tilde{g}^{ji} (\nabla r)^k (\nabla \psi)_i \rangle \right] \\ &= \frac{1}{2} \langle p_i^* \{ \tilde{g}^{ij} (\nabla^2 r)_j^k + (\nabla^2 r)^i_j \tilde{g}^{jk} - (\nabla^r \tilde{g})^{ik} \} p_k \rangle_\psi - \frac{1}{2} \langle p_j^* (\Delta r) \tilde{g}^{ji} p_i \rangle_\psi. \end{aligned} \quad (2.24)$$

Hence by (2.23) and (2.24) the assertion follows.  $\square$

*Proof of Lemma 2.2.* By (1.9) and (1.3) we can compute

$$\begin{aligned} [H, iA]_\Theta &= \operatorname{Im}(A \Theta A \tilde{\eta} A) + \operatorname{Im}(A \Theta L) + 2 \operatorname{Im}(A \Theta q) - \frac{1}{2} \operatorname{Im}((\nabla^r \tilde{\eta})(\Delta r) \Theta A) \\ &= \frac{1}{2} A \eta \Theta' A - \frac{1}{2} A (\nabla^r \tilde{\eta}) \Theta A + \frac{1}{2} [p_i^* \Theta \ell^{ij} p_j, iA] + \operatorname{Re}(A(1 - \eta) \Theta' p^r) \\ &\quad - |\operatorname{dr}|^2 q_1 \Theta' - (\nabla^r q_1) \Theta - 2 \operatorname{Im}(q_2 \Theta A) - \frac{1}{2} \operatorname{Im}((\nabla^r \tilde{\eta})(\Delta r) \Theta A). \end{aligned} \quad (2.25)$$

To compute the third term on the right-hand side of (2.25) we apply Lemma 2.4 with  $\tilde{g} = \Theta \ell$ . We also use (1.3), (2.6a) and (1.8), and then we can combine the third and eighth terms of (2.25) as

$$\begin{aligned} &\frac{1}{2} [p_i^* \Theta \ell^{ij} p_j, iA] - \frac{1}{2} \operatorname{Im}((\nabla^r \tilde{\eta})(\Delta r) \Theta A) \\ &= \frac{1}{2} p_i^* \left( \Theta \ell^i_j (\nabla^2 r)^{jk} + \Theta (\nabla^2 r)^i_j \ell^{jk} - (\nabla^r \Theta \ell)^{ik} \right) p_k \\ &\quad + \frac{1}{4} (p_i^* \Theta \ell^{ij} p_j \Delta r) - \frac{1}{2} \operatorname{Im}((\nabla^r \tilde{\eta})(\Delta r) \Theta A) \\ &= \frac{1}{2} p_i^* \left( 2 \Theta (\nabla^2 r)^{ik} - |\operatorname{dr}|^2 \Theta' \ell^{ik} + (\nabla^r \tilde{\eta}) \Theta (\operatorname{dr} \otimes \operatorname{dr})^{ik} \right) p_k \\ &\quad - \frac{1}{2} \operatorname{Im}(\Theta (\nabla_i \Delta r) \ell^{ij} p_i) - \frac{1}{4} (1 - \eta) (\nabla^r \Delta r) \Theta' \\ &\quad - \frac{1}{2} \operatorname{Im}((\nabla^r \tilde{\eta})(\Delta r) \Theta p^r) + \frac{1}{4} (\nabla^r \tilde{\eta})(\Delta r)^2 \Theta \\ &= p_i^* \Theta (\nabla^2 r)^{ik} p_k - \frac{1}{2} p_i^* |\operatorname{dr}|^2 \Theta' \ell^{ik} p_k + \frac{1}{2} A (\nabla^r \tilde{\eta}) \Theta A \\ &\quad - \frac{1}{2} \operatorname{Im}(\Theta (\nabla_i \Delta r) \ell^{ij} p_i) - \frac{1}{4} (1 - \eta) (\nabla^r \Delta r) \Theta' + \frac{1}{8} (\nabla^r \tilde{\eta})(\Delta r)^2 \Theta. \end{aligned} \quad (2.26)$$

By (1.8) we write the fourth and seventh terms of (2.25) as

$$\begin{aligned} & \operatorname{Re}(A(1-\eta)\Theta'p^r) - 2\operatorname{Im}(q_2\Theta A) \\ &= A(1-\eta)\Theta'A - \frac{1}{4}(\nabla^r\eta)(\Delta r)\Theta' + \frac{1}{4}(1-\eta)(\nabla^r\Delta r)\Theta' \\ & \quad + \frac{1}{4}(1-\eta)|dr|^2(\Delta r)\Theta'' + q_2(\Delta r)\Theta - \operatorname{Im}(2q_2\Theta p^r). \end{aligned} \quad (2.27)$$

By (2.25), (2.26) and (2.27) it follows that

$$\begin{aligned} [H, iA]_{\Theta} &= \frac{1}{2}A(2-\eta)\Theta'A + p_i^*\Theta(\nabla^2 r)^{ik}p_k - \frac{1}{2}p_i^*|dr|^2\Theta'\ell^{ik}p_k - (\nabla^r q_1)\Theta \\ & \quad + q_2(\Delta r)\Theta - \frac{1}{2}\operatorname{Im}(\Theta(\nabla_i\Delta r)\ell^{ij}p_j) + \frac{1}{8}(\nabla^r\tilde{\eta})(\Delta r)^2\Theta - |dr|^2q_1\Theta' \\ & \quad - \frac{1}{4}(\nabla^r\eta)(\Delta r)\Theta' + \frac{1}{4}(1-\eta)|dr|^2(\Delta r)\Theta'' - \operatorname{Im}(2q_2\Theta p^r). \end{aligned} \quad (2.28)$$

We combine a part of the first term, the third and eighth terms of (2.28) in the following manner. We make use of the expressions (1.8), (1.9), (1.3) and (2.6a), and then obtain

$$\begin{aligned} & -\frac{1}{2}A\eta\Theta'A - \frac{1}{2}p_i^*|dr|^2\Theta'\ell^{ik}p_k - |dr|^2q_1\Theta' \\ &= -\frac{1}{2}\operatorname{Im}\left[(\nabla^r|dr|^2\Theta')\tilde{\eta}A + (\nabla_i|dr|^2\Theta')\ell^{ik}p_k\right] \\ & \quad - \frac{1}{2}\operatorname{Re}\left[|dr|^2\Theta'A\tilde{\eta}A + |dr|^2\Theta'p_i^*\ell^{ij}p_j\right] - |dr|^2q_1\Theta' \\ &= -\frac{1}{2}\operatorname{Im}\left[(\nabla_i|dr|^2\Theta')g^{ij}p_j\right] + \frac{1}{4}(\nabla^r|dr|^2\Theta')\tilde{\eta}(\Delta r) \\ & \quad - \operatorname{Re}(|dr|^2\Theta'H) + \frac{1}{4}|dr|^2(\nabla^r\tilde{\eta})(\Delta r)\Theta' + |dr|^2q_2\Theta' \\ &= -\operatorname{Im}\left[\Theta'(dr)_i(\nabla^2 r)^{ij}p_j\right] - \frac{1}{4}(\nabla^r|dr|^2)\Theta'' - \frac{1}{4}|dr|^4\Theta''' \\ & \quad - \frac{1}{4}(1-\eta)|dr|^2(\Delta r)\Theta'' - \operatorname{Re}(|dr|^2\Theta'H) + \frac{1}{4}(\nabla^r\eta)(\Delta r)\Theta' + |dr|^2q_2\Theta' \end{aligned} \quad (2.29)$$

If we substitute (2.29) into (2.28), then the expression (2.20) follows.

It remains to show the boundedness of  $[H, iA]_{\Theta}$  as an operator  $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ , but it is obvious by (2.28) or (2.20) and Conditions 1.1–1.3.  $\square$

**2.4. Doing and undoing commutators.** In the previous subsection we defined the weighted commutator  $[H, iA]_{\Theta}$  as a quadratic form on  $\mathcal{H}^1$  by the extension from  $C_c^\infty(M)$ . On the other hand, throughout the paper, we shall use the notation

$$\operatorname{Im}(A\Theta H) = \frac{1}{2i}(A\Theta H - H\Theta A) \quad (2.30)$$

as a quadratic form defined on  $\mathcal{D}(H)$ , i.e. for  $\psi \in \mathcal{D}(H)$

$$\langle \operatorname{Im}(A\Theta H) \rangle_{\psi} = \frac{1}{2i}(\langle A\psi, \Theta H\psi \rangle - \langle H\psi, \Theta A\psi \rangle).$$

Obviously the quadratic forms  $[H, iA]_{\Theta}$  and  $2\operatorname{Im}(A\Theta H)$  coincide on  $C_c^\infty(M)$ , but they do not in general on  $\mathcal{D}(H)$ . This is due to the third order terms in (2.30). Although the third order terms themselves finally cancel out after integrations by parts, the remaining “boundary contribution” is not negligible, see Remark 2.9. Nonetheless, fortunately, these boundary contributions have sign, and Lemma 2.5 below allows us to “do” and “undo” the commutator to some extent.

**Lemma 2.5.** *Suppose Conditions 1.1–1.3, and let  $\Theta$  be a non-negative smooth function of  $r$  with bounded derivatives (2.19). Then, as quadratic form on  $\mathcal{D}(H)$ ,*

$$[H, iA]_{\Theta} \leq 2\operatorname{Im}(A\Theta H). \quad (2.31)$$

In this subsection we prove Lemma 2.5. Let us denote the Dirichlet self-adjoint realization of the operator

$$H_\Theta = \frac{1}{2}p_i^* \Theta g^{ij} p_j + \Theta V \quad \text{on } C_c^\infty(M)$$

by the same symbol  $H_\Theta$ . We denote its operator and form domains by  $\mathcal{D}(H_\Theta)$  and  $\mathcal{H}_\Theta^1$ , respectively, and then obviously we have

$$\mathcal{H}^1 \subseteq \mathcal{H}_\Theta^1, \quad \mathcal{D}(H) \subseteq \mathcal{D}(H_\Theta), \quad (2.32)$$

cf. Subsection 1.1. It follows that, as quadratic forms on  $\mathcal{D}(H)$ ,

$$2 \operatorname{Im}(A\Theta H) = 2 \operatorname{Im}(AH_\Theta) + \operatorname{Re}(A\Theta' p^r) \quad (2.33)$$

and that, as quadratic forms on  $\mathcal{H}^1$  (as extensions from  $C_c^\infty(M)$ ),

$$[H, iA]_\Theta = [H_\Theta, iA] + \operatorname{Re}(A\Theta' p^r). \quad (2.34)$$

The second operators on the right-hand sides of (2.33) and (2.34) clearly coincide on  $\mathcal{D}(H)$ , and hence the proof of Lemma 2.5 reduces to that of

$$[H_\Theta, iA] \leq 2 \operatorname{Im}(AH_\Theta), \quad (2.35)$$

as quadratic forms on  $\mathcal{D}(H)$ .

In order to prove (2.35) we first see regularity properties of the flow (2.9).

**Lemma 2.6.** *Suppose Conditions 1.1–1.3. Then for any  $t \geq 0$  one has the natural bounded extension/restriction  $T(\pm t): \mathcal{H}^{\mp 1} \rightarrow \mathcal{H}^{\mp 1}$ , and*

$$\sup_{t \in [0,1]} \|T(\pm t)\|_{\mathcal{B}(\mathcal{H}^{\mp 1})} < \infty, \quad (2.36)$$

respectively. Moreover,  $T(\pm t) \in \mathcal{B}(\mathcal{H}^{\mp 1})$  are strongly continuous in  $t \geq 0$ , respectively.

*Proof.* It suffices to prove the assertions for  $T(-t)$ ,  $t \geq 0$ , since those for  $T(t)$ ,  $t \geq 0$ , follow by taking the adjoint, cf. [HP, Theorem 10.6.5]. For any  $\psi \in C_c^\infty(M)$  we have by (2.9) and standard regularity properties for flows that

$$\begin{aligned} p_i(T(-t)\psi)(x) &= [\partial_i y^\alpha(-t, x)](T(-t)p_\alpha \psi)(x) \\ &\quad + \left( \int_0^{-t} \frac{1}{2i} [\partial_i y^\alpha(s, x)] (\partial_\alpha \Delta r)(y(s, x)) \, ds \right) (T(-t)\psi)(x) \end{aligned} \quad (2.37)$$

if  $(-t, x) \in \mathcal{M}$ , and  $p_i(T(-t)\psi)(x) = 0$  otherwise. Here we slightly abused notation writing  $(T(-t)p_\alpha \psi)(x)$  rather than the expression  $e^{f \cdots} (p_\alpha \psi)(y(-t, x))$ . We note that by definition for  $(-t, x) \notin \mathcal{M}$  the factor  $(T(-t)\psi)(x) = 0$ . Repeated differentiation leads to the conclusion that  $T(-t)\psi \in C_c^\infty(M) \subseteq \mathcal{H}^1$ .

It follows readily from (2.9) and (2.37) that the  $\mathcal{H}^1$ -valued function  $T(-t)\psi$  for  $\psi \in C_c^\infty(M)$  is continuous in  $t \geq 0$ . Given the boundedness (2.36) we would then obtain the strong continuity of  $T(-t) \in \mathcal{B}(\mathcal{H}^1)$  by a density argument. Hence it remains to show (2.36) for  $T(-t)$ . We shall prove

$$\langle H_0 + 1 \rangle_{T(-t)\psi} \leq C_1$$

independently of  $t \in [0, 1]$  and  $\psi \in C_c^\infty(M)$  with  $\|\psi\|_{\mathcal{H}^1} = 1$ , and for that it suffices to bound

$$f(t) := \langle H + C_2 \rangle_{T(-t)\psi} \geq \langle H_0 + 1 \rangle_{T(-t)\psi}; \quad C_2 = 1 + \|V\|_{L^\infty},$$

above. By Lemma 2.2 we indeed have  $C_3 := \|[H, iA]\|_{\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})} < \infty$ , and then

$$f'(t) = -\langle [H, iA] \rangle_{T(-t)\psi} \leq C_3 \|T(-t)\psi\|_{\mathcal{H}^1}^2 \leq C_4 f(t).$$

This estimate leads to  $f(t) \leq f(0)e^{tC_4}$ , and we are done.  $\square$

**Lemma 2.7.** *Under Conditions 1.1–1.3 there exists  $C > 0$  such that for any  $t \in [0, 1]$*

$$\|H_\Theta - T(t)H_\Theta T(-t)\|_{\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})} \leq Ct.$$

*Proof.* By the inclusion (2.16), as quadratic forms on  $C_c^\infty(M)$ ,

$$H_\Theta - T(t)H_\Theta T(-t) = \int_0^t T(s)[H_\Theta, iA]T(-s) ds.$$

By (2.32), Lemma 2.6 and the denseness of  $C_c^\infty(M) \subseteq \mathcal{H}^1$  the assertion follows.  $\square$

**Lemma 2.8.** *Under Conditions 1.1–1.3 the commutator  $[H_\Theta, iA]$  has the expression*

$$[H_\Theta, iA] = \text{s-}\lim_{t \rightarrow 0^+} t^{-1}(H_\Theta - T(t)H_\Theta T(-t)) \quad \text{in } \mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1}). \quad (2.38)$$

*Proof.* By the inclusion (2.16) for any  $\psi \in C_c^\infty(M)$

$$\begin{aligned} & t^{-1}(H_\Theta - T(t)H_\Theta T(-t))\psi - [H_\Theta, iA]\psi \\ &= t^{-1} \int_0^t \{T(s)[H_\Theta, iA]T(-s) - [H_\Theta, iA]\} \psi ds. \end{aligned}$$

Then we obtain (2.38) on  $C_c^\infty(M)$  due to the strong continuity of  $T(\pm t)$  stated in Lemma 2.6. Then by Lemma 2.7 and the density argument, the strong limit to the right of (2.38) exists in  $\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})$  and the equality holds.  $\square$

*Proof of Lemma 2.5.* It suffices to show (2.35) on  $\mathcal{D}(H)$ . Due to the non-negativity of  $\Theta$ , we have the inequality, as quadratic forms on  $\mathcal{H}^1$ ,

$$\begin{aligned} & H_\Theta - T(t)H_\Theta T(-t) \\ &= H_\Theta(1 - T(-t)) + (1 - T(-t))^* H_\Theta - (1 - T(-t))^* H_\Theta(1 - T(-t)) \\ &\leq H_\Theta(1 - T(-t)) + (1 - T(-t))^* H_\Theta - (1 - T(-t))^* \Theta V(1 - T(-t)). \end{aligned}$$

We evaluate this inequality in the state  $\psi \in \mathcal{D}(H) \subseteq \mathcal{H}_1$ , divide it by  $t > 0$  and take the limit  $t \rightarrow 0^+$  using Lemma 2.8 and (2.16). Then we obtain

$$\begin{aligned} \langle [H_\Theta, iA] \rangle_\psi &= \text{s-}\lim_{t \rightarrow 0^+} t^{-1} \langle H_\Theta - T(t)H_\Theta T(-t) \rangle_\psi \\ &\leq \text{s-}\lim_{t \rightarrow 0^+} t^{-1} \left\{ \langle H_\Theta \psi, (1 - T(-t))\psi \rangle + \langle (1 - T(-t))\psi, H_\Theta \psi \rangle \right. \\ &\quad \left. - \langle (1 - T(-t))\psi, \Theta V(1 - T(-t))\psi \rangle \right\} \\ &= \langle H_\Theta \psi, iA\psi \rangle + \langle iA\psi, H_\Theta \psi \rangle. \end{aligned} \quad (2.39)$$

Hence we are done.  $\square$

**Remark 2.9.** We can also prove that, if the gradient vector field  $\omega$  is both forward and backward complete, then the equality holds in (2.39), and hence also in (2.31). In fact we can prove this by using, instead of (2.38), an alternative expression

$$[H_\Theta, iA] = \text{s-}\lim_{t \rightarrow 0} t^{-1}(H_\Theta T(t) - T(t)H_\Theta) \quad \text{in } \mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})$$

holding true if  $T(t)$  is unitary. As we can see in the proof, the gap of the two quantities on the both sides of (2.39) is given by

$$\text{s-}\lim_{t \rightarrow 0^+} t^{-1} \|\sqrt{\Theta} p(1 - T(-t))\psi\|^2. \quad (2.40)$$

Assuming only forward completeness on  $\omega$  it does not vanish in general. Actually the gap (2.40) corresponds formally to a “boundary contribution” appearing from integrations by parts. In cases where somehow we can realize a smooth boundary  $\partial M$  of  $M$ , such as the open half-space of the Euclidean space, indeed we can explicitly compute the gap (2.40), and it is the square of a weighted  $L^2$ -norm of the normal derivative of  $\psi$  on  $\partial M$ . Of course, the weight vanishes if  $\omega$  is both forward and backward complete, i.e. parallel to this boundary.

### 3. RELICH’S THEOREM

Our proof of Theorem 1.4 shares features of the standard scheme [FHH20, FH, IS2] used for showing absence of “ $L^2$ -eigenvalues”. However there are notable differences. Our main novelty is the use of the conjugate operator  $A$  associated with  $r$  rather than the one associated with  $r^2$ , cf. Subsections 1.1 and 2.2. For such  $A$  the formal commutator  $[H, iA]$  has only a weaker and partial positivity (in the spherical direction), but refined arguments finally provide a stronger result. For the Euclidean space our result overlaps with [L1, L2] and in particular with [Hö, Section 30.2].

Basically the proof consists of two steps, a priori super-exponential decay estimates and the absence of super-exponentially decaying eigenfunctions. Obviously, Theorem 1.4 follows immediately as a combination of the following propositions. Throughout the section we suppose Conditions 1.1–1.3.

**Proposition 3.1.** *Let  $\lambda > \lambda_0$ . If a function  $\phi \in \mathcal{H}_{\text{loc}}$  satisfies for some  $m_0 \geq 0$ :*

- (1)  $(H - \lambda)\phi = 0$  in the distributional sense,
- (2)  $\bar{\chi}_{m_0}\phi \in B_0^*$  and  $\chi_{m_0, n}\phi \in \mathcal{H}^1$  for any  $n > m_0$ ,

then  $\bar{\chi}_{m_0}e^{\alpha r}\phi \in B_0^*$  for any  $\alpha \geq 0$ .

**Proposition 3.2.** *Let  $\lambda > \lambda_0$ . If a function  $\phi \in \mathcal{H}_{\text{loc}}$  satisfies for some  $m_0 \geq 0$ :*

- (1)  $(H - \lambda)\phi = 0$  in the distributional sense,
- (2)  $\bar{\chi}_{m_0}e^{\alpha r}\phi \in B_0^*$  for any  $\alpha \geq 0$  and  $\chi_{m_0, n}\phi \in \mathcal{H}^1$  for any  $n > m_0$ ,

then  $\phi(x) = 0$  in  $M$ .

We prove Propositions 3.1 and 3.2 in Subsections 3.1 and 3.2, respectively. The proofs are quite similar to each other, and both are dependent on commutator estimates with a particular form of weight inside. Let us introduce the regularized weights

$$\Theta = \Theta_{m, n, \nu}^{\alpha, \beta, \delta} = \chi_{m, n}e^{\theta}; \quad n > m \geq 0, \quad (3.1)$$

with exponents

$$\theta = \theta_{\nu}^{\alpha, \beta, \delta} = 2\alpha r + 2\beta \int_0^r (1 + s/R_{\nu})^{-1-\delta} ds; \quad \alpha, \beta \geq 0, \quad \delta > 0, \quad \nu \geq 0.$$

Denote their derivatives in  $r$  by primes, e.g., if we set for notational simplicity

$$\theta_0 = 1 + r/R_{\nu},$$

then

$$\theta' = 2\alpha + 2\beta\theta_0^{-1-\delta}, \quad \theta'' = -2(1 + \delta)\beta R_\nu^{-1}\theta_0^{-2-\delta}, \quad \dots$$

In particular, since  $R_\nu^{-1}\theta_0^{-1} \leq r^{-1}$ , we have

$$|\theta^{(k)}| \leq C_{\delta,k}\beta r^{1-k}\theta_0^{-1-\delta}; \quad k = 2, 3, \dots$$

**3.1. A priori super-exponential decay estimates.** In this subsection we prove Proposition 3.1. The following commutator estimate is a key:

**Lemma 3.3.** *Let  $\lambda > \lambda_0$ , and fix any  $\alpha_0 \geq 0$  and  $\delta \in (0, \min\{1, \rho', \tau/2\})$  in the definition (3.1) of  $\Theta$ . Then there exist  $\beta, c, C > 0$  and  $n_0 \geq 0$  such that uniformly in  $\alpha \in [0, \alpha_0]$ ,  $n > m \geq n_0$  and  $\nu \geq n_0$ , as quadratic forms on  $\mathcal{D}(H)$ ,*

$$\begin{aligned} \operatorname{Im}(A\Theta(H - \lambda)) &\geq cr^{-1}\theta_0^{-\delta}\Theta - C(\chi_{m-1, m+1}^2 + \chi_{n-1, n+1}^2)r^{-1}e^\theta \\ &\quad - \operatorname{Re}(\gamma(H - \lambda)), \end{aligned} \quad (3.2)$$

where  $\gamma = \gamma_{m, n, \nu}$  is a certain function satisfying  $\operatorname{supp} \gamma \subseteq \operatorname{supp} \chi_{m, n}$  and  $|\gamma| \leq Ce^\theta$ .

*Proof.* We are going to prove the lemma by computing and bounding the quadratic form on the left-hand side of (3.2). Fix  $\lambda > \lambda_0$  and  $\delta \in (0, \min\{1, \rho', \tau/2\})$  as in the assertion. First we derive uniform estimates in the parameters  $\alpha \geq 0$ ,  $\beta \in [0, 1]$ ,  $n > m \geq 0$  and  $\nu \geq 0$ . Then in the second step we shall restrict ranges of these parameters to obtain the assertion.

Recall the notation (1.16). Then by Lemmas 2.5, 2.2, (1.8), the Cauchy–Schwarz inequality and (2.6b) we can estimate

$$\begin{aligned} &\operatorname{Im}(A\Theta(H - \lambda)) \\ &\geq \frac{1}{2}A\theta'\Theta A + \frac{1}{2}p_i^*\Theta h^{ik}p_k - \frac{1}{2}\operatorname{Im}(\theta'\Theta(dr)_i h^{ij}p_j) \\ &\quad - \frac{1}{8}|dr|^4\theta'^3\Theta - \frac{3}{8}|dr|^4\theta'\theta''\Theta - \frac{1}{2}\operatorname{Re}(|dr|^2\Theta'(H - \lambda)) - C_1Q \\ &\geq \frac{1}{2}c_1A\tilde{\eta}r^{-1}\theta_0^{-\delta}\Theta A + \frac{1}{2}c_1p_i^*r^{-1}\theta_0^{-\delta}\Theta\ell^{ij}p_j \\ &\quad + \frac{1}{2}A(\theta' - c_1\tilde{\eta}r^{-1}\theta_0^{-\delta})\Theta A + \frac{1}{4}p_i^*\Theta(h^{ij} - 2c_1r^{-1}\theta_0^{-\delta}\ell^{ij})p_j \\ &\quad - \frac{1}{8}|dr|^4\theta'^3\Theta - \frac{3}{8}|dr|^4\theta'\theta''\Theta - \frac{1}{2}\operatorname{Re}(|dr|^2\Theta'(H - \lambda)) - C_2Q, \end{aligned} \quad (3.3)$$

where  $c_1 > 0$  is a small constant such that the fourth term on the right-hand side of (3.3) is non-negative, and we have introduced for simplicity

$$\begin{aligned} Q &= \left( (1 + \alpha^2)r^{-1-\min\{1, \rho', \tau/2\}}\chi_{m, n} + (1 + \alpha^2)|\chi'_{m, n}| + (1 + \alpha)|\chi''_{m, n}| \right. \\ &\quad \left. + |\chi'''_{m, n}| \right) e^\theta + p_i^* \left( r^{-1-\min\{1, \rho', \tau/2\}}\chi_{m, n} + |\chi'_{m, n}| \right) e^\theta g^{ij}p_j. \end{aligned} \quad (3.4)$$

Let us further compute and estimate the terms on the right-hand side of (3.3). Using the expressions (1.8) and (1.9) we estimate the first and second terms of (3.3) by

$$\begin{aligned} &\frac{1}{2}A\tilde{\eta}r^{-1}\theta_0^{-\delta}\Theta A + \frac{1}{2}p_i^*r^{-1}\theta_0^{-\delta}\Theta\ell^{ij}p_j \\ &\geq \frac{1}{2}\operatorname{Im}(\eta r^{-1}\theta_0^{-\delta}\theta'\Theta A) + \frac{1}{2}\operatorname{Re}\left[r^{-1}\theta_0^{-\delta}\Theta A\tilde{\eta}A + r^{-1}\theta_0^{-\delta}\Theta L\right] - C_3Q \\ &\geq (\lambda - q_1)r^{-1}\theta_0^{-\delta}\Theta + \frac{1}{4}\eta|dr|^2r^{-1}\theta_0^{-\delta}\theta'^2\Theta + \operatorname{Re}\left[r^{-1}\theta_0^{-\delta}\Theta(H - \lambda)\right] - C_4Q. \end{aligned} \quad (3.5)$$

We combine the third, fifth and sixth terms of (3.3) as

$$\begin{aligned} & \frac{1}{2}A(\theta' - c_1\tilde{\eta}r^{-1}\theta_0^{-\delta})\Theta A - \frac{1}{8}|dr|^4\theta'^3\Theta - \frac{3}{8}|dr|^4\theta'\theta''\Theta \\ & \geq \frac{1}{2}(A + \frac{i}{2}|dr|^2\theta')(\theta' - c_1\tilde{\eta}r^{-1}\theta_0^{-\delta})\Theta(A - \frac{i}{2}|dr|^2\theta') \\ & \quad - \frac{1}{8}c_1\eta|dr|^2r^{-1}\theta_0^{-\delta}\theta'^2\Theta + \frac{1}{8}|dr|^4\theta'\theta''\Theta - C_5Q. \end{aligned} \quad (3.6)$$

Substitute (3.5) and (3.6) into (3.3), and then it follows that

$$\begin{aligned} \operatorname{Im}(A\Theta(H - \lambda)) & \geq c_1(\lambda - q_1)r^{-1}\theta_0^{-\delta}\Theta + \frac{1}{8}c_1\eta|dr|^2r^{-1}\theta_0^{-\delta}\theta'^2\Theta + \frac{1}{8}|dr|^4\theta'\theta''\Theta \\ & \quad + \frac{1}{2}(A + \frac{i}{2}|dr|^2\theta')(\theta' - c_1\tilde{\eta}r^{-1}\theta_0^{-\delta})\Theta(A - \frac{i}{2}|dr|^2\theta') \\ & \quad + \operatorname{Re}\left[(c_1r^{-1}\theta_0^{-\delta}\Theta - \frac{i}{2}|dr|^2\theta')\Theta(H - \lambda)\right] - C_6Q. \end{aligned} \quad (3.7)$$

Using the formula (2.22) we rewrite and bound the remainder operator (3.4) as

$$\begin{aligned} Q & \leq C_7(1 + \alpha^2)r^{-1-\min\{1,\rho',\tau/2\}}\Theta + C_7(1 + \alpha^2)(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)r^{-1}e^\theta \\ & \quad + 2\operatorname{Re}\left[(r^{-1-\min\{1,\rho',\tau/2\}}\chi_{m,n} + |\chi'_{m,n}|)e^\theta(H - \lambda)\right]. \end{aligned} \quad (3.8)$$

Hence we obtain by (3.7) and (3.8)

$$\begin{aligned} & \operatorname{Im}(A\Theta(H - \lambda)) \\ & \geq \left(c_1(\lambda - q_1)r^{-1}\theta_0^{-\delta} + \frac{1}{8}c_1|dr|^2r^{-1}\theta_0^{-\delta}\theta'^2 + \frac{1}{8}|dr|^4\theta'\theta'' \right. \\ & \quad \left. - C_8(1 + \alpha^2)r^{-1-\min\{1,\rho',\tau/2\}}\Theta\right) \\ & \quad + \frac{1}{2}(A + \frac{i}{2}|dr|^2\theta')(\theta' - c_1\tilde{\eta}r^{-1}\theta_0^{-\delta})\Theta(A - \frac{i}{2}|dr|^2\theta') \\ & \quad - C_8(1 + \alpha^2)(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)r^{-1}e^\theta + \operatorname{Re}(\gamma(H - \lambda)), \end{aligned} \quad (3.9)$$

where

$$\gamma = c_1r^{-1}\theta_0^{-\delta}\Theta - \frac{i}{2}|dr|^2\Theta' - 2C_6r^{-1-\min\{1,\rho',\tau/2\}}\Theta - 2C_6|\chi'_{m,n}|e^\theta.$$

Now we restrict parameters. Fix any  $\alpha_0 \geq 0$ . Then uniformly in  $\alpha \in [0, \alpha_0]$

$$\begin{aligned} & c_1(\lambda - q_1)r^{-1}\theta_0^{-\delta} + \frac{1}{8}c_1|dr|^2r^{-1}\theta_0^{-\delta}\theta'^2 + \frac{1}{8}|dr|^4\theta'\theta'' - C_8(1 + \alpha^2)r^{-1-\min\{1,\rho',\tau/2\}} \\ & \geq c_2r^{-1}\theta_0^{-\delta} - C_9\beta r^{-1}\theta_0^{-1-\delta} - C_9r^{-1-\min\{1,\rho',\tau/2\}}, \end{aligned}$$

and hence, if we choose sufficiently small  $\beta \in (0, 1]$  and sufficiently large  $n_0 \geq 0$ , the first term of (3.9) is bounded below uniformly in  $\alpha \in [0, \alpha_0]$ ,  $n > m \geq n_0$  and  $\nu \geq 0$  as

$$\begin{aligned} & \left(c_1(\lambda - q_1)r^{-1}\theta_0^{-\delta} + \frac{1}{8}c_1|dr|^2r^{-1}\theta_0^{-\delta}\theta'^2 + \frac{1}{8}|dr|^4\theta'\theta'' - C_8(1 + \alpha^2)r^{-1-\min\{1,\rho',\tau/2\}}\Theta\right) \\ & \geq c_3r^{-1}\theta_0^{-\delta}\Theta. \end{aligned}$$

Since

$$\theta' - c_1\tilde{\eta}r^{-1}\theta_0^{-\delta} \geq 2\beta\theta_0^{-1-\delta} - C_{10}r^{-1}\theta_0^{-\delta},$$

by retaking  $n_0 \geq 0$  larger, if necessary, the second term of (3.9) is non-negative for any  $\alpha \in [0, \alpha_0]$ ,  $n > m \geq n_0$  and  $\nu \geq n_0$ . Hence the desired estimate (3.2) follows.  $\square$



*Proof of Proposition 3.1.* Let  $\lambda > \lambda_0$ ,  $\phi \in \mathcal{H}_{\text{loc}}$  and  $m_0 \in \mathbb{N}$  be as in the assertion, and set

$$\alpha_0 = \sup\{\alpha \geq 0 \mid \bar{\chi}_{m_0} e^{\alpha r} \phi \in B_0^*\}.$$

Assume  $\alpha_0 < \infty$ , and we shall find a contradiction. Fix any  $\delta \in (0, \min\{1, \rho', \tau/2\})$ , and choose  $\beta > 0$  and  $n_0 \geq 0$  in agreement with Lemma 3.3. Note that we may assume  $n_0 \geq m_0 + 3$ , so that for all  $n > m \geq n_0$

$$\chi_{m-2, n+2} \phi \in \mathcal{D}(H).$$

If  $\alpha_0 = 0$ , let  $\alpha = 0$  so that automatically  $\alpha + \beta > \alpha_0$ . Otherwise, we choose  $\alpha \in [0, \alpha_0)$  such that  $\alpha + \beta > \alpha_0$ . With these values of  $\alpha$  and  $\beta$  we evaluate the inequality (3.2) in the state  $\chi_{m-2, n+2} \phi \in \mathcal{D}(H)$ , and then obtain for any  $n > m \geq n_0$  and  $\nu \geq n_0$

$$\|(r^{-1} \theta_0^{-\delta} \Theta)^{1/2} \phi\|^2 \leq C_m \|\chi_{m-1, m+1} \phi\|^2 + C_\nu R_n^{-1} \|\chi_{n-1, n+1} e^{\alpha r} \phi\|^2. \quad (3.10)$$

The second term to the right of (3.10) vanishes when  $n \rightarrow \infty$  since  $\bar{\chi}_{m_0} e^{\alpha r} \phi \in B_0^*$ , and consequently by Lebesgue's monotone convergence theorem

$$\|(\bar{\chi}_m r^{-1} \theta_0^{-\delta} e^\theta)^{1/2} \phi\|^2 \leq C_m \|\chi_{m-1, m+1} \phi\|^2. \quad (3.11)$$

Next we let  $\nu \rightarrow \infty$  in (3.11) invoking again Lebesgue's monotone convergence theorem, and then it follows that

$$\bar{\chi}_m^{1/2} r^{-1/2} e^{(\alpha+\beta)r} \phi \in \mathcal{H}.$$

Consequently this implies  $\bar{\chi}_m^{1/2} e^{\kappa r} \phi \in B_0^*$  for any  $\kappa \in (0, \alpha + \beta)$ . But this is a contradiction, since  $\alpha + \beta > \alpha_0$ . We are done.  $\square$

**3.2. Absence of super-exponentially decaying eigenstates.** In this subsection we prove Proposition 3.2.

**Lemma 3.4.** *Let  $\lambda > \lambda_0$  and  $\alpha_0 > 0$ , and fix  $\beta = 0$  in the definition (3.1) of  $\Theta$ , i.e.  $\Theta = \chi_{m,n} e^{2\alpha r}$ . Then there exist  $c, C > 0$  and  $n_0 \geq 0$  such that uniformly in  $\alpha > \alpha_0$  and  $n > m \geq n_0$ , as quadratic forms on  $\mathcal{D}(H)$ ,*

$$\begin{aligned} \text{Im}(A\Theta(H - \lambda)) &\geq c\alpha^2 r^{-1} \Theta - C\alpha^2 (\chi_{m-1, m+1}^2 + \chi_{n-1, n+1}^2) r^{-1} e^{2\alpha r} \\ &\quad + \text{Re}(\gamma(H - \lambda)), \end{aligned} \quad (3.12)$$

where  $\gamma = \gamma_{m,n}$  is a certain function satisfying  $\text{supp } \gamma \subseteq \text{supp } \chi_{m,n}$  and  $|\gamma| \leq C\alpha e^{2\alpha r}$ .

*Proof.* Fix any  $\lambda > \lambda_0$  and  $\delta \in (0, \min\{1, \rho', \tau/2\})$ . Then, repeating the arguments of the proof of Lemma 3.3, we have the bound (3.9) uniformly in  $\alpha \geq 0$ ,  $\beta \in [0, 1]$ ,  $n > m \geq 0$  and  $\nu \geq 0$ . There we fix any  $\alpha_0 > 0$ , let  $\beta = 0$  and  $\nu \rightarrow \infty$ , and choose sufficiently large  $n_0 \geq 0$ . Consequently we can easily verify the asserted inequality (3.12) uniformly in  $\alpha > \alpha_0$  and  $n > m \geq n_0$ . Hence we are done.  $\square$

*Proof of Proposition 3.2.* Let  $\lambda > \lambda_0$ ,  $\phi \in \mathcal{H}_{\text{loc}}$  and  $m_0 \in \mathbb{N}$  be fixed as in the proposition. Fix any  $\alpha_0 > 0$  and  $\beta = 0$ , and choose  $n_0 \geq 0$  in agreement with Lemma 3.4. We may assume that  $n_0 \geq m_0 + 3$ , so that for all  $n > m \geq n_0$

$$\chi_{m-2, n+2} \phi \in \mathcal{D}(H).$$

Let us evaluate the inequality (3.12) in the state  $\chi_{m-2, n+2} \phi \in \mathcal{D}(H)$ . Then it follows that for any  $\alpha > \alpha_0$  and  $n > m \geq n_0$

$$\|r^{-1/2} \Theta \phi\| \leq C_1 \|\chi_{m-1, m+1} e^{\alpha r} \phi\|^2 + C_1 R_n^{-1} \|\chi_{n-1, n+1} e^{\alpha r} \phi\|^2 \quad (3.13)$$

The second term to the right of (3.13) vanishes when  $n \rightarrow \infty$ , and hence by Lebesgue's monotone convergence theorem we obtain

$$\|\bar{\chi}_m^{1/2} r^{-1/2} e^{\alpha r} \phi\|^2 \leq C_1 \|\chi_{m-1, m+1} e^{\alpha r} \phi\|^2,$$

or

$$\|\bar{\chi}_m^{1/2} r^{-1/2} e^{\alpha(r-4R_m)} \phi\|^2 \leq C_1 \|\chi_{m-1, m+1} \phi\|^2. \quad (3.14)$$

Now assume  $\bar{\chi}_{m+2} \phi \not\equiv 0$ . The left-hand side of (3.14) grows exponentially as  $\alpha \rightarrow \infty$  whereas the right-hand side remains bounded. This is a contradiction. Thus  $\bar{\chi}_{m+2} \phi \equiv 0$ . By invoking the unique continuation property for the second order elliptic operator  $H$  we conclude that  $\phi \equiv 0$  globally on  $M$ .  $\square$

#### 4. BESOV BOUND

In this section we discuss the locally uniform Besov bound for the resolvent  $R(z)$ , and prove Theorem 1.7. A key to the proof is a kind of single commutator estimate with weight inside stated in Lemma 4.3, or its direct consequence, Proposition 4.1.

Here let us state a slightly technical but main proposition of the section that follows from Lemma 4.3. We introduce the regularized weight

$$\Theta = \Theta_\nu^\delta = \int_0^{r/R_\nu} (1+s)^{-1-\delta} ds = [1 - (1+r/R_\nu)^{-\delta}] / \delta; \quad \delta > 0, \nu \geq 0, \quad (4.1)$$

and compute derivatives in  $r$ :

$$\Theta' = (1+r/R_\nu)^{-1-\delta} / R_\nu, \quad \Theta'' = -(1+\delta)(1+r/R_\nu)^{-2-\delta} / R_\nu^2. \quad (4.2)$$

Recall the notation defined right before Theorem 1.7.

**Proposition 4.1.** *Suppose Conditions 1.1–1.3, let  $I \subseteq \mathcal{I}$  be any relatively compact open subset, and fix any  $\delta \in (0, \min\{1, \rho', \tau/2\})$  in the definition (4.1) of  $\Theta$ . Then there exist  $C > 0$  and  $n \geq 0$  such that for all  $\phi = R(z)\psi$  with  $z \in I_\pm$  and  $\psi \in B$  and for all  $\nu \geq 0$*

$$\begin{aligned} & \|\Theta'^{1/2} \phi\|^2 + \|\Theta'^{1/2} A \phi\|^2 + \langle p_i^* \Theta h^{ij} p_j \rangle_\phi \\ & \leq C \left( \|\phi\|_{B^*} \|\psi\|_B + \|A \phi\|_{B^*} \|\psi\|_B + \|\chi_n \Theta^{1/2} \phi\|^2 \right). \end{aligned} \quad (4.3)$$

In Subsection 4.1 we prove Lemma 4.3 and, as a result, Proposition 4.1 too. In Subsection 4.2, combining Proposition 4.1 and Condition 1.6, we prove Theorem 1.7 by contradiction.

**4.1. Commutator estimate.** We first note properties of  $\Theta$  defined by (4.1).

**Lemma 4.2.** *Suppose Condition 1.1, and fix any  $\delta > 0$  in the definition (4.1) of  $\Theta$ . Then there exist  $c, C, C_k > 0$ ,  $k = 2, 3, \dots$ , such that for any  $k = 2, 3, \dots$  and uniformly in  $\nu \geq 0$*

$$\begin{aligned} c/R_\nu & \leq \Theta \leq \min\{C, r/R_\nu\}, \\ c(\min\{R_\nu, r\})^\delta r^{-1-\delta} \Theta & \leq \Theta' \leq r^{-1} \Theta, \\ 0 & \leq (-1)^{k-1} \Theta^{(k)} \leq C_k r^{-k} \Theta. \end{aligned}$$

*Proof.* All the asserted estimates are straightforward except possibly for the first estimate in the second line. But by using the last estimate of the first line this estimate follows if we can bound

$$\left(\min\{R_\nu, r\}\right)^\delta r^{-1-\delta} \left(\min\{R_\nu, r\}/R_\nu\right) \left((1+r/R_\nu)^{1+\delta} R_\nu\right) \leq C,$$

which obviously is correct for the case  $r \leq R_\nu$  as well as for the case  $r > R_\nu$ .  $\square$

Now we state and prove our key lemma:

**Lemma 4.3.** *Suppose Conditions 1.1–1.3, let  $I \subseteq \mathcal{I}$  be any relatively compact open subset, and fix any  $\delta \in (0, \min\{1, \rho', \tau/2\})$  in the definition (4.1) of  $\Theta$ . Then there exist  $c, C > 0$  and  $n \geq 0$  such that uniformly in  $z \in I_\pm$  and  $\nu \geq 0$ , as quadratic forms on  $\mathcal{D}(H)$ ,*

$$\operatorname{Im}(A\Theta(H-z)) \geq c\Theta' + cA\Theta'A + cp_i^* \Theta h^{ij} p_j - C\chi_n^2 \Theta - \operatorname{Re}(\gamma(H-z)), \quad (4.4)$$

where  $\gamma = \gamma_{z,\nu}$  is a uniformly bounded complex-valued function:  $|\gamma| \leq C$ .

*Proof.* Let  $I$  and  $\delta$  be as in the assertion. We are going to prove the lemma by bounding the form on the left-hand side of (4.4). First by Lemmas 2.5, 2.2, 4.2, (1.8), the Cauchy–Schwarz inequality and (2.6b) we can bound uniformly in  $z = \lambda \pm i\Gamma \in I_\pm$  and  $\nu \geq 0$

$$\begin{aligned} \operatorname{Im}(A\Theta(H-z)) &\geq \frac{1}{2}A\Theta'A + \frac{1}{2}p_i^* \Theta h^{ik} p_k - \frac{1}{2} \operatorname{Im}(\Theta'(dr)_i h^{ij} p_j) \\ &\quad \mp \Gamma \operatorname{Re}(A\Theta) - \frac{1}{2} \operatorname{Re}(|dr|^2 \Theta'(H-\lambda)) - C_1 Q \\ &\geq \frac{1}{2}c_1 A \tilde{\eta} \Theta' A + \frac{1}{2}c_1 p_i^* \Theta' \ell^{ik} p_k \\ &\quad + \frac{1}{2}A(1-c_1 \tilde{\eta}) \Theta' A + \frac{1}{4}p_i^* (\Theta h^{ik} - 2c_1 \Theta' \ell^{ik}) p_k \\ &\quad \mp \Gamma \Theta^{1/2} A \Theta^{1/2} - \frac{1}{2} \operatorname{Re}(|dr|^2 \Theta'(H-z)) - C_2 Q, \end{aligned} \quad (4.5)$$

where  $c_1 > 0$  is a small constant and

$$Q = r^{-1-\min\{1, \rho', \tau/2\}} \Theta + p_i^* r^{-1-\min\{1, \rho', \tau/2\}} \Theta g^{ij} p_j.$$

We further estimate the terms on the right-hand side of (4.5) as follows. By Lemma 4.2 we can choose and fix  $c_1 > 0$  in (4.5) small enough that the third and fourth terms on the right-hand side satisfy

$$\frac{1}{2}A(1-c_1 \tilde{\eta}) \Theta' A + \frac{1}{4}p_i^* (\Theta h^{ik} - 2c_1 \Theta' \ell^{ik}) p_k \geq c_2 A \Theta' A + c_2 p_i^* \Theta h^{ik} p_k. \quad (4.6)$$

Using Lemma 4.2, the Cauchy–Schwarz inequality and (1.9) we can rewrite and bound the first and second terms of (4.5) as

$$\begin{aligned} \frac{1}{2}c_1 A \tilde{\eta} \Theta' A + \frac{1}{2}c_1 p_i^* \Theta' \ell^{ij} p_j &\geq \frac{1}{2}c_1 \operatorname{Re} \left[ \Theta' A \tilde{\eta} A + \Theta' L \right] - C_3 Q \\ &\geq c_1 (\lambda - q_1) \Theta' + \operatorname{Re}(c_1 \Theta'(H-z)) - C_4 Q. \end{aligned} \quad (4.7)$$

To the fifth term of (4.5) we apply the Cauchy–Schwarz inequality, Lemma 4.2 and the general identity holding for any real functions  $f, h \in C^1(M)$ :

$$h \operatorname{Re}(f H_0) h = \operatorname{Re}(h^2 f H_0) + \frac{1}{2}(\partial_i h) g^{ij} (\partial_j h f). \quad (4.8)$$

Then it follows that

$$\begin{aligned} \mp \Gamma \Theta^{1/2} A \Theta^{1/2} &\geq -C_5 \Gamma \Theta^{1/2} (H-\lambda) \Theta^{1/2} - C_6 \Gamma \\ &\geq -C_5 \Gamma \operatorname{Re}(\Theta^{1/2} (H-z) \Theta^{1/2}) \pm C_6 \operatorname{Im}(H-z) \\ &\geq -C_7 Q - \operatorname{Re}((C_5 \Gamma \Theta \pm i C_6)(H-z)). \end{aligned} \quad (4.9)$$

We substitute the estimates (4.6), (4.7) and (4.9) into (4.5), and obtain

$$\begin{aligned} \operatorname{Im}(A\Theta(H-z)) &\geq c_3\Theta' + c_2A\Theta'A + c_2p_i^*\Theta h^{ij}p_j - C_8Q \\ &\quad - \operatorname{Re}\left(\left[\left(\frac{1}{2}|dr|^2 - c_1\right)\Theta' + C_5\Gamma\Theta \pm iC_6\right](H-z)\right). \end{aligned} \quad (4.10)$$

Finally we can use (2.22) and Lemma 4.2 to combine and estimate the first and fourth terms of (4.10): For small  $c_4 > 0$  and large  $n \geq 0$

$$c_3\Theta' - C_8Q \geq c_4\Theta' - C_9\chi_n^2\Theta - 2C_8 \operatorname{Re}(r^{-1-\min\{1,\rho',\tau/2\}}\Theta(H-z)). \quad (4.11)$$

Hence by (4.10) and (4.11), if we set

$$\gamma = \left(\frac{1}{2}|dr|^2 - c_1\right)\Theta' + C_5\Gamma\Theta \pm iC_6 + 2C_8r^{-1-\min\{1,\rho',\tau/2\}}\Theta,$$

then the desired inequality (4.4) follows.  $\square$

*Proof of Proposition 4.1.* The assertion follows immediately from Lemma 4.3.  $\square$

**4.2. Compactness and contradiction.** Now we suppose Condition 1.6, and prove Theorem 1.7 by Proposition 4.1 and contradiction.

*Proof of Theorem 1.7.* Let  $I \subseteq \mathcal{I}$  be any relatively compact open subset. We prove the assertion only for the upper sign.

*Step I.* We first reduce the bound (1.17) to the single bound

$$\|\phi\|_{B^*} \leq C_1\|\psi\|_B. \quad (4.12)$$

In fact, assume (4.12). Then the last term of the left-hand side of (1.17) clearly satisfies the desired estimate by the identity

$$H_0\phi = \psi - (V-z)\phi$$

and the fact that  $V$  is bounded by Conditions 1.1–1.3. Hence it suffice to consider the second and third terms of (1.17). Fix any  $\delta \in (0, \min\{1, \rho', \tau/2\})$ . Then by Proposition 4.1 and (4.12) there exists  $C_2 > 0$  such that for any  $\phi = R(z)\psi$  with  $z \in I_+$  and  $\psi \in B$  uniformly in  $\epsilon_1 \in (0, 1)$  and  $\nu \geq 0$

$$\|\Theta^{1/2}A\phi\|^2 + \langle p_i^*\Theta h^{ij}p_j \rangle_\phi \leq \epsilon_1^{-1}C_2\|\psi\|_B^2 + \epsilon_1\|A\phi\|_{B^*}^2. \quad (4.13)$$

In the first term on the left-hand side of (4.13) for each  $\nu \geq 0$ , noting the expression of  $\Theta'$  in (4.2), we restrict the integral region to  $B_{R_{\nu+1}} \setminus B_{R_\nu}$ . As for the second term on the same side we look at the estimate (4.13) for any fixed  $\nu \geq 0$ , say  $\nu = 0$ . Then we can deduce from (4.13) that

$$c_1\|A\phi\|_{B^*}^2 + c_1\langle p_i^*h^{ij}p_j \rangle_\phi \leq 2\epsilon_1^{-1}C_2\|\psi\|_B^2 + 2\epsilon_1\|A\phi\|_{B^*}^2.$$

If we let  $\epsilon_1 \in (0, c_1/2)$ , the rest of (1.17) follows from this estimate and (1.8). Hence (1.17) reduces to (4.12).

*Step II.* We prove (4.12) by contradiction. Assume the opposite, and let  $z_k \in I_+$  and  $\psi_k \in B$  be such that

$$\lim_{k \rightarrow \infty} \|\psi_k\|_B = 0, \quad \|\phi_k\|_{B^*} = 1; \quad \phi_k = R(z_k)\psi_k. \quad (4.14)$$

Note that then it automatically follows that

$$\|p\phi_k\|_{B^*} + \|H_0\phi_k\|_{B^*} \leq C_3. \quad (4.15)$$

In fact, arguing similarly to Step I, we can deduce from (4.14) and Proposition 4.1 that

$$\|A\phi_k\|_{B^*}^2 + \langle p_i^* h^{ij} p_j \rangle_{\phi_k} \leq C_4, \quad H_0 \phi_k = \psi_k - (V - z_k) \phi_k,$$

and these combined Condition 1.2, (1.8) and (4.14) imply (4.15). Now, choosing a subsequence and retaking  $I \subseteq \mathcal{I}$  slightly larger, we may assume that  $z_k \in I_+$  converges to some  $z \in I \cup I_+$ . If the limit  $z$  belongs to  $I_+$ , the bounds

$$\|\phi_k\|_{B^*} \leq \|\phi_k\|_{\mathcal{H}} \leq \|R(z_k)\|_{\mathcal{B}(\mathcal{H})} \|\psi_k\|_{\mathcal{H}} \leq C_5 \|R(z_k)\|_{\mathcal{B}(\mathcal{H})} \|\psi_k\|_B$$

and (4.14) contradict the norm continuity of  $R(z) \in \mathcal{B}(\mathcal{H})$  in  $z \in I_+$ . Hence we have the limit

$$\lim_{k \rightarrow \infty} z_k = z = \lambda \in I. \quad (4.16)$$

Let  $s > 1/2$ . By choosing a further subsequence we may assume that  $\phi_k$  converges weakly to some  $\phi$  in  $\mathcal{H}_{-s}$ , cf. [Yo]. But then  $\phi_k$  actually converges strongly in  $\mathcal{H}_{-s}$ . To see this let us fix  $s' \in (1/2, s)$  and  $f \in C_0^\infty(\mathcal{I})$  with  $f = 1$  on a neighborhood of  $I$ , and decompose for any  $n \geq 0$

$$\begin{aligned} r^{-s} \phi_k &= r^{-s} f(H) (\chi_n r^s) (r^{-s} \phi_k) + (r^{-s} f(H) r^s) (\bar{\chi}_n r^{s'-s}) (r^{-s'} \phi_k) \\ &\quad + r^{-s} (1 - f(H)) R(z_k) \psi_k. \end{aligned}$$

The last term on the right-hand side converges to 0 in  $\mathcal{H}$  due to (4.14), and the second term can be taken arbitrarily small in  $\mathcal{H}$  by choosing  $n \geq 0$  sufficiently large since  $r^{-s} f(H) r^s$  is a bounded operator. It follows from Condition 1.6 that  $r^{-s} f(H)$  is compact. Whence for fixed  $n \geq 0$  the first term converges strongly in  $\mathcal{H}$ . So indeed  $\phi_k$  converges to  $\phi$  strongly in  $\mathcal{H}_{-s}$ . By using (4.15) we can see that  $p\phi \in \mathcal{H}_{-s}$ , and then by using (2.22), or alternatively the first resolvent equation, we can see that the sequence  $\{p\phi_k\}$  is a Cauchy sequence in  $\mathcal{H}_{-s}$ . Whence we have

$$\lim_{k \rightarrow \infty} \phi_k = \phi \quad \text{in } \mathcal{H}_{-s}, \quad \lim_{k \rightarrow \infty} p\phi_k = p\phi \quad \text{in } \mathcal{H}_{-s}. \quad (4.17)$$

By (4.14), (4.16) and (4.17) it follows that

$$\phi \in \mathcal{N}, \quad (H - \lambda)\phi = 0 \text{ in the distributional sense.} \quad (4.18)$$

In addition, we can verify  $\phi \in B_0^*$ . In fact, let us apply Proposition 4.1 with  $\delta = 2s - 1 > 0$  to  $\phi_k = R(z_k)\psi_k$ , and take the limit  $k \rightarrow \infty$  using (4.17), (4.14), (4.15) and Lemma 4.2. We obtain for all  $\nu \geq 0$

$$\|\Theta^{1/2}\phi\| \leq C_6 \|\chi_n \Theta^{1/2}\phi\| \leq C_6 R_\nu^{-1/2} \|\chi_n r^{1/2}\phi\|. \quad (4.19)$$

Letting  $\nu \rightarrow \infty$  in (4.19), we obtain  $\phi \in B_0^*$ , and then we conclude  $\phi = 0$  by (4.18) and Theorem 1.4. But this is a contradiction, because similarly to Step I we have

$$1 = \|\phi_k\|_{B^*}^2 \leq C_7 (\|\psi_k\|_B + \|\chi_n \phi_k\|^2),$$

and, as  $k \rightarrow \infty$ , the right-hand side converges to 0. Hence (4.12) holds.  $\square$

## 5. RADIATION CONDITION

In this section we discuss the radiation condition bounds and their relevant consequences. In Subsection 5.1 we state and prove the main key commutator estimate of the section, which is somewhat similar to that of Section 4. In Subsection 5.2, using this key estimate, we prove Theorem 1.10. Corollaries 1.11–1.13 are also proved in the same subsection.

Throughout the section we suppose Condition 1.9, and prove the statements only for the upper sign for simplicity.

## 5.1. Commutator estimate.

**Lemma 5.1.** *Let  $I \subseteq \mathcal{I}$  be any relatively compact open subset. There exists  $C > 0$  such that uniformly in  $z \in I \cup I_+$*

$$\begin{aligned} |a| &\leq C, \quad \operatorname{Im} a \geq -Cr^{-1-\min\{\rho', \rho/2\}}, \quad q_{21} \operatorname{Im} a \geq -C(\Gamma + r^{-1})r^{-\rho}, \\ |p^r a + a^2 - 2|\operatorname{dr}|^2(z - q_1)| + |\ell^{\bullet i} \nabla_i a| &\leq Cr^{-1-\min\{\rho/2, \tau/2\}}. \end{aligned}$$

*Proof.* It is clear by the definition (1.19) that the function  $a$  is bounded. For the second estimate it suffices to note that by Condition 1.9.

$$\nabla^r q_{11} = \nabla^r q_1 - \nabla^r q_{12} \leq C_1 r^{-1-\min\{\rho', \rho/2\}}.$$

The third estimate is also clear by Condition 1.9. Since we can write

$$\begin{aligned} p^r a + a^2 - 2|\operatorname{dr}|^2(z - q_1) &= (p^r \eta_\lambda |\operatorname{dr}|) \sqrt{2(z - q_1)} + \frac{1}{4}(p^r \eta_\lambda p^r q_{11}) / (z - q_1) \\ &\quad + \frac{1}{4} \eta_\lambda (1 + \frac{1}{4} \eta_\lambda) (p^r q_{11})^2 / (z - q_1)^2 + \frac{1}{4} \eta_\lambda (p^r q_{11})(p^r q_{12}) / (z - q_1)^2 \\ &\quad - \eta_\lambda |\operatorname{dr}| (p^r q_1 - \eta_\lambda p^r q_{11}) / \sqrt{2(z - q_1)} - 2(1 - \eta_\lambda^2) |\operatorname{dr}|^2 (z - q_1), \end{aligned}$$

it is clear that this quantity satisfies the assertion by Condition 1.9. The last bound is also clear.  $\square$

To simplify a commutator computation in Lemma 5.4 we decompose  $H - z$  into the radial and non-radial directions like (1.9).

**Lemma 5.2.** *Let  $I \subseteq \mathcal{I}$  be any relatively compact open subset. Then there exist a complex-valued function  $q_3$  and a constant  $C > 0$  such that uniformly in  $z \in I \cup I_+$ , as quadratic forms on  $\mathcal{H}^1$ ,*

$$H - z = \frac{1}{2}(A + a)\tilde{\eta}(A - a) + \frac{1}{2}L + q_{21} + q_3; \quad |q_3| \leq Cr^{-1-\min\{\rho/2, \tau/2\}}.$$

*Proof.* Using the expression (1.9) we can write

$$H - z = \frac{1}{2}(A + a)\tilde{\eta}(A - a) + \frac{1}{2}(p^r \tilde{\eta} a) + \frac{1}{2} \tilde{\eta} a^2 + \frac{1}{2}L + q_1 + q_2 + \frac{1}{4}(\nabla^r \tilde{\eta})(\Delta r) - z.$$

Hence the desired identity is obtained by setting

$$\begin{aligned} q_3 &= \frac{1}{2} \tilde{\eta} [(p^r a) + a^2 - 2|\operatorname{dr}|^2(z - q_1)] - (1 - \eta)(z - q_1) \\ &\quad + q_{22} - \frac{1}{2}(\nabla^r \tilde{\eta})a + \frac{1}{4}(\nabla^r \tilde{\eta})(\Delta r) \end{aligned}$$

and applying Lemma 5.1.  $\square$

**Remark 5.3.** The decay rate of  $q_3$  depends very much on how accurately we can construct an approximate solution to the radial Riccati equation (1.20). We note that the change of variable  $a = \pm(p^r b)/b$  reduces the equation (1.20) to the second-order linear differential equation

$$(p^r)^2 b - 2|dr|^2(z - q_1)b = 0,$$

which is nothing but a one-dimensional Schrödinger eigenequation with a long-range perturbation.

Next we state and prove the key commutator estimate of the section that is needed for our proof of the radiation condition bounds. Let us introduce the regularized weight

$$\Theta = \Theta_\nu^\delta = \int_0^{r/R_\nu} (1+s)^{-1-\delta} ds = [1 - (1+r/R_\nu)^{-\delta}]/\delta; \quad \delta > 0, \nu \geq 0,$$

which is the same weight as (4.1) introduced in Section 4. We denote its derivatives in  $r$  by primes such as (4.2). Again we shall use Lemma 4.2 although we are going to choose  $\delta > 0$  differently from Section 4.

**Lemma 5.4.** *Let  $I \subseteq \mathcal{I}$  be any relatively compact open subset, and fix any  $\delta \in (0, \min\{\rho'/2, \rho/4, \tau/4\}]$  and  $\beta \in (0, \sigma/2)$ . Then there exist  $c, C > 0$  such that uniformly in  $z \in I \cup I_+$  and  $\nu \geq 0$ , as quadratic forms on  $\mathcal{D}(H)$*

$$\begin{aligned} \operatorname{Im}((A-a)^* \Theta^{2\beta} (H-z)) &\geq c(A-a)^* \Theta' \Theta^{2\beta-1} (A-a) + cp_i^* \Theta^{2\beta} h^{ij} p_j \\ &\quad - Cr^{-1-\min\{\rho, \tau\}+2\delta} \Theta^{2\beta} - \operatorname{Re}(\gamma \Theta^{2\beta} (H-z)), \end{aligned}$$

where  $\gamma$  is a complex-valued function satisfying  $|\gamma| \leq Cr^{-\min\{\rho, \tau\}+2\delta}$ .

*Proof.* Let  $I$ ,  $\delta$  and  $\beta$  be as in the statement, and we are going to expand and bound the left-hand side of the asserted inequality. By Lemma 2.5 we may compute it formally on  $C_c^\infty(M)$ , as long as we bound it below. For the practical computations below we proceed similarly to the proof of Lemma 2.2 employing Lemma 5.2 instead of (1.9). By Lemmas 5.2, 5.1 and 4.2 and the Cauchy–Schwarz inequality it follows that uniformly in  $z \in I \cup I_+$  and  $\nu \geq 0$

$$\begin{aligned} &\operatorname{Im}((A-a)^* \Theta^{2\beta} (H-z)) \\ &= \frac{1}{2} \operatorname{Im}((A-a)^* \Theta^{2\beta} (A+a) \tilde{\eta} (A-a)) + \frac{1}{2} \operatorname{Im}((A-a)^* \Theta^{2\beta} L) \\ &\quad + \operatorname{Im}((A-a)^* \Theta^{2\beta} q_{21}) + \operatorname{Im}((A-a)^* \Theta^{2\beta} q_3) \\ &\geq \frac{1}{2} (A-a)^* (\beta \Theta' - C_1 r^{-1-2\delta} \Theta) \Theta^{2\beta-1} (A-a) \\ &\quad + \frac{1}{2} \operatorname{Im}(A \Theta^{2\beta} L) - \frac{1}{2} \operatorname{Im}(a^* \Theta^{2\beta} L) - C_1 \Gamma r^{-\rho} \Theta^{2\beta} - C_1 Q, \end{aligned} \tag{5.1}$$

where

$$Q = r^{-1-\min\{\rho, \tau\}+2\delta} \Theta^{2\beta} + p_i^* r^{-1-\min\{\rho, \tau\}+2\delta} \Theta^{2\beta} g^{ij} p_j.$$

We further estimate the terms on the right-hand side of (5.1). By Lemma 4.2 the first term of (5.1) can be bounded as

$$\begin{aligned} &\frac{1}{2} (A-a)^* (\beta \Theta' - C_1 r^{-1-2\delta} \Theta) \Theta^{2\beta-1} (A-a) \\ &\geq c_1 (A-a)^* \Theta' \Theta^{2\beta-1} (A-a) - C_2 Q. \end{aligned} \tag{5.2}$$

Reusing parts of computations in (2.25), (2.26) and (2.27), we can write and bound the second term of (5.1) as, for any  $\epsilon \in (0, 1)$ ,

$$\begin{aligned}
\frac{1}{2} \operatorname{Im}(A\Theta^{2\beta}L) &= \frac{1}{4} [p_i^* \Theta^{2\beta} \ell^{ij} p_j, iA] + \frac{1}{2} \operatorname{Re}(A(1-\eta)(\Theta^{2\beta})' p^r) \\
&= \frac{1}{4} p_i^* \Theta^{2\beta} \left( 2(\nabla^2 r)^{ik} + (\nabla^r \tilde{\eta})(dr \otimes dr)^{ik} \right) p_k \\
&\quad - \frac{1}{4} p_i^* |dr|^2 (\Theta^{2\beta})' \ell^{ik} p_k - \frac{1}{4} \operatorname{Im}(\Theta^{2\beta}(\nabla_i \Delta r) \ell^{ij} p_i) \\
&\quad + \frac{1}{2} A(1-\eta)(\Theta^{2\beta})' A - \frac{1}{8} (\nabla^r \eta)(\Delta r)(\Theta^{2\beta})' \\
&\quad + \frac{1}{8} (1-\eta) |dr|^2 (\Delta r)(\Theta^{2\beta})'' \\
&\geq \frac{1}{2} p_i^* \Theta^{2\beta} h^{ik} p_k - \frac{1}{2} \beta p_i^* |dr|^2 \Theta' \Theta^{2\beta-1} \ell^{ik} p_k \\
&\quad - \epsilon p_i^* r^{-1} \Theta^{2\beta} \ell^{ij} p_j - \epsilon^{-1} C_3 Q.
\end{aligned} \tag{5.3}$$

As for the third term of (5.1) use (1.4), Lemma 5.1 and the Cauchy–Schwarz inequality, and then we obtain for the same  $\epsilon \in (0, 1)$  as above

$$\begin{aligned}
& - \frac{1}{2} \operatorname{Im}(a^* \Theta^{2\beta} L) \\
&= \frac{1}{2} p_i^* (\operatorname{Im} a) \Theta^{2\beta} \ell^{ij} p_j - \frac{1}{2} \operatorname{Re}((\nabla_i a)^* \Theta^{2\beta} \ell^{ij} p_j) - \beta \operatorname{Re}((1-\eta)a^* \Theta' \Theta^{2\beta-1} p^r) \\
&\geq -\epsilon p_i^* r^{-1} \Theta^{2\beta} \ell^{ij} p_j - \epsilon^{-1} C_4 Q.
\end{aligned} \tag{5.4}$$

As for the fourth term of (5.1), we have by the Cauchy–Schwarz inequality

$$\begin{aligned}
-C_1 \Gamma r^{-\rho} \Theta^{2\beta} &= -\frac{1}{2} C_1 [H, i r^{-\rho} \Theta^{2\beta}] + C_1 \operatorname{Im}(r^{-\rho} \Theta^{2\beta} (H - z)) \\
&\geq -C_9 Q + C_1 \operatorname{Im}(r^{-\rho} \Theta^{2\beta} (H - z)).
\end{aligned} \tag{5.5}$$

Now we substitute the bounds (5.2), (5.3), (5.4), and (5.5) into (5.1), and obtain

$$\begin{aligned}
\operatorname{Im}((A - a)^* \Theta^{2\beta} (H - z)) &\geq c_1 (A - a)^* \Theta' \Theta^{2\beta-1} (A - a) \\
&\quad + \frac{1}{2} p_i^* \Theta^{2\beta-1} \left( h^{ij} - 4\epsilon r^{-1} \Theta \ell^{ij} - \beta |dr|^2 \Theta' \ell^{ij} \right) p_j \\
&\quad - \epsilon^{-1} C_5 Q + C_1 \operatorname{Im}(r^{-\rho} \Theta^{2\beta} (H - z)).
\end{aligned} \tag{5.6}$$

If we choose  $\epsilon > 0$  small enough, we have for the second term of (5.6)

$$\frac{1}{2} p_i^* \Theta^{2\beta-1} \left( h^{ij} - 4\epsilon r^{-1} \Theta \ell^{ij} - \beta |dr|^2 \Theta' \ell^{ij} \right) p_j \geq c_2 p_i^* \Theta^{2\beta} h^{ij} p_j. \tag{5.7}$$

Hence it finally remains to bound  $-Q$  below, but by (2.22) we can estimate it as

$$-Q \geq -C_6 r^{-1-\min\{\rho, \tau\}+2\delta} \Theta^{2\beta} - 2 \operatorname{Re}(r^{-1-\min\{\rho, \tau\}+2\delta} \Theta^{2\beta} (H - z)). \tag{5.8}$$

By (5.6), (5.7) and (5.8), if we set

$$\gamma = iC_1 r^{-\rho} + 2\epsilon^{-1} C_5 r^{-1-\min\{\rho, \tau\}+2\delta},$$

then the assertion follows.  $\square$

**5.2. Applications.** Now we are going to prove Theorem 1.10 and Corollaries 1.11–1.13 in this order.



5.2.1. *Radiation condition bounds for complex spectral parameters.*

*Proof of Theorem 1.10.* Let  $I \subseteq \mathcal{I}$  be any relative compact open subset. For  $\beta = 0$  the assertion is obvious by Theorem 1.7, and hence we may let  $\beta \in (0, \beta_c)$ . We take any

$$\delta \in (0, \min\{\rho'/2, \rho/4, \tau/4\}] \cap (0, \min\{\rho/2, \tau/2\} - \beta).$$

By Lemma 5.4, the Cauchy–Schwarz inequality and Theorem 1.7 there exists  $C_1 > 0$  such that for any state  $\phi = R(z)\psi$  with  $\psi \in C_c^\infty(M)$  and  $z \in I_+$

$$\begin{aligned} & \|\Theta^{1/2}\Theta^{\beta-1/2}(A-a)\phi\|^2 + \langle p_i^* \Theta^{2\beta} h^{ij} p_j \rangle_\phi \\ & \leq C_1 \left[ \|\Theta^\beta(A-a)\phi\|_{B^*} \|\Theta^\beta\psi\|_B + \|r^{-(1+\min\{\rho,\tau\})/2+\delta}\Theta^\beta\phi\|^2 \right. \\ & \quad \left. + \|r^{(1-\min\{\rho,\tau\})/2+\delta}\Theta^\beta\psi\|^2 \right] \\ & \leq C_2 R_\nu^{-2\beta} \left[ \|r^\beta(A-a)\phi\|_{B^*} \|r^\beta\psi\|_B + \|r^\beta\psi\|_B^2 \right]. \end{aligned} \tag{5.9}$$

Here we note that  $r^\beta(A-a)\phi \in B^*$  for each  $z \in I_+$  and hence the quantity on the right-hand side of (5.9) is finite. In fact, this can be verified by commuting  $R(z)$  and powers of  $r$  sufficiently many times and using the fact that  $\psi \in C_c^\infty(M)$ . Then by (5.9) it follows

$$\begin{aligned} & R_\nu^{2\beta} \|\Theta^{1/2}\Theta^{\beta-1/2}(A-a)\phi\|^2 + R_\nu^{2\beta} \langle p_i^* \Theta^{2\beta} h^{ij} p_j \rangle_\phi \\ & \leq C_2 \left[ \|r^\beta(A-a)\phi\|_{B^*} \|r^\beta\psi\|_B + \|r^\beta\psi\|_B^2 \right]. \end{aligned} \tag{5.10}$$

In the first term on the left-hand side of (5.10) we take the supremum in  $\nu \geq 0$  noting (4.2), and then obtain

$$c_1 \|r^\beta(A-a)\phi\|_{B^*}^2 \leq C_2 \left[ \|r^\beta(A-a)\phi\|_{B^*} \|r^\beta\psi\|_B + \|r^\beta\psi\|_B^2 \right],$$

which implies

$$\|r^\beta(A-a)\phi\|_{B^*}^2 \leq C_3 \|r^\beta\psi\|_B^2. \tag{5.11}$$

As for the second term on the left-hand side of (5.10) we use (5.11), the concavity of  $\Theta$  and Lebesgue's monotone convergence theorem and then obtain by letting  $\nu \rightarrow \infty$

$$\langle p_i^* r^{2\beta} h^{ij} p_j \rangle_\phi \leq C_4 \|r^\beta\psi\|_B^2.$$

Hence we are done.  $\square$

5.2.2. *Limiting absorption principle.*

*Proof of Corollary 1.11.* Let  $s > 1/2$  be as in the assertion. Throughout the proof let us fix any  $s' \in (1/2, s)$  and  $\beta \in [0, \beta_c)$ . We decompose for  $m \geq 0$  and  $z, z' \in I_+$

$$\begin{aligned} R(z) - R(z') &= \chi_m R(z) \chi_m - \chi_m R(z') \chi_m \\ &+ (R(z) - \chi_m R(z) \chi_m) - (R(z') - \chi_m R(z') \chi_m). \end{aligned} \tag{5.12}$$

We estimate the last two terms of (5.12) as follows: By Theorem 1.7 we have uniformly in  $m \geq 0$  and  $z, z' \in I_+$

$$\begin{aligned} & \|R(z) - \chi_m R(z) \chi_m\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \\ & \leq \|r^{-s} \bar{\chi}_m R(z) \bar{\chi}_m r^{-s}\|_{\mathcal{B}(\mathcal{H}, \mathcal{H})} + \|r^{-s} \bar{\chi}_m R(z) \chi_m r^{-s}\|_{\mathcal{B}(\mathcal{H}, \mathcal{H})} \\ & \quad + \|r^{-s} \chi_m R(z) \bar{\chi}_m r^{-s}\|_{\mathcal{B}(\mathcal{H}, \mathcal{H})} \\ & \leq C_1 R_m^{s'-s}, \end{aligned} \tag{5.13}$$

and, similarly,

$$\|R(z') - \chi_m R(z') \chi_m\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_2 R_m^{s'-s}. \tag{5.14}$$

As for the first and second terms on the right-hand side of (5.12), using the expressions (1.8) and

$$i[H, \chi_n] = \operatorname{Re}(\chi_n' p^r) = \operatorname{Re}(\chi_n' A) \tag{5.15}$$

and noting the identity  $\overline{a_{\bar{z}}} = a_z$ , we write for  $n > m$

$$\begin{aligned} & \chi_m R(z) \chi_m - \chi_m R(z') \chi_m \\ & = \chi_m R(z) \{ \chi_n (H - z') - (H - z) \chi_n \} R(z') \chi_m \\ & = \chi_m R(z) \left\{ (z - z') \chi_n - \frac{i}{2} (a_z - a_{z'}) \chi_n' \right\} R(z') \chi_m \\ & \quad + \frac{i}{2} \chi_m R(z) \chi_n' (A - a_{z'}) R(z') \chi_m + \frac{i}{2} \chi_m R(z) (A + a_{\bar{z}})^* \chi_n' R(z') \chi_m. \end{aligned}$$

Then by Theorems 1.7 and 1.10 we have uniformly in  $n > m \geq 0$  and  $z, z' \in I_+$

$$\|\chi_m R(z) \chi_m - \chi_m R(z') \chi_m\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_3 R_n |z - z'| + C_4 R_n^{-\beta}. \tag{5.16}$$

Summing up (5.12)–(5.16), we obtain uniformly in  $n > m \geq 0$  and  $z, z' \in I_+$

$$\|R(z) - R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} = C_5 R_m^{s'-s} + C_3 R_n |z - z'| + C_4 R_n^{-\beta}.$$

Now we choose  $n = m + 1$  and  $R_m \leq |z - z'|^{-1/\min\{s-s'+1, \beta+1\}} < R_n$ , and then obtain uniformly in  $z, z' \in I_+$

$$\|R(z) - R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_6 |z - z'|^\epsilon \tag{5.17}$$

with  $\epsilon = \min\{(s - s')/(s - s' + 1), \beta/(\beta + 1)\}$ . The Hölder continuity (1.23) for  $R(z)$  follows from (5.17). The Hölder continuity (1.23) for  $pR(z)$  follows by using in addition (2.22) or alternatively the first resolvent equation.

The existence of the limits of (1.24) is an immediate consequence of (1.23). By Theorem 1.7 the limits  $p^\alpha R(\lambda + i0)$  actually map into  $B^*$ , and moreover they extend continuously to maps  $B \rightarrow B^*$  by a density argument. Finally, since  $R(z)$  for  $z \in I_+$  maps into  $\mathcal{N}$ , it follows by (1.23) and approximation arguments that  $R(\lambda \pm i0)$  map into  $\mathcal{N}$ . Hence we are done.  $\square$

### 5.2.3. Radiation condition bounds for real spectral parameters.

*Proof of Corollary 1.12.* The corollary follows from Theorem 1.10, Corollary 1.11 and approximation arguments. Note the elementary property

$$\|\psi\|_{B^*} = \sup_{n \geq 0} \|\chi_n \psi\|_{B^*}; \quad \psi \in B^*.$$

Hence we are done.  $\square$

5.2.4. *Sommerfeld uniqueness result.*

*Proof of Corollary 1.13.* Let  $\lambda \in \mathcal{I}$ ,  $\phi \in \mathcal{H}_{\text{loc}}$  and  $\psi \in r^{-\beta}B$  with  $\beta \in [0, \beta_c)$ . We first assume  $\phi = R(\lambda + i0)\psi$ . Then (i) and (ii) of the corollary obviously hold by Corollaries 1.11 and 1.12. Conversely, assume (i) and (ii) of the corollary, and let

$$\phi' = \phi - R(\lambda + i0)\psi.$$

Then by Corollaries 1.11 and 1.12 it follows that  $\phi'$  satisfies (i) and (ii) of the corollary with  $\psi = 0$ . In addition, we can verify  $\phi' \in B_0^*$  by the virial-type argument. In fact noting the identity

$$2\text{Im}(\chi_\nu(H - \lambda)) = (\text{Re } a)\chi'_\nu + \text{Re}(\chi'_\nu(A - a)),$$

cf. (1.8) and (5.15), we conclude that

$$0 \leq \langle (\text{Re } a)\bar{\chi}'_\nu \rangle_{\phi'} \leq \text{Re} \langle \chi'_\nu(A - a) \rangle_{\phi'}. \quad (5.18)$$

Taking the limit  $\nu \rightarrow \infty$  and using  $\phi' \in r^\beta B$  and  $(A - a)\phi' \in r^{-\beta}B_0^*$  in (5.18), indeed we obtain  $\phi \in B_0^*$ . By Theorem 1.4 it follows  $\phi' = 0$ , and hence  $\phi = R(\lambda + i0)\psi$ .  $\square$

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