# STATIONARY SCATTERING THEORY ON MANIFOLDS, II

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Abstract. Based on our previous study [IS2] we develop fully the stationary scattering theory for the Schrödinger operator on a manifold possessing an escape function. A particular class of examples are manifolds with Euclidean and/or hyperbolic ends, possibly with unbounded and non-smooth obstacles. We develop the theory largely along the classical lines [Sa, Co] and derive in particular WKB-asymptotics of a class of minimal generalized eigenfunctions. As an application we prove a conjecture of [HPW] on cross-ends transmissions in its natural and strong form within the framework of our theory.

Contents



## 1. Introduction

Let $(M,g)$ be a connected Riemannian manifold. In this paper we study stationary scattering theory for the geometric Schrödinger operator

$$H = H_0 + V; \quad H_0 = -\tfrac{1}{2}\Delta = \tfrac{1}{2}p_i^* g^{ij} p_j,\ p_i = -\mathrm{i}\partial_i,$$

on the Hilbert space $\mathcal{H} = L^2(M)$. The potential $V$ is real-valued and bounded, and the self-adjointness of $H$ is realized by the Dirichlet boundary condition. We shall develop a long-range stationary scattering theory to a large extent along the lines of [Sa, Co] on $\mathbb{R}^d$ or exterior domains of $\mathbb{R}^d$. In particular our theory relies on the existence of an intrinsic escape function. For other previous works on scattering with

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unbounded obstacles we refer to [Il1, Il2, Il3]. For previous time-dependent short-range scattering theories on manifolds we refer to [IN, IS1]. We shall develop the time-dependent scattering theory from the stationary theory of this paper elsewhere [IS3].

Our main results are asymptotic completeness Theorem 1.17, i.e. the existence of unitarily diagonalizing distorted Fourier transforms, and a characterization of an associated class of minimal generalized eigenfunctions in terms of (zeroth order) WKB-asymptotics Theorem 1.19. As an application we prove a conjecture of [HPW] on cross-ends transmissions. It is stated in a strong form as Corollary 1.20. The results of the paper are obtained in terms of an intrinsic escape function geometrically controlled by parameters. At the border of our parameter constraints we construct an example for which the minimal generalized eigenfunctions do not have WKB-asymptotics Example 1.18. Whence our somewhat technical conditions are more natural than a first reading might indicate and in a sense optimal.

1.1. **Setting and results from** [IS2]**.** Our paper is a direct continuation of [IS2], and we start by recalling the setting and various results there partly to fix notation and terminologies. This subsection exhibits only a minimal review, and we refer to [IS2, Subsection 1.1] for more details and to [IS2, Subsection 1.2] for several examples of manifolds satisfying the abstract conditions appearing below.

1.1.1. *Basic setting.* We assume an *end* structure on $M$ in a somewhat disguised form.

**Condition 1.1.** Let $(M, g)$ be a connected Riemannian manifold of dimension $d \geq 1$. There exist a function $r \in C^\infty(M)$ with image $r(M) = [1, \infty)$ and constants $c > 0$ and $r_0 \geq 2$ such that:

1. (1) The gradient vector field $\omega = \operatorname{grad} r \in \mathfrak{X}(M)$ is forward complete in the sense that the integral curve of $\omega$ is defined for any initial point $x \in M$ and any non-negative time parameter $t \geq 0$.
2. (2) The bound $|\mathrm{d}r| = |\omega| \geq c$ holds on $\{x \in M \,|\, r(x) > r_0/2\}$.

Under Condition 1.1 each component of the subset $E = \{x \in M \,|\, r(x) > r_0\}$ is called an *end* of $M$, and, along with Condition 1.2 below, the function $r$ may model a distance function there. We note that by Condition 1.1 (2) and the implicit function theorem the $r$-spheres

$$S_R = \{x \in M \,|\, r(x) = R\}; \quad R > r_0/2,$$

are submanifolds of $M$. We will construct the *spherical coordinates* on $E$ in Subsection 1.2.

Let us impose more conditions on the geometry of $E$ in terms of the radius function $r$. Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2, \end{cases} \qquad \chi \geq 0, \quad \chi' \leq 0, \quad \sqrt{1-\chi} \in C^\infty, \tag{1.1}$$

and set

$$\eta = 1 - \chi(2r/r_0), \quad \tilde{\eta} = |\mathrm{d}r|^{-2}\eta = |\mathrm{d}r|^{-2}\bigl(1 - \chi(2r/r_0)\bigr). \tag{1.2}$$

We introduce a "radial" differential operator $A$:

$$A = \operatorname{Re} p^r = \tfrac{1}{2}\bigl(p^r + (p^r)^*\bigr); \quad p^r = -\mathrm{i}\nabla^r, \ \nabla^r = \nabla_\omega = g^{ij}(\nabla_i r)\nabla_j, \tag{1.3}$$

and also the "spherical" tensor $\ell$ and the associated differential operator $L$:

$$\ell = g - \tilde\eta\,\mathrm{d}r\otimes\mathrm{d}r,\quad L = p_i^*\ell^{ij}p_j. \tag{1.4}$$

As we can see easily in the spherical coordinates introduced in Subsection 1.2, the tensor $\ell$ may be identified with the pull-back of $g$ to the $r$-spheres. We call $L$ the spherical part of $-\Delta$. Note that if $|\mathrm{d}r| = 1$ then $-L$ acts as the Laplace–Beltrami operator on $S_r$ (in general as a kind of perturbation of this operator, see (2.11)). We remark that the tensor $\ell$ clearly satisfies

$$0\le \ell\le g,\quad \ell^{\bullet i}(\nabla r)_i = (1-\eta)\mathrm{d}r, \tag{1.5}$$

where the first bounds of (1.5) are understood as quadratic form estimates on the fibers of the tangent bundle of $M$. The quantities of (1.4) will play a major role in this paper.

Let us recall a local expression of the Levi–Civita connection $\nabla$: If we denote the Christoffel symbol by $\Gamma^k_{ij} = \frac12 g^{kl}(\partial_i g_{lj}+\partial_j g_{li}-\partial_l g_{ij})$, then for any smooth function $f$ on $M$

$$(\nabla f)_i = (\nabla_i f) = (\mathrm{d}f)_i = \partial_i f,\quad (\nabla^2 f)_{ij} = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f. \tag{1.6}$$

Note that $\nabla^2 f$ is the geometric Hessian of $f$.

**Condition 1.2.** There exist constants $\tau, C>0$ such that globally on $M$

$$|\nabla|\mathrm{d}r|^2|\le Cr^{-1-\tau/2},\quad |\nabla^k r|\le C \text{ for } k\in\{1,2\},\quad \bigl|\ell^{\bullet i}\nabla_i\Delta r\bigr|\le Cr^{-1-\tau/2}. \tag{1.7a}$$

There exists $\sigma'>0$ such that for all $R>r_0/2$, and as quadratic forms on fibers of the tangent bundle of $S_R$,

$$R\,\iota_R^*\nabla^2 r\ge \tfrac12\sigma'|\mathrm{d}r|^2\iota_R^* g, \tag{1.7b}$$

where $\iota_R\colon S_R\hookrightarrow M$ is the inclusion map.

We note that Condition 1.2 and the identity

$$(\nabla^2 r)^{ij}(\nabla r)_j = \tfrac12(\nabla|\mathrm{d}r|^2)^i \tag{1.8}$$

was used in [IS2] to obtain the more practical version of (1.7b): For any $\sigma\in(0,\sigma')$ and $\tau$ as in Condition 1.2 there exists $C>0$ such that globally on $M$

$$r\bigl(\nabla^2 r - \tfrac12\tilde\eta^2(\nabla^r|\mathrm{d}r|^2)\mathrm{d}r\otimes\mathrm{d}r\bigr)\ge \tfrac12\sigma|\mathrm{d}r|^2\ell - Cr^{-\tau}g. \tag{1.9}$$

Next we introduce an effective potential:

$$q = V + \tfrac18\tilde\eta\bigl[(\Delta r)^2 + 2\nabla^r\Delta r\bigr]. \tag{1.10}$$

**Condition 1.3.** There exists a splitting by real-valued functions:

$$q = q_1+q_2;\quad q_1\in C^1(M)\cap L^\infty(M),\ q_2\in L^\infty(M),$$

such that for some $\rho', C>0$ the following bounds hold globally on $M$:

$$\nabla^r q_1\le Cr^{-1-\rho'},\quad |q_2|\le Cr^{-1-\rho'}. \tag{1.11}$$

We remark that in this paper only derivatives of $r$ of order at most five are used quantitatively.

Now let us mention the self-adjoint realizations of $H$ and $H_0$. Since $(M,g)$ can be incomplete, the operators $H$ and $H_0$ are not necessarily essentially self-adjoint on $C^\infty_{\mathrm c}(M)$. We realize $H_0$ as a self-adjoint operator by imposing the Dirichlet boundary condition, i.e. $H_0$ is the unique self-adjoint operator associated with the closure of the quadratic form

$$\langle H_0\rangle_\psi = \langle \psi, -\tfrac{1}{2}\Delta\psi\rangle, \quad \psi \in C^\infty_{\mathrm{c}}(M).$$

We denote the form closure and the self-adjoint realization by the same symbol $H_0$. Define the associated Sobolev spaces $\mathcal{H}^s$ by

$$\mathcal{H}^s = (H_0+1)^{-s/2}\mathcal{H}, \quad s \in \mathbb{R}. \tag{1.12}$$

Then $H_0$ may be understood as a closed quadratic form on $Q(H_0) = \mathcal{H}^1$. Equivalently, $H_0$ makes sense also as a bounded operator $\mathcal{H}^1 \to \mathcal{H}^{-1}$, whose action coincides with that for distributions. By the definition of the Friedrichs extension the self-adjoint realization of $H_0$ is the restriction of such distributional $H_0\colon \mathcal{H}^1 \to \mathcal{H}^{-1}$ to the domain:

$$\mathcal{D}(H_0) = \{\psi \in \mathcal{H}^1 \,|\, H_0\psi \in \mathcal{H}\} \subseteq \mathcal{H}.$$

Since $V$ is bounded and self-adjoint by Conditions 1.1–1.3, we can realize the self-adjoint operator $H = H_0 + V$ simply as

$$H = H_0 + V, \quad \mathcal{D}(H) = \mathcal{D}(H_0).$$

In contrast to (1.12) we introduce the Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_{s\pm}$ with configuration weights:

$$\mathcal{H}_s = r^{-s}\mathcal{H}, \quad \mathcal{H}_{s+} = \bigcup_{s'>s}\mathcal{H}_{s'}, \quad \mathcal{H}_{s-} = \bigcap_{s'<s}\mathcal{H}_{s'}, \quad s \in \mathbb{R}.$$

We consider the $r$-balls $B_R = \{r(x) < R\}$ and the characteristic functions

$$F_\nu = F(B_{R_{\nu+1}} \setminus B_{R_\nu}),\ R_\nu = 2^\nu,\ \nu \geq 0, \tag{1.13}$$

where $F(\Omega)$ is used for sharp characteristic function of a subset $\Omega \subseteq M$. Define the associated Besov spaces $B$ and $B^*$ by

$$\begin{aligned} B &= \{\psi \in L^2_{\mathrm{loc}}(M) \,|\, \|\psi\|_B < \infty\}, \quad \|\psi\|_B = \sum_{\nu=0}^\infty R_\nu^{1/2}\|F_\nu\psi\|_{\mathcal{H}}, \\ B^* &= \{\psi \in L^2_{\mathrm{loc}}(M) \,|\, \|\psi\|_{B^*} < \infty\}, \quad \|\psi\|_{B^*} = \sup_{\nu\geq 0} R_\nu^{-1/2}\|F_\nu\psi\|_{\mathcal{H}}, \end{aligned} \tag{1.14}$$

respectively. We also define $B^*_0$ to be the closure of $C^\infty_{\mathrm{c}}(M)$ in $B^*$. Recall the nesting:

$$\mathcal{H}_{1/2+} \subsetneq B \subsetneq \mathcal{H}_{1/2} \subsetneq \mathcal{H} \subsetneq \mathcal{H}_{-1/2} \subsetneq B^*_0 \subsetneq B^* \subsetneq \mathcal{H}_{-1/2-}.$$

Using the function $\chi \in C^\infty(\mathbb{R})$ of (1.1), define $\chi_n, \bar\chi_n, \chi_{m,n} \in C^\infty(M)$ for $n > m \geq 0$ by

$$\chi_n = \chi(r/R_n), \quad \bar\chi_n = 1 - \chi_n, \quad \chi_{m,n} = \bar\chi_m\chi_n. \tag{1.15}$$

Let us introduce an auxiliary space:

$$\mathcal{N} = \{\psi \in L^2_{\mathrm{loc}}(M) \,|\, \chi_n\psi \in \mathcal{H}^1 \text{ for all } n \geq 0\}.$$

This is a space of functions that satisfy the Dirichlet boundary condition, possibly with infinite $\mathcal{H}^1$-norm on $M$. Note that under Conditions 1.1–1.3 the manifold $M$ may be, e.g. a half-space in the Euclidean space, and there could be a “boundary” even for large $r$, which is “invisible” from inside $M$.

1.1.2. *Review of the previous results.* Now we gather and review the main results of [IS2]. Note that all the theorems in this subsection are already proved there.

Our first theorem is Rellich's theorem, the absence of $B_0^*$-eigenfunctions with eigenvalues above a certain "critical energy" $\lambda_0 \in \mathbb{R}$ defined by

$$\lambda_0 = \limsup_{r\to\infty} q_1 = \lim_{R\to\infty} \bigl(\sup\{q_1(x) \,|\, r(x) \geq R\}\bigr). \tag{1.16}$$

For the Euclidean and the hyperbolic spaces and many other examples the critical energy $\lambda_0$ can be computed explicitly, and the essential spectrum is given by $\sigma_{\mathrm{ess}}(H) = [\lambda_0, \infty)$. The latter is usually seen in terms of Weyl sequences, see [Ku1].

**Theorem 1.4.** *Suppose Conditions 1.1–1.3, and let $\lambda > \lambda_0$. If a function $\phi \in L^2_{\mathrm{loc}}(M)$ satisfies that*

*(1) $(H - \lambda)\phi = 0$ in the distributional sense,*

*(2) $\bar{\chi}_m \phi \in \mathcal{N} \cap B_0^*$ for all $m \geq 0$ large enough,*

*then $\phi = 0$ in $M$.*

Next we discuss the limiting absorption principle and the radiation condition related to the resolvent $R(z) = (H - z)^{-1}$. We state a locally uniform bound for the resolvent as a map: $B \to B^*$. For that we need a compactness condition.

**Condition 1.5.** In addition to Conditions 1.1–1.3, there exists an open subset $\mathcal{I} \subseteq (\lambda_0, \infty)$ such that for any $n \geq 0$ and compact interval $I \subseteq \mathcal{I}$ the mapping

$$\chi_n P_H(I) \colon \mathcal{H} \to \mathcal{H}$$

is compact, where $P_H(I)$ denotes the spectral projection onto $I$ for $H$.

Due to Rellich's compact embedding theorem [RS, Theorem XIII.65], "boundedness" of $r$-balls provides a criterion for Condition 1.5: If each $r$-ball $B_R$, $R \geq 1$, is isometric to a bounded subset of a complete manifold, Condition 1.5 is satisfied for $\mathcal{I} = (\lambda_0, \infty)$. Condition 1.5 in fact includes more general situations where $M$ has multiple ends of different critical energies and $r$-balls are unbounded as in [Ku2].

We fix any $\sigma \in (0, \sigma')$ and then large enough $C > 0$ in agreement with (1.9), and introduce the positive quadratic form

$$h := \nabla^2 r - \tfrac{1}{2}\tilde{\eta}^2(\nabla^r |\mathrm{d}r|^2)\mathrm{d}r \otimes \mathrm{d}r + 2Cr^{-1-\tau} g \geq \tfrac{1}{2}\sigma r^{-1} |\mathrm{d}r|^2 \ell + Cr^{-1-\tau} g.$$

For any subset $I \subseteq \mathcal{I}$ we denote

$$I_\pm = \{z = \lambda \pm \mathrm{i}\Gamma \in \mathbb{C} \,|\, \lambda \in I,\ \Gamma \in (0, 1)\},$$

respectively. We also use the notation $\langle T \rangle_\phi = \langle \phi, T\phi \rangle$.

**Theorem 1.6.** *Suppose Condition 1.5 and let $I \subseteq \mathcal{I}$ be a compact interval. Then there exists $C > 0$ such that for any $\phi = R(z)\psi$ with $z \in I_\pm$ and $\psi \in B$*

$$\|\phi\|_{B^*} + \|p^r \phi\|_{B^*} + \langle p_i^* h^{ij} p_j \rangle_\phi^{1/2} + \|H_0 \phi\|_{B^*} \leq C\|\psi\|_B. \tag{1.17}$$

In our theory the Besov boundedness (1.17) does not immediately imply the limiting absorption principle, and for the latter we need also radiation condition bounds implied by minor additional regularity conditions.

**Condition 1.7.** In addition to Condition 1.5 there exist splittings $q_1 = q_{11} + q_{12}$ and $q_2 = q_{21} + q_{22}$ by real-valued functions

$$q_{11} \in C^2(M) \cap L^\infty(M), \quad q_{12}, q_{21} \in C^1(M) \cap L^\infty(M), \quad q_{22} \in L^\infty(M)$$

and constants $\rho, C > 0$ such that for $\alpha = 0, 1$

$$|\nabla^r q_{11}| \le Cr^{-(1+\rho/2)/2}, \qquad |\ell^{\bullet i}\nabla_i q_{11}| \le Cr^{-1-\rho/2}, \qquad |\mathrm{d}\nabla^r q_{11}| \le Cr^{-1-\rho/2},$$
$$|\mathrm{d}q_{12}| \le Cr^{-1-\rho/2}, \qquad |(\nabla^r)^\alpha q_{21}| \le Cr^{-\alpha-\rho}, \qquad q_{21}\nabla^r q_{11} \le Cr^{-1-\rho},$$
$$|q_{22}| \le Cr^{-1-\rho/2}.$$

Our radiation condition bounds are stated in terms of the distributional radial differential operator $A$ defined in (1.3) and an asymptotic complex phase $a$ given below. Pick a smooth decreasing function $r_\lambda \ge 2r_0$ of $\lambda > \lambda_0$ such that

$$\lambda + \lambda_0 - 2q_1 \ge 0 \text{ for } r \ge r_\lambda/2, \tag{1.18}$$

and that $r_\lambda = r_0$ for all $\lambda$ large enough. Then we set

$$\eta_\lambda = 1 - \chi(2r/r_\lambda),$$

and for $z = \lambda \pm \mathrm{i}\Gamma \in \mathcal{I} \cup \mathcal{I}_\pm$

$$b = \eta_\lambda |\mathrm{d}r| \sqrt{2(z - q_1)}, \qquad \tilde{b} = \tilde{\eta} b, \tag{1.19a}$$
$$a = b \pm \tfrac{1}{4}\eta_\lambda (p^r q_{11})\big/(z - q_1), \quad \tilde{a} = \tilde{\eta} a, \tag{1.19b}$$

respectively, where the branch of square root is chosen such that $\mathrm{Re}\sqrt{w} > 0$ for $w \in \mathbb{C} \setminus (-\infty, 0]$. Note that for $z \in \mathcal{I}$ there are two values of $a$ (and similarly of course for $\tilde{a}$) which could be denoted $a_\pm$. For convenience we prefer to use the shorter notation. Note also that the phase $a$ of (1.19b) is an approximate solution to the radial Riccati equation

$$\pm p^r a + a^2 - 2|\mathrm{d}r|^2 (z - q_1) = 0 \tag{1.20}$$

in the sense that it makes the quantity on the left-hand side of (1.20) small for large $r \ge 1$. The quantity $b$ of (1.19a) alone already gives an approximate solution to the same equation, however with the second term of (1.19b) a better approximation is obtained, cf. Lemma 2.2. Set

$$\beta_c = \tfrac{1}{2}\min\{\sigma, \tau, \rho\}. \tag{1.21}$$

Here and whenceforth we consider $\sigma \in (0, \sigma')$ as a fixed parameter.

**Theorem 1.8.** *Suppose Condition 1.7, and let $I \subseteq \mathcal{I}$ be a compact interval. Then for all $\beta \in [0, \beta_c)$ there exists $C > 0$ such that for any $\phi = R(z)\psi$ with $\psi \in r^{-\beta}B$ and $z \in I_\pm$*

$$\|r^\beta (A \mp a)\phi\|_{B^*} + \langle p_i^* r^{2\beta} h^{ij} p_j \rangle_\phi^{1/2} \le C\|r^\beta \psi\|_B, \tag{1.22}$$

*respectively.*

The limiting absorption principle reads.

**Corollary 1.9.** *Suppose Condition 1.7, and let $I \subseteq \mathcal{I}$ be a compact interval. For any $s > 1/2$ and $\epsilon \in (0, \min\{(2s-1)/(2s+1), \beta_c/(\beta_c+1)\})$ there exists $C > 0$ such that for $\alpha = 0, 1$ and any $z, z' \in I_+$ or $z, z' \in I_-$*

$$\|p^\alpha R(z) - p^\alpha R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \le C|z - z'|^\epsilon. \tag{1.23}$$

*In particular, the operators $p^\alpha R(z)$, $\alpha = 0, 1$, attain uniform limits as $I_\pm \ni z \to \lambda \in I$ in the norm topology of $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$, say denoted*

$$p^\alpha R(\lambda \pm \mathrm{i}0) := \lim_{I_\pm \ni z \to \lambda} p^\alpha R(z), \quad \lambda \in I, \tag{1.24}$$

*respectively. These limits* $p^\alpha R(\lambda\pm \mathrm{i}0)\in\mathcal{B}(B,B^*)$*, and* $R(\lambda\pm\mathrm{i}0): B\to\mathcal{N}\cap B^*$.

Given the limiting resolvents $R(\lambda\pm\mathrm{i}0)$ the radiation condition bounds for real spectral parameters follow directly from Theorem 1.8.

**Corollary 1.10.** *Suppose Condition 1.7, and let* $I\subseteq\mathcal{I}$ *be a compact interval. Then for all* $\beta\in[0,\beta_c)$ *there exists* $C>0$ *such that for any* $\phi=R(\lambda\pm\mathrm{i}0)\psi$ *with* $\psi\in r^{-\beta}B$ *and* $\lambda\in I$

$$\|r^\beta(A\mp a)\phi\|_{B^*}+\langle p_i^* r^{2\beta}h^{ij}p_j\rangle_\phi^{1/2}\le C\|r^\beta\psi\|_B, \tag{1.25}$$

*respectively.*

For the Euclidean and the hyperbolic spaces without potential $V$ we have $\beta_c\ge 1$. Hence in these cases the bound (1.25) hold for any $\beta\in[0,1)$. We remark that for the Euclidean space and a sufficiently regular potential the bound (1.25) is well-known for $\beta\in[0,1)$, cf. [Is, Sa, HS1]. However in this case one can actually allow $\beta\in[1,2)$, cf. [HS1]. If $\beta>1$ is allowed the existence of the distorted Fourier transform follows easy, cf. [HS1, HS2, Sk]. This is demonstrated in Subsection 3.1.

As another application of the radiation condition bounds we have characterized the limiting resolvents $R(\lambda\pm\mathrm{i}0)$. For the Euclidean space such characterization is usually referred to as the Sommerfeld uniqueness result, see for example [Is].

**Corollary 1.11.** *Suppose Condition 1.7, and let* $\lambda\in\mathcal{I}$, $\phi\in L^2_{\mathrm{loc}}(M)$ *and* $\psi\in r^{-\beta}B$ *with* $\beta\in[0,\beta_c)$. *Then* $\phi=R(\lambda\pm\mathrm{i}0)\psi$ *holds if and only if both of the following conditions hold:*

*(i)* $(H-\lambda)\phi=\psi$ *in the distributional sense.*

*(ii)* $\phi\in\mathcal{N}\cap r^\beta B^*$ *and* $(A\mp a)\phi\in r^{-\beta}B_0^*$.

1.2. **Limiting Hilbert space.** To state the main results of the paper in Subsection 1.3 here we introduce the spherical coordinates and the limiting Hilbert space.

1.2.1. *Abstract construction.* Let us begin with an abstract theory. We construct the spherical coordinates on $E$ under Condition 1.1 Using $\tilde\eta$ of (1.2), define the *normalized* gradient vector field $\tilde\omega\in\mathfrak{X}(M)$ by

$$\tilde\omega=\tilde\eta\omega.$$

We let

$$\tilde y\colon\widetilde{\mathcal{M}}\to M,\ (t,x)\mapsto\tilde y(t,x)=\exp(t\tilde\omega)(x), \tag{1.26}$$

denote the maximal flow generated by the vector field $\tilde\omega$ (whence by Condition 1.1 $[0,\infty)\times M\subseteq\widetilde{\mathcal{M}}$). By definition it satisfies, in local coordinates,

$$\partial_t\tilde y^i(t,x)=\tilde\omega^i(\tilde y(t,x))=(\tilde\eta g^{ij}\nabla_j r)(\tilde y(t,x)),\quad \tilde y(0,x)=x.$$

This implies in particular that for $r(x)\ge r_0$ and $t\ge 0$

$$r(\tilde y(t,x))=r(x)+t, \tag{1.27}$$

and hence the semigroup (1.26) induces a family of diffeomorphic embeddings

$$\iota_{R,R'}=\tilde y(R'-R,\ \cdot\ )_{|S_R}\colon S_R\to S_{R'};\quad r_0\le R\le R', \tag{1.28}$$

satisfying

$$\iota_{R',R''}\circ\iota_{R,R'}=\iota_{R,R''};\quad r_0\le R\le R'\le R''. \tag{1.29}$$

Through (1.28) and (1.29) we may regard $S_R \subseteq S_{R'}$ for any $R \leq R'$ in a well-defined manner. Such inclusions naturally induce a manifold structure on the union

$$S = \bigcup_{R>r_0} S_R. \tag{1.30}$$

In fact, the manifold $S$ can be attained as an inductive limit, but we do not get into technical details since we are going to use a concrete simple procedure (which is facilitated by an additional condition). The manifold $S$ may in any case be considered as a boundary of $M$ at infinity. Let $\sigma$ be any local coordinates on $S$. We can define $\sigma(x)$ for $x \in E$ by considering $x \in S_{r(x)} \subseteq S$. Then the spherical coordinates of a point $x \in E$, written slightly inconsistently, are the components of $(r, \sigma) = (r(x), \sigma(x)) \in (r_0, \infty) \times S$. We shall refer to $r$ as the *radius function*, and $S_R \subseteq S$ as the *angular* or *spherical manifolds*. Note that in such coordinates $E$ is identified with an open subset of the half-infinite cylinder $(r_0, \infty) \times S$ whose $r$-sections are monotonically increasing and exhausting $S$.

Regarding $r \geq r_0$ just as a parameter and letting $\mathrm{d}\mathcal{A}_r$ be the naturally induced measure on $S_r$, we introduce the Hilbert space

$$\mathcal{G}_r = L^2(S_r, \mathrm{d}\tilde{\mathcal{A}}_r); \quad \mathrm{d}\tilde{\mathcal{A}}_r = |\mathrm{d}r|^{-1}\mathrm{d}\mathcal{A}_r = (\det g)^{1/2}\,\mathrm{d}\sigma^2 \cdots \mathrm{d}\sigma^d, \tag{1.31}$$

where the last equality holds in the spherical coordinates with any local coordinates for $S_r \subseteq S$. As for the measure $\mathrm{d}\tilde{\mathcal{A}}_r$ we note that the co-area formula (cf. [E, Theorem C.5]) is valid for all integrable functions $\phi$ supported in $E$: In the spherical coordinates

$$\int_E \phi(x)\big(\det g(x)\big)^{1/2}\,\mathrm{d}x = \int_{r_0}^\infty \mathrm{d}r \int_{S_r} \phi(r,\sigma)\,\mathrm{d}\tilde{\mathcal{A}}_r(\sigma). \tag{1.32}$$

Noting (1.32), we can construct isometric embeddings

$$i_{r,r'} \colon \mathcal{G}_r \to \mathcal{G}_{r'} \quad \text{for } r_0 \leq r \leq r' \tag{1.33}$$

as follows. For any $\xi_r \in \mathcal{G}_r$ we define $i_{r,r'}\xi_r = \xi_{r'} \in \mathcal{G}_{r'}$ by letting, in the spherical coordinates,

$$\xi_{r'}(\sigma) = \big(\det g(r,\sigma)/\det g(r',\sigma)\big)^{1/4}\,\xi_r(\sigma) \quad \text{for } (r,\sigma) \in E, \tag{1.34}$$

and $\xi_{r'}(\sigma) = 0$ for $(r,\sigma) \notin E$. Indeed (1.33) are isometric embeddings satisfying

$$i_{r',r''} \circ i_{r,r'} = i_{r,r''} \quad \text{for } r_0 \leq r \leq r' \leq r''.$$

Then in parallel to (1.30) we may regard $\mathcal{G}_r \subseteq \mathcal{G}_{r'}$ for $r_0 \leq r \leq r'$, and these inclusions naturally induce a pre-Hilbert space structure on the union

$$\mathcal{G}_\infty = \bigcup_{r>r_0} \mathcal{G}_r.$$

We can define the "limiting Hilbert space" $\mathcal{G}$ as the completion of $\mathcal{G}_\infty$.

We remark that if $\omega$ is also backward complete all the above embeddings are in fact equalities, i.e. $S_r \cong S_{r'}$ and $\mathcal{G}_r \cong \mathcal{G}_{r'}$ for $r_0 \leq r \leq r'$, and we may construct the limiting objects just by letting $S = S_{r_0}$ and $\mathcal{G} = \mathcal{G}_{r_0}$. This is a motivation for the following concrete construction.

1.2.2. *Concrete construction.* In this paper we impose an additional geometric condition under which the set $S$ and the associated limiting Hilbert space $\mathcal{G}$ can be realized more concretely than above.

**Condition 1.12.** There exists an extended Riemannian manifold $(M^{\mathrm{ex}}, g^{\mathrm{ex}})$ of dimension $d$ in which $(M, g)$ is isometrically embedded. The previous Condition 1.1 is also fulfilled for $(M^{\mathrm{ex}}, g^{\mathrm{ex}})$ (with the same constants $c$ and $r_0$) possibly without the connectedness assumption. In addition the extended vector field, say denoted by $\omega^{\mathrm{ex}}$, is backward complete in $M^{\mathrm{ex}}$ (that is complete in $M^{\mathrm{ex}}$).

Under Condition 1.12 we define

$$\begin{aligned} S^{\mathrm{ex}}(M) &= \big\{x \in S^{\mathrm{ex}}_{r_0} \,\big|\, \{\tilde{y}^{\mathrm{ex}}(t,x) \,|\, t \geq 0\} \cap M \neq \emptyset\big\}, \\ \mathcal{G}^{\mathrm{ex}} &= L^2(S^{\mathrm{ex}}(M), \mathrm{d}\tilde{\mathcal{A}}^{\mathrm{ex}}_{r_0}) \subseteq L^2(S^{\mathrm{ex}}_{r_0}, \mathrm{d}\tilde{\mathcal{A}}^{\mathrm{ex}}_{r_0}). \end{aligned} \tag{1.35}$$

This leads to the isometrical embedding $\mathcal{G}_r \subseteq \mathcal{G}^{\mathrm{ex}}$, $r \geq r_0$, by mapping $\mathcal{G}_r \ni \xi \to \xi^{\mathrm{ex}} \in \mathcal{G}^{\mathrm{ex}}$ given in the spherical coordinates by

$$\xi^{\mathrm{ex}}(\sigma) = \big(\det g^{\mathrm{ex}}(r,\sigma)/\det g^{\mathrm{ex}}(r_0,\sigma)\big)^{1/4}\,\xi(\sigma) \quad \text{for } (r,\sigma) \in S_r, \tag{1.36}$$

and $\xi^{\mathrm{ex}} = 0$ at other points in $S^{\mathrm{ex}}(M)$.

If $\omega$ is forward and backward complete obviously we do not need extended objects and $\mathcal{G} = L^2(S_{r_0}, \mathrm{d}\tilde{\mathcal{A}}_{r_0})$.

To study scattering on $M$ it is convenient (although not necessary for all of our results) to use $\mathcal{G}^{\mathrm{ex}}$ and integrals of extended orbits (as appearing above). Although we shall not elaborate our methods should have the potential of some similar results as in this paper with different conditions than Condition 1.12. For convenience we shall in the paper drop the superscript "ex" and write $\mathcal{G} = \mathcal{G}^{\mathrm{ex}}$, $\tilde{y}(t,x) = \tilde{y}^{\mathrm{ex}}(t,x)$, etc., whenever it follows from the context that these objects are "extended". The notation $S$ and $\mathrm{d}\tilde{\mathcal{A}}$ will exclusively be used for $S^{\mathrm{ex}}(M)$ and $\mathrm{d}\tilde{\mathcal{A}}^{\mathrm{ex}}_{r_0}$, respectively. Whence $\mathcal{G} = L^2(S, \mathrm{d}\tilde{\mathcal{A}})$.

The above formulas (1.34) and (1.36) can be understood in terms of translations on $\mathcal{H}$ or on $\mathcal{H}^{\mathrm{ex}}$. We introduce *normalized radial translations* $\tilde{T}(t)\colon \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}$, as follows. Recall the notation of the normalized maximal flow (1.26). Then $\tilde{T}(t)\psi$, $\psi \in \mathcal{H}$, is defined by

$$\begin{aligned} (\tilde{T}(t)\psi)(x) &= \tilde{J}(t,x)^{1/2}\,\big(\det g(\tilde{y}(t,x))/\det g(x)\big)^{1/4}\,\psi(\tilde{y}(t,x)) \\ &= \exp\left(\int_0^t \tfrac{1}{2}(\operatorname{div}\tilde{\omega})(\tilde{y}(s,x))\,\mathrm{d}s\right)\psi(\tilde{y}(t,x)) \end{aligned} \tag{1.37}$$

if $(t,x) \in \widetilde{\mathcal{M}}$, and $(\tilde{T}(t)\psi)(x) = 0$ otherwise, where $\tilde{J}(t,\,\cdot\,)$ is the Jacobian of the mapping $\tilde{y}(t,\,\cdot\,)\colon M \to M$ and $\operatorname{div}\tilde{\omega} = \operatorname{tr}\nabla\tilde{\omega}$. Note that $\tilde{J} = 1$ in the spherical coordinates, clearly showing a relationship to (1.34). The well-definedness and the equivalence of the two expressions in (1.37) can be verified similarly to the unnormalized flow in [IS2]. Here we only note that for any $\psi \in \mathcal{H}$ the former expression of (1.37) and a change of variables imply

$$\|\tilde{T}(t)\psi\| = \left(\int_{\tilde{M}(t)} |\psi(x)|^2\big(\det g(x)\big)^{1/2}\,\mathrm{d}x\right)^{1/2}; \quad \tilde{M}(t) = \tilde{y}(\max\{t,0\}, M). \tag{1.38}$$

Hence the operators $\tilde{T}(t) = \mathrm{e}^{\mathrm{i}t\tilde{A}_+}$, $t \geq 0$, and the operators $\tilde{T}(-t) = \mathrm{e}^{-\mathrm{i}t\tilde{A}_-}$, $t \geq 0$, form strongly continuous one-parameter semigroups of surjective partial isometries and isometries, respectively. The operators are the adjoints of each other in the sense that $\tilde T(t)^* = \tilde T(-t)$, however in general the family $(\tilde T(t))_{t\in\mathbb R}$ does not form a one-parameter group. This is in contrast to the similarly defined quantity for $M^{\mathrm{ex}}$, say denoted $(\tilde T^{\mathrm{ex}}(t))_{t\in\mathbb R} = (\mathrm e^{\mathrm it\tilde A^{\mathrm{ex}}})_{t\in\mathbb R}$, which indeed is a group on $\mathcal H^{\mathrm{ex}}$ with self-adjoint generator

$$\tilde A^{\mathrm{ex}} = \operatorname{Re}\big(-\mathrm i\nabla_{\tilde\omega^{\mathrm{ex}}}\big),$$

where

$$\nabla_{\tilde\omega^{\mathrm{ex}}} = (\tilde\omega^{\mathrm{ex}})^i\nabla_i,\quad \tilde\omega^{\mathrm{ex}} = \tilde\eta^{\mathrm{ex}}(\nabla r^{\mathrm{ex}}),\quad \tilde\eta^{\mathrm{ex}} = \eta^{\mathrm{ex}}|\mathrm dr^{\mathrm{ex}}|^{-2}.$$

Note the relationship $\tilde T(t) = 1_M\tilde T^{\mathrm{ex}}(t)1_M$ for all $t\in\mathbb R$. We shall use the notation $\tilde A$ as a generic notation for the generators $\tilde A_+, \tilde A_-$ and $\tilde A^{\mathrm{ex}}$ without distinction, if it is not confusing from the context.

## 1.3. **Main results.**

1.3.1. *Distorted Fourier transform.* We need additional assumptions. The following one suffices for constructing the *distorted Fourier transform.*

**Condition 1.13.** Along with Condition 1.12, Condition 1.7 holds with

$$2\beta_c = \min\{\sigma,\tau,\rho\} > 1. \tag{1.39}$$

The function $\tilde b = \tilde b(\lambda,x)$ has a real $C^1$-extension to $\mathcal I\times M^{\mathrm{ex}}$, say denoted by $\tilde b^{\mathrm{ex}}$ (or by $\tilde b$ again for short). The following bound holds uniformly in $x\in E$ and locally uniformly in $\lambda\in\mathcal I$:

$$\sup_{r_0\le\check r\le r(x)}\biggl|\nabla'\int_{\check r-r(x)}^0 \tilde b^{\mathrm{ex}}(\tilde y^{\mathrm{ex}}(t,x))\,\mathrm dt\biggr| \le Cr(x)^{-1/2}, \tag{1.40}$$

where $\nabla' = \ell^{\bullet i}\nabla_i$ denotes the covariant derivative for the $r$-sphere $S_{r(x)}$ (with induced Riemannian metric).

**Remarks.** If $M^{\mathrm{ex}} = M$ the technical bound (1.40) follows from Lemma 3.5. More generally we can *verify* (1.40) assuming Condition 1.12 and that various of the requirements in Conditions 1.2 and 1.7 for quantities on $M$ hold as well for the extended quantities (on $M^{\mathrm{ex}}$) with (1.39). This follows from our proof of Lemma 3.5. The bound is only used in the proof of Lemma 3.7, and we note that it is not needed if we impose the strengthening (1.46) of (1.39) (however we do need it for the alternative Condition 1.16 (2)).

For any $\psi\in\mathcal H_{1+}$ and $r\ge r_0$ we introduce a function $\xi(r)\in\mathcal G$ using the mapping (1.36) (and omitting the superscript "ex") and noting the expression in (1.37). We let

$$\xi(r)(\sigma) = \exp\biggl(\int_{r_0}^r \Bigl(\mp\mathrm i\tilde b + \tfrac12\operatorname{div}\tilde\omega\Bigr)(s,\sigma)\,\mathrm ds\biggr)\bigl[\sqrt{b}R(\lambda\pm\mathrm i0)\psi\bigr](r,\sigma), \tag{1.41}$$

or, alternatively,

$$\xi(r) = \mathrm e^{\mathrm i(r-r_0)(\tilde A^{\mathrm{ex}}\mp\tilde b^{\mathrm{ex}})}\bigl[\sqrt{b}R(\lambda\pm\mathrm i0)\psi\bigr]_{|S_r} = \mathrm e^{\mathrm i(r-r_0)(\tilde A\mp\tilde b)}\bigl[\sqrt{b}R(\lambda\pm\mathrm i0)\psi\bigr]_{|S_r}. \tag{1.42}$$

Then we would like to define the "distorted Fourier transform" by

$$F^\pm(\lambda)\psi = \mathcal G\text{-}\lim_{r\to\infty}\xi(r);\quad \psi\in\mathcal H_{1+}. \tag{1.43}$$

By definition the function $F^{\pm}(\lambda)\psi \in \mathcal{G} = L^2(S, \mathrm{d}\tilde{\mathcal{A}})$, and we note that our construction of $F^{\pm}(\lambda)\psi$ is non-canonical primarily due to the freedom in choosing $\mathcal{G}$. In fact for $M^{\mathrm{ex}} = M$ the only non-canonical feature comes from the dependence of $r_0$ (determining $\mathcal{G}$ in that case), while in general there is an additional freedom in choosing extended functions.

Of course we need to justify the definition (1.43).

**Theorem 1.14.** *Suppose Condition 1.13. Then for any $\psi \in \mathcal{H}_{1+}$ there exist the limits* (1.43). *The maps $\mathcal{I} \ni \lambda \mapsto F^{\pm}(\lambda)\psi \in \mathcal{G}$ are continuous. Moreover the identities*

$$\|F^{\pm}(\lambda)\psi\|^2 = 2\pi\langle\psi, \delta(H-\lambda)\psi\rangle; \quad \delta(H-\lambda) := \pi^{-1}\operatorname{Im} R(\lambda + \mathrm{i}0), \tag{1.44}$$

*hold.*

Due to (1.44) the operators $F^{\pm}(\lambda)$ extend as continuous operators $B \to \mathcal{G}$, and for any $\psi \in B$ the maps $F^{\pm}(\,\cdot\,)\psi \in \mathcal{G}$ are continuous. In Proposition 1.15 stated below we give a formula for these extensions.

Introduce

$$\mathcal{H}_{\mathcal{I}} = P_H(\mathcal{I})\mathcal{H}, \quad \widetilde{\mathcal{H}}_{\mathcal{I}} = L^2(\mathcal{I}, (2\pi)^{-1}\mathrm{d}\lambda; \mathcal{G}),$$

set $H_{\mathcal{I}} = HP_H(\mathcal{I})$ and let $M_\lambda$ be the operator of multiplication by $\lambda$ on $\widetilde{\mathcal{H}}_{\mathcal{I}}$. We define

$$F^{\pm} = \int_{\mathcal{I}} \oplus F^{\pm}(\lambda)\,\mathrm{d}\lambda \colon B \to C(\mathcal{I}; \mathcal{G}).$$

These operators can be extended to proper spaces which is stated as the first part of the following result.

**Proposition 1.15.** *Suppose Condition 1.13. The operators $F^{\pm}$ considered as maps $B \cap \mathcal{H}_{\mathcal{I}} \to \widetilde{\mathcal{H}}_{\mathcal{I}}$ extend uniquely to isometries $\mathcal{H}_{\mathcal{I}} \to \widetilde{\mathcal{H}}_{\mathcal{I}}$. These extensions obey $F^{\pm}H_{\mathcal{I}} \subseteq M_\lambda F^{\pm}$. Moreover for any $\psi \in B$ the vectors $F^{\pm}(\lambda)\psi$ are given as averaged limits. More precisely introducing for any such $\psi$ the integral $⨍_R \xi(r)\,\mathrm{d}r := R^{-1}\int_R^{2R}\xi(r)\,\mathrm{d}r$, these vectors are given as*

$$\begin{aligned} F^{\pm}(\lambda)\psi &= \mathcal{G}\text{–}\lim_{R\to\infty} ⨍_R \xi(r)\,\mathrm{d}r \\ &= \mathcal{G}\text{–}\lim_{R\to\infty} ⨍_R \exp\Big(\int_{r_0}^{r}\Big(\mp\mathrm{i}\tilde{b} + \tfrac{1}{2}\operatorname{div}\tilde{\omega}\Big)(s,\,\cdot\,)\,\mathrm{d}s\Big)[\sqrt{b}R(\lambda \pm \mathrm{i}0)\psi](r,\,\cdot\,)\,\mathrm{d}r, \end{aligned} \tag{1.45}$$

*and the limits* (1.45) *are attained locally uniformly in $\lambda \in \mathcal{I}$.*

The above extended isometries $F^{\pm}\colon \mathcal{H}_{\mathcal{I}} \to \widetilde{\mathcal{H}}_{\mathcal{I}}$ are actually unitary under an additional condition, and for this reason we call them the Fourier transforms associated with $H_{\mathcal{I}}$. The new condition consists of two alternatives. The first one is a partial strengthening of Condition 1.13. The other one is primarily a set of bounds on higher order derivatives of various quantities defined on $M$.

**Condition 1.16.** In addition to Condition 1.13 one of the following properties holds:

(1)

$$\min\{\sigma, \tau, \rho\} > 2. \tag{1.46}$$

(2) The extension $\tilde b^{\mathrm{ex}}$ of Condition 1.13 is in $C^2$. The restriction $q_{1|S_r}$ belongs to $C^2(S_r)$ for $r\ge r_0$, and there exists $C>0$ such that

$$\bigl|\iota_R^*\nabla^3 r\bigr|\le CR^{-1-\tau/2}\text{ for }R\ge r_0,\tag{1.47a}$$

$$|\nabla'^2 q_{1|S_r}|\le Cr^{-1-\rho}\text{ for }r\ge r_0,\tag{1.47b}$$

and

$$\begin{gathered}\bigl|\nabla'^2|\mathrm{d}r|^2_{|S_r}\bigr|\le Cr^{-1-\tau},\quad \bigl|\nabla'^2(\nabla_\omega|\mathrm{d}r|^2)_{|S_r}\bigr|\le Cr^{-1-\tau},\\ \bigl|\nabla'^2(\Delta r)_{|S_r}\bigr|\le Cr^{-1-\tau}\text{ for }r\ge r_0,\end{gathered}\tag{1.47c}$$

where $\nabla'$ denotes the Levi–Civita connection associated with the induced Riemannian metric $\iota_r^*g$ on the $r$-sphere $S_r$.

We remark that for any $f\in C^\infty(M)$ the Hessian $\nabla'^2(f_{|S_R})=\iota_R^*t_f$ where $t_f=\nabla^2 f-(\tilde\omega^j\partial_j f)\nabla^2 r$. The bounds (1.47b) and (1.47c) allow us to estimate $\nabla'^2\bigl(\pm\mathrm{i}\tilde b-\tfrac12\operatorname{div}\tilde\omega\bigr)=O(r^{-1-\min\{\tau\,\rho\}})$ (to be used in our verification of (3.31)).

**Theorem 1.17.** *Suppose Condition 1.16. Then the operators $F^\pm\colon\mathcal H_{\mathcal I}\to\widetilde{\mathcal H}_{\mathcal I}$ are unitarily diagonalizing transforms for $H_{\mathcal I}$, that is, they are unitary and*

$$F^\pm H_{\mathcal I}=M_\lambda F^\pm,$$

*respectively.*

**Remark.** For the conclusion of Theorem 1.17 it suffices to assume Condition 1.13 and (3.31) of Lemma 3.10. In fact, Condition 1.16 is here and henceforth used only for the verification of (3.31).

1.3.2. *Scattering matrix and generalized eigenfunctions.* Next for any $\xi\in\mathcal G$ let us introduce purely outgoing/incoming approximate generalized eigenfunctions $\phi^\pm[\xi]\in B^*$ by, using the spherical coordinates,

$$\begin{aligned}\phi^\pm[\xi](r,\sigma)&=\eta_\lambda\bigl[2|\mathrm{d}r|^2(\lambda-q_1)\bigr]^{-1/4}\\&\quad\cdot\exp\biggl(\int_{r_0}^r\Bigl(\pm\mathrm{i}\tilde b-\tfrac12\operatorname{div}\tilde\omega\Bigr)(s,\sigma)\,\mathrm{d}s\biggr)\xi(\sigma),\end{aligned}\tag{1.48}$$

cf. Theorem 1.14. We remark that formulas like (1.48) in the context of Schrödinger operators are referred to as (zeroth order) WKB-approximations. If we denote the oscillatory part of the phase by $S(x)=\pm\int_{r_0}^{r(x)}\tilde b(s,\sigma(x))\,\mathrm{d}s$ then Condition 1.7 and (1.39) of Condition 1.13 imply that for $M^{\mathrm{ex}}=M$

$$\tfrac12|\mathrm{d}S(x)|^2+q_1-\lambda=O(r^{-1-\epsilon})\text{ for all }\epsilon<2\beta_c-1,$$

see Lemma 3.5. Whence in this case $S(\cdot)$ is an approximate solution to the eikonal equation with the effective potential $q_1$ and a short-range error. In the general case of Condition 1.13 the bound (1.40) is barely too weak to give a uniform short-range error, however due to Lemma 3.5 we still have pointwise short-range bounds (i.e. short-range bounds that are not uniform in $\sigma\in S$). Although such property is basic for the WKB-method (in particular for obtaining higher order expansions) it will only be used in a disguised form in this paper. We remark that under Condition 1.13 for any $\xi\in C_{\mathrm c}^\infty(S)\subseteq\mathcal G$ the vectors $\phi^\pm[\xi]\in\mathcal N$ (here possibly needed cutoff further at infinity), and under Condition 1.16 they are approximate generalized eigenfunctions in the sense $R(\mathrm i)(H-\lambda)\phi^\pm[\xi]\in B\cap\mathcal H^1$ which is a consequence of (3.31) (cf. the proof of Lemma 3.11).

**Example 1.18.** Consider a subset $M \subseteq \mathbb{R}^2$ equipped with the Euclidean metric and given with an end bounded by the "interior" of a parabola, say $x^2 < y$, and $r^2 := x^2/2 + y^2 > r_0^2$. We consider only $V = 0$. The orbits of $\omega = \operatorname{grad} r$ are the branches of parabolas $cy^{1/2} = x$ where $-1 < c < 1$, and Condition 1.7 is fulfilled with $\sigma = \tau = 1$ and $\rho = 2$ (and similarly for Condition 1.16 (2)), in particular $\beta_c = 1/2$ is fulfilled. However the barely stronger condition (1.39) is not fulfilled and whence the example is not covered by the theory of this paper (in contrast to [IS2]). Moreover we can in fact show that the generalized eigenfunctions in $\mathcal{N} \cap B^*$ are *not* of WKB-type as in the theorem stated below, see Subsection 3.5. Let us here note, as an indication of this result, that for any $0 \neq \xi \in C_c^\infty(S) \subseteq \mathcal{G}$

$$(H - \lambda)\phi^\pm[\xi] \in \mathcal{H}_{1/2-} \setminus B,$$

which technically prohibits us to construct WKB-solutions.

For some examples for which our theory applies we refer the reader to [IS2, Subsection 1.2].

Under Condition 1.16 and for any $\lambda \in \mathcal{I}$ the *scattering matrix* $S(\lambda)\colon \mathcal{G} \to \mathcal{G}$ is defined by the identity

$$F^+(\lambda)\psi = S(\lambda)F^-(\lambda)\psi; \quad \psi \in B. \tag{1.49}$$

It follows from (3.35b) that $C_c^\infty(S) \subseteq \operatorname{Ran} F^\pm(\lambda)$, and hence, with Theorem 1.14, Proposition 1.15 and a density argument, $S(\cdot)$ is a well-defined strongly continuous unitary operator. We obtain a characterization of the generalized eigenfunctions in $\mathcal{N} \cap B^*$, i.e. the elements of

$$\mathcal{E}_\lambda := \{\phi \in \mathcal{N} \cap B^* \mid (H - \lambda)\phi = 0\}.$$

Due to Theorem 1.4 these eigenfunctions may be called *minimal*.

**Theorem 1.19.** *Suppose Condition 1.16. Then for any $\lambda \in \mathcal{I}$ the following assertions hold.*

*(i) For any one of $\xi_\pm \in \mathcal{G}$ or $\phi \in \mathcal{E}_\lambda$ the two other quantities in $\{\xi_-, \xi_+, \phi\}$ uniquely exist such that*

$$\phi - \phi^+[\xi_+] + \phi^-[\xi_-] \in B_0^*. \tag{1.50a}$$

*(ii) The correspondences in* (1.50a) *are given by the formulas (recall* (1.42)*)*

$$\phi = \mathrm{i}F^\pm(\lambda)^*\xi_\pm, \quad \xi_+ = S(\lambda)\xi_-, \tag{1.50b}$$

$$\xi_\pm = 2^{-1}\,\mathcal{G}\text{-}\lim_{R\to\infty} \fint_R \mathrm{e}^{\mathrm{i}(r-r_0)(\tilde{A}^{\mathrm{ex}} \mp \tilde{b}^{\mathrm{ex}})}\big[b^{-1/2}(A \pm b)\phi\big]_{|S_r}\,\mathrm{d}r. \tag{1.50c}$$

*In particular the wave matrices $F^\pm(\lambda)^*\colon \mathcal{G} \to \mathcal{E}_\lambda$ are linear isomorphisms.*

*(iii) The wave matrices $F^\pm(\lambda)^*\colon \mathcal{G} \to \mathcal{E}_\lambda\ (\subseteq B^*)$ are bi-continuous. In fact*

$$2\|\xi_\pm\|_{\mathcal{G}}^2 = \lim_{R\to\infty} R^{-1} \int_{B_{2R}\setminus B_R} |b^{1/2}\phi|^2\,(\det g)^{1/2}\mathrm{d}x. \tag{1.50d}$$

*(iv) The operators $F^\pm(\lambda)\colon B \to \mathcal{G}$ and $\delta(H - \lambda)\colon B \to \mathcal{E}_\lambda$ are onto.*

We remark that parts of this theorem overlap with [ACH, AH, Co, GY, Me, Va].

Finally we give an application of our results to channel scattering theory addressed, but treated very differently, in [HPW]. Suppose $M^{\mathrm{ex}}$ has $N \geq 2$ number of ends, i.e. $E^{\mathrm{ex}} = \{x \in M^{\mathrm{ex}} \,|\, r^{\mathrm{ex}}(x) > r_0\}$ has $N \geq 2$ components $E_i$, $i = 1, \dots, N$. Then the Hilbert space $\mathcal{G}$ splits as

$$\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_N; \quad \mathcal{G}_i = L^2(S_i),\ S_i = S \cap \overline{E_i},$$

and, accordingly, the scattering matrix $S(\lambda)$ has a matrix representation

$$S(\lambda) = (S_{ij}(\lambda))_{1 \leq i,j \leq N}, \quad S_{ij}(\lambda) \in \mathcal{B}(\mathcal{G}_j, \mathcal{G}_i).$$

**Corollary 1.20.** *Suppose under Condition 1.16 that $E^{\mathrm{ex}}$ has $N$ number of ends. Decomposing as above for any $\lambda \in \mathcal{I}$ the scattering matrix $S(\lambda)$ into components the off-diagonal ones, $S_{ij}(\lambda)$ with $i \neq j$, are one-to-one mappings.*

*Proof.* If $\xi_- = (\xi_-^1, \dots, \xi_-^N) \in \mathcal{G}$ is given with $\xi_-^j = 0$ for $j \neq 2$ and $\xi_+^1 = 0$ then $\phi = \mathrm{i}F^+(\lambda)^*\xi_-$ obeys that $1_{E_1 \cap M}\phi \in B_0^*$. By using a suitable cutoff of the function $r$ (essentially defined by making it vanish in $E_j \cap M$ for $j \geq 2$) we then obtain from Theorem 1.4 that $\phi = 0$. For example we could redefine $r$ and $r_0$ of Condition 1.1 as follows (using the notation (1.1)): First replace $r$ by the function $r1_{E_1 \cap M}\big(1 - \chi(r/r_0)\big)$ and then replace the parameter $r_0$ by $4r_0$. With these modifications Conditions 1.1–1.3 are fulfilled (with the other parameters there unchanged), and therefore indeed Theorem 1.4 applies. In particular we deduce that $\xi_-^2 = 0$, showing that $\ker S_{12}(\lambda) = \{0\}$. We can argue in the same way for all other off-diagonal components of the scattering matrix. □

We note that Corollary 1.20 may be seen as a stationary solution to conjectures of [HPW], see [HPW, Remark 5.7]. We shall develop the time-dependent version of our results in [IS3]. In particular this includes a time-dependent version of Corollary 1.20 directly proving conjectures of [HPW] in a strong form.

## 2. Preliminaries

2.1. **Elementary tensor analysis.** Here we fix our convention for the covariant derivatives. We formulate and use them always in local expressions, but for a coordinate-independent representation, see [Ch, p. 34].

2.1.1. *Derivatives of functions.* We shall denote two tensors by the same symbol if they are related to each other through the canonical identification $TM \cong T^*M$, and distinguish them by super- and subscripts. We denote $TM \cong T^*M$ by $T$ for short, and set $T^p = T^{\otimes p}$. The covariant derivative $\nabla$ acts as a linear operator $\Gamma(T^p) \to \Gamma(T^{p+1})$ and is defined for $t \in \Gamma(T^p)$ by

$$(\nabla t)_{ji_1 \cdots i_p} = \nabla_j t_{i_1 \cdots i_p} = \partial_j t_{i_1 \cdots i_p} - \sum_{s=1}^{p} \Gamma^k_{ji_s} t_{i_1 \cdots k \cdots i_p}. \tag{2.1}$$

Here $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$ is the Christoffel symbol and $t$ is considered as a section of the $p$-fold cotangent bundle, and we adopt the convention that a new subscript is always added to the left as in (2.1). By the identification $TM \cong T^*M$ it suffices to discuss an expression only for the subscripts. In fact, we have the compatibility condition

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} = 0, \tag{2.2}$$

and then by (2.1) and (2.2) the covariant derivative can be computed for the tensors of any type. For example, for $t \in \Gamma(T) = \Gamma(T^1)$

$$\begin{aligned}(\nabla t)_j{}^i &= g^{ik}(\nabla t)_{jk} = g^{ik}\big(\partial_j t_k - \Gamma^l_{jk} t_l\big)\\ &= g^{ik}\big(\partial_j g_{kl} t^l - \Gamma^l_{jk} g_{lm} t^m\big) = \partial_j t^i + \Gamma^i_{jk} t^k,\end{aligned} \tag{2.3}$$

and this extends to the general case with ease. The covariant derivative acts as a derivation with respect to tensor product, i.e. for $t \in \Gamma(T^p)$ and $u \in \Gamma(T^q)$

$$(\nabla(t\otimes u))_{ji_1\cdots i_{p+q}} = (\nabla t)_{ji_1\cdots i_p} u_{i_{p+1}\cdots i_{p+q}} + t_{i_1\cdots i_p}(\nabla u)_{ji_{p+1}\cdots i_{p+q}}. \tag{2.4}$$

The formal adjoint $\nabla^*\colon \Gamma(T^{p+1}) \to \Gamma(T^p)$ is defined to satisfy

$$\int \overline{u_{ji_1\cdots i_p}}(\nabla t)^{ji_1\cdots i_p}(\det g)^{1/2}\,\mathrm{d}x = \int \overline{(\nabla^* u)_{i_1\cdots i_p}} t^{i_1\cdots i_p}(\det g)^{1/2}\,\mathrm{d}x$$

for $u \in \Gamma(T^{p+1})$ and $t \in \Gamma(T^p)$ compactly supported in a coordinate neighbourhood. Actually we can write it in a divergence form: For $u \in \Gamma(T^{p+1})$

$$(\nabla^* u)_{i_1\cdots i_p} = -(\operatorname{div} u)_{i_1\cdots i_p} = -(\nabla u)_j{}^j{}_{i_1\cdots i_p} = -g^{jk}(\nabla u)_{jki_1\cdots i_p}.$$

Finally let us give several remarks. It is clear that for any function $f \in \Gamma(T^0) = C^\infty(M)$ the second covariant derivative $\nabla^2 f = \nabla\nabla f$ is symmetric, i.e.

$$(\nabla^2 f)_{ij} = (\nabla^2 f)_{ji} = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f, \tag{2.5}$$

and we have expressions for the Laplace–Beltrami operator $\Delta$:

$$\Delta f = (\nabla^2 f)_i{}^i = g^{ij}(\nabla^2 f)_{ij} = \operatorname{tr}\nabla^2 f = \operatorname{div}\nabla f.$$

We note that covariant differentiation and contraction are commuting operations. Whence we have, for example, for $t \in \Gamma(T)$ and $u \in \Gamma(T^{p+1})$

$$\begin{aligned}\nabla_k t^j u_{ji_1\cdots i_p} &= (\nabla t)_k{}^j u_{ji_1\cdots i_p} + t^j(\nabla u)_{kji_1\cdots i_p},\\ \nabla_j(\nabla t)_i{}^i &= (\nabla^2 t)_{ji}{}^i = g^{ik}(\nabla^2 t)_{jik}.\end{aligned} \tag{2.6}$$

2.1.2. *Derivatives of mappings.* Next let us present a short description of the derivatives of a mapping (not of a function). Let $y\colon M \to N$ be a general mapping from a Riemannian manifold $(M, g)$ to another $(N, h)$. In geometric literatures the $k$-th derivatives $\nabla^k y$, $k = 1, 2, \ldots$, are defined to satisfy the "chain rule". For instance, the derivatives $\nabla y$ and $\nabla^2 y$ are required to satisfy in local coordinates that for any function $f \in C^\infty(N)$

$$\begin{aligned}[\nabla(f(y))]_i &= (\nabla y)^\alpha{}_i(\nabla f)_\alpha(y),\\ [\nabla^2(f(y))]_{ij} &= (\nabla^2 y)^\alpha{}_{ij}(\nabla f)_\alpha(y) + (\nabla y)^\alpha{}_i(\nabla y)^\beta{}_j(\nabla^2 f)_{\alpha\beta}(y).\end{aligned}$$

Here we used the Roman and the Greek alphabets to denote the indices of coordinates $x \in M$ and $y = y(x) \in N$, respectively. Although we are not going to verify this, the above definition is indeed well-justified, and we have the following local expressions for such derivatives:

$$(\nabla y)^\alpha{}_i = \partial_i y^\alpha, \quad (\nabla^2 y)^\alpha{}_{ij} = \partial_i\partial_j y^\alpha - \Gamma^k_{ij}\partial_k y^\alpha + \Gamma^\alpha_{\beta\gamma}(\partial_i y^\beta)(\partial_j y^\gamma). \tag{2.7}$$

Note that we adopted the same convention on the Roman and Greek indices as above: In particular, $\Gamma^k_{ij}$ and $\Gamma^\alpha_{\beta\gamma}$ denote the Christoffel symbols for $(M, g)$ and $(N, g)$, respectively.

2.2. **Decomposition of Hamiltonian.** Throughout the remaining part of the paper we extensively use the notation

$$\kappa = \min\{1 + \tau/2, 1 + \rho/2, \rho\}$$

and $\tilde{\eta}$ of (1.2). Let us recall two results, [IS2, (1.9)] and [IS2, Lemma 5.1], respectively. (Recall for Lemma 2.2 that $a$ has two values for $z \in I$, say $a = a_\pm$.)

**Lemma 2.1.** *Suppose Conditions 1.1–1.3. Then, as quadratic forms on $\mathcal{H}^1$,*

$$H = \tfrac{1}{2} A\tilde{\eta}A + \tfrac{1}{2}L + q_1 + q_4; \quad q_4 = q_2 + \tfrac{1}{4}(\nabla^r\tilde{\eta})(\Delta r).$$

**Lemma 2.2.** *Let $I \subseteq \mathcal{I}$ be a compact interval. There exist $C > 0$ such that uniformly in $z \in I \cup I_+$ or $z \in I \cup I_-$*

$$|a| \le C, \quad \big|\pm p^r a + a^2 - 2|\mathrm{d}r|^2(z - q_1)\big| + \big|\ell^{\bullet i}\nabla_i a\big| \le Cr^{-\kappa}.$$

We may consider Lemma 2.1 as a decomposition of $H$ into a sum of radial and spherical components (see the discussion at the end of the section). In the next section we shall use similar decompositions:

**Lemma 2.3.** *Let $I \subseteq \mathcal{I}$ be a compact interval. Then as a quadratic form on $\bar{\chi}_n\mathcal{H}^1 \subseteq \mathcal{H}^1$ for any large $n$ and uniformly in $z = \lambda \pm \mathrm{i}\Gamma \in I \cup I_\pm$*

$$H - z = \tfrac{1}{2}(A \pm a)\tilde{\eta}(A \mp a) + \tfrac{1}{2}L + O(r^{-\kappa}), \tag{2.8a}$$

$$H - z = \tfrac{1}{2}b^{1/2}(\tilde{A} \pm \tilde{b})b^{-1/2}(A \mp a) + \tfrac{1}{2}L + O(r^{-\kappa})(A \mp a) + O(r^{-\kappa}), \tag{2.8b}$$

$$H - z = \tfrac{1}{2}\tfrac{a}{\sqrt{b}}(\tilde{A} \pm \tilde{b})\tfrac{\sqrt{b}}{a}(A \mp a) + \tfrac{1}{2}L + O(r^{-\kappa})(A \mp a) + O(r^{-\kappa}). \tag{2.8c}$$

*Proof.* Using Lemma 2.1 we can write

$$\begin{aligned} H - z = {}& \tfrac{1}{2}(A \pm a)\tilde{\eta}(A \mp a) \pm \tfrac{1}{2}(p^r\tilde{\eta}a) + \tfrac{1}{2}\tilde{\eta}a^2 \\ & + \tfrac{1}{2}L + q_1 + q_2 + \tfrac{1}{4}(\nabla^r\tilde{\eta})(\Delta r) - z. \end{aligned}$$

Hence the first identity (2.8a) is obtained applying Lemma 2.2 to the remainder written

$$\begin{aligned} & \tfrac{1}{2}\tilde{\eta}\big[\pm(p^r a) + a^2 - 2|\mathrm{d}r|^2(z - q_1)\big] \\ & \quad - (1 - \eta)(z - q_1) + q_2 + \tfrac{1}{4}(\nabla^r\tilde{\eta})(\Delta r \mp 2\mathrm{i}a) = O(r^{-\kappa}). \end{aligned}$$

This is valid with or without the factor $\bar{\chi}_n$. However for (2.8b) and (2.8c) we need this factor to avoid dividing by zero. We use (2.8a) and the identities

$$A\tilde{\eta} = \tilde{A} - \tfrac{\mathrm{i}}{2}(\nabla^r\tilde{\eta}), \tag{2.9a}$$

$$(A \pm a)\tilde{\eta} = b^{1/2}(\tilde{A} \pm \tilde{b})b^{-1/2} + O(r^{-\kappa}), \tag{2.9b}$$

$$(A \pm a)\tilde{\eta} = ab^{-1/2}(\tilde{A} \pm \tilde{b})a^{-1}b^{1/2} + O(r^{-\kappa}). \tag{2.9c}$$

□

The identities

$$(A \mp b)b^{1/2} = b^{1/2}(A \mp b - \tfrac{\mathrm{i}}{2}\nabla^r \ln b) = b^{1/2}(A \mp a + O(r^{-\kappa})) \tag{2.10}$$

would provide more symmetric versions (2.8b) and (2.8c), however these are not useful under our conditions.

We note the natural identification in spherical coordinates, cf. (1.32),

$$L^2(E) \cong L^2([r_0,\infty)_r;\mathcal{G}_r),\quad \langle\check\phi,\phi\rangle_{L^2(E)} = \int_{r_0}^\infty \langle\check\phi,\phi\rangle_{\mathcal{G}_r}\,\mathrm{d}r.$$

Recalling (1.4), i.e.

$$L = p_i^*\ell^{ij}p_j,\quad \ell = g - \tilde\eta\,\mathrm{d}r\otimes\mathrm{d}r \in \Gamma(T^2),$$

we can write correspondingly (in the form sense)

$$L \cong \int_{r_0}^\infty \oplus L_r\,\mathrm{d}r.$$

Note here the orthogonal splittings $g = \operatorname{diag}(|\mathrm{d}r|^{-2}, g')$ and $\ell = \operatorname{diag}(0, g')$, where $g' = g_r$ is the induced metric on $S_r \subseteq M$. Indeed explicitly

$$\langle\check\phi, L_r\phi\rangle_{\mathcal{G}_r} = \int_{S_r} \overline{(p_i\check\phi)}g_r^{ij}(p_j\phi)\,\mathrm{d}\tilde{\mathcal{A}}_r = \langle p'\check\phi, p'\phi\rangle_{\mathcal{G}_r},$$

where $\mathrm{i}p'$ is to the covariant derivative on $S_r$. With Condition 1.12 we can at this point use local coordinates of $S$ to define and do the integral, in any case clearly the radial derivative $\partial_r$ does not enter.

We may consider $L_r$ as an operator, more precisely as the operator defined by the Friedrichs extension from $C_\mathrm{c}^\infty(S_r) \subseteq \mathcal{G}_r$ of the expression

$$L_r = -|\mathrm{d}r|(\det g_r)^{-1/2}\partial_i|\mathrm{d}r|^{-1}(\det g_r)^{1/2}g_r^{ij}\partial_j = -\Delta_r + \tfrac12(\partial_i \ln|\mathrm{d}r|^2)\ell^{ij}\partial_j, \tag{2.11}$$

where

$$\Delta_r = \Delta_r^{\mathrm{LB}} = (\det g_r)^{-1/2}\partial_i(\det g_r)^{1/2}g_r^{ij}\partial_j$$

denotes the Laplace–Beltrami operator on $S_r$.

By an approximation argument it follows that for any $\phi \in \mathcal{H}^1$ the restriction $\phi_{|S_r} \in \mathcal{D}(L_r^{1/2}) = \mathcal{D}(p')$ for almost all $r \geq r_0$, in fact for all $r \geq r_0$

$$\int_{r_0}^r \|p'\phi\|_{\mathcal{G}_s}^2\,\mathrm{d}s = \int_{B_r\setminus B_{r_0}} \overline{(p_i\phi)}\ell^{ij}(p_j\phi)(\det g)^{1/2}\,\mathrm{d}x \leq \|\phi\|_{\mathcal{H}^1}^2.$$

## 3. Distorted Fourier transform and stationary scattering theory

In this section we impose Condition 1.13. Recall from Section 1.2 that $\mathrm{e}^{\mathrm{i}t\tilde A}$, $t \leq 0$, naturally induces an isometry

$$\mathrm{e}^{\mathrm{i}t\tilde A}\colon \mathcal{G}_{r+t} \to \mathcal{G}_r;\quad \mathcal{G}_r = L^2(S_r, \mathrm{d}\tilde{\mathcal{A}}_r),\ r+t \geq r_0.$$

For $t \geq 0$ this operator is in general only a partial isometry. Let us modify the exponent and consider the semigroups $\mathrm{e}^{\mathrm{i}t(\tilde A\mp\tilde b)}$ generated by the operators $\tilde A \mp \tilde b$, respectively. For any $\lambda > \lambda_0$ we have, cf. (1.19a),

$$\tilde b = \tilde b_\lambda = \eta_\lambda|\mathrm{d}r|^{-1}\sqrt{2(\lambda - q_1)} = \tilde\eta b_\lambda. \tag{3.1}$$

With the expressions (1.37) it is easy to verify that

$$(\mathrm{e}^{\mathrm{i}t(\tilde A\mp\tilde b)}\phi)(x) = \exp\left(\mp\mathrm{i}\int_0^t \tilde b(\tilde y(s,x))\,\mathrm{d}s\right)(\mathrm{e}^{\mathrm{i}t\tilde A}\phi)(x). \tag{3.2}$$

A similar formula holds for $(\mathrm{e}^{\mathrm{i}t(\tilde A^{\mathrm{ex}}\mp\tilde b^{\mathrm{ex}})}$. It follows from (3.2) that the operators $\mathrm{e}^{\mathrm{i}(r-r')(\tilde A\mp\tilde b)}$, $r'\geq r\geq r_0$, induce isometries $\mathcal G_r\to\mathcal G_{r'}$.

$$\xi(r):=\exp\biggl(\int_{r_0}^r\Bigl(\mp\mathrm{i}\tilde b^{\mathrm{ex}}+\tfrac12\operatorname{div}\tilde\omega^{\mathrm{ex}}\Bigr)(s,\,\cdot\,)\,\mathrm{d}s\biggr)[\sqrt{b}R(\lambda\pm\mathrm{i}0)\psi](r,\cdot)\in\mathcal G,\tag{3.3}$$

and that the distorted Fourier transform is "given" by

$$F^{\pm}(\lambda)\psi=\mathcal G\text{–}\lim_{r\to\infty}\xi(r).\tag{3.4}$$

The first problem of justifying this is to show that for each fixed $r$ indeed $\xi(r)\in\mathcal G$. This is in fact doable for $\psi\in B$. More generally under Condition 1.13 the following four results are valid with $\phi=R(\lambda\pm\mathrm{i}0)\psi$ for any $\lambda>\lambda_0$ and $\psi\in B$.

**Lemma 3.1.** *For all $\psi\in B$ and $r\geq r_0$ the quantity $\xi(r)\in\mathcal G$.*

*Proof.* Introduce $\xi\in C_{\mathrm c}^\infty(S_r)$, $0\leq\xi\leq1$ and look at the push-forward given by $\xi_{r'}=\xi(\tilde y(r-r',\,\cdot\,))\in C_{\mathrm c}^\infty(S_{r'})$, $r'\geq r\geq r_0$.

Note that the $\mathcal G_r$-valued function

$$u(r'):=\mathrm{e}^{\mathrm{i}(r'-r)\tilde A}\bigl\{\xi_{r'}[\sqrt b\phi]_{|S_{r'}}\bigr\};\ r'\in[r,\infty),$$

is a well-defined absolutely continuous function. In particular by the fundamental theorem of calculus

$$u(r)=\int_r^{r+1}u(s)\mathrm{d}s-\int_r^{r+1}\int_r^s\tfrac{\mathrm d}{\mathrm{d}r'}u(r')\mathrm{d}r'\mathrm{d}s,$$

yielding upon computing the derivative

$$\tfrac{\mathrm d}{\mathrm{d}r'}u(r')=\mathrm{e}^{\mathrm{i}(r'-r)\tilde A}\bigl\{\xi_{r'}[\mathrm{i}\tilde A\sqrt b\phi]_{|S_{r'}}\bigr\},$$

taking the norm inside and using the Cauchy-Schwarz inequality the bound

$$\|\xi[\sqrt b\phi]_{|S_r}\|_{\mathcal G_r}\leq\|1_{B_{r+1}}\sqrt b\phi\|+3^{-1/2}\|1_{B_{r+1}}\tilde A\sqrt b\phi\|.\tag{3.5}$$

By taking $\xi\nearrow1$ we obtain a concrete bound of the trace $[\sqrt b\phi]_{|S_r}\in\mathcal G_r$. □

In the above proof we only used the property that $\check\phi:=\sqrt b\phi\in\mathcal N$ which follows from the fact that $\phi\in\mathcal N$. The latter property suffices for for the next result too. Note that for any such $\check\phi$ and $r\geq r_0$ we may for any $R_\nu>r+1$ approximate $\chi_\nu\check\phi\in\mathcal H^1$ by a sequence $(\check\phi_n)\subseteq C_{\mathrm c}^\infty(M)\subseteq\mathcal H^1$. Then it follows from (3.5) that $[\check\phi_n]_{|S_r}\to\check\phi_{|S_r}$ in $\mathcal G_r$ for $n\to\infty$.

**Lemma 3.2.** *The quantity $\xi(\cdot)\in\mathcal G$ (possibly considered for an arbitrary $\phi\in\mathcal N$) is an absolutely continuous $\mathcal G$-valued function on $[r_0,\infty)$.*

*Proof.* We fix any $r_1>r_0$ and write for $r\in[r_0,r_1]$

$$\xi(r)=\mathrm{e}^{\mathrm{i}(r-r_0)(\tilde A^{\mathrm{ex}}\mp\tilde b^{\mathrm{ex}})}\bigl[\sqrt bR(\lambda\pm\mathrm{i}0)\psi\bigr]_{|S_r}=\mathrm{e}^{\mathrm{i}(r_1-r_0)(\tilde A^{\mathrm{ex}}\mp\tilde b^{\mathrm{ex}})}\mathrm{e}^{\mathrm{i}(r-r_1)(\tilde A\mp\tilde b)}\bigl[\sqrt b\phi\bigr]_{|S_r}.$$

It suffices to show that the $\mathcal G_{r_1}$-valued function

$$[r_0,r_1]\ni r\to v(r)=\mathrm{e}^{\mathrm{i}(r-r_1)(\tilde A\mp\tilde b)}\bigl[\sqrt b\phi\bigr]_{|S_r}$$

is absolutely continuous. Formally

$$v'(r)=\mathrm{e}^{\mathrm{i}(r-r_1)(\tilde A\mp\tilde b)}\bigl[\mathrm{i}(\tilde A\mp\tilde b)\sqrt b\phi\bigr]_{|S_r},\tag{3.6}$$

i.e. $v(r) = v(r_1) - \int_r^{r_1} v'(s)\,\mathrm{d}s$ with $v'(s)$ given by this formula. If we replace $\sqrt{b}\phi =: \check{\phi} \in \mathcal{N}$ by an approximating sequence $(\check{\phi}_n) \subseteq C_c^\infty(M)$ as in the remark preceding the lemma (used with $r = r_1$) indeed (3.6) holds true for all $n$. Therefore (3.6) also holds in the limit $n \to \infty$. □

The above proof gives the following formula for the derivative (omitting "ex"):

$$\xi'(r) = \tfrac{\mathrm{d}}{\mathrm{d}r}\xi(r) = \exp\Big(\int_{r_0}^{r} \Big( \mp \mathrm{i}\tilde{b} + \tfrac{1}{2}\operatorname{div}\tilde{\omega}\Big)(s,\,\cdot\,)\,\mathrm{d}s\Big)[\mathrm{i}(\tilde{A} \mp \tilde{b})\sqrt{b}\phi](r,\cdot) \in \mathcal{G}. \quad (3.7)$$

**Lemma 3.3.** *The following limit exists and is given as*

$$\lim_{R\to\infty} \fint_R \|[\sqrt{b}\phi]_{|S_r}\|^2_{\mathcal{G}_r}\,\mathrm{d}r = \pm 2\operatorname{Im}\langle\psi,\phi\rangle. \quad (3.8)$$

*Proof.* Consider for convenience only the upper sign.

$$\int_R^{2R} \|[\sqrt{b}\phi]_{|S_r}\|^2_{\mathcal{G}_r}\,\mathrm{d}r = \int_R^{2R} \big(\operatorname{Re}\langle\phi,(b-A)\phi\rangle_{\mathcal{G}_r} + \operatorname{Im}\langle\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}\big)\,\mathrm{d}r.$$

By Corollaries 1.9 and 1.10

$$\fint_R \operatorname{Re}\langle\phi,(b-A)\phi\rangle_{\mathcal{G}_r}\,\mathrm{d}r \to 0.$$

Next we introduce for any $R \geq r_0$ a smooth approximation of the characteristic function of the ball $B_R$ of the form (employing (1.1))

$$\chi_{\epsilon,s}(r) = \chi((r-R-s)/\epsilon); \quad \epsilon > 0,\ s \in [0,R].$$

We compute a Green's identity

$$\begin{aligned}
&\int_R^{2R} \operatorname{Im}\langle\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}\,\mathrm{d}r\\
&= \lim_{\epsilon\to 0}\operatorname{Im}\int_{r_0}^{\infty}\Big(\chi(r-2R)/\epsilon) - \chi((r-R)/\epsilon\Big)\langle\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}\,\mathrm{d}r\\
&= -\lim_{\epsilon\to 0}\operatorname{Im}\int_{r_0}^{\infty}\Big(\int_0^R \chi'_{\epsilon,s}(r)\,\mathrm{d}s\Big)\langle\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}\,\mathrm{d}r\\
&= -\lim_{\epsilon\to 0}\operatorname{Im}\int_0^R\int_M (\partial_i\chi_{\epsilon,s})\bar{\phi}g^{ij}(\partial_j\phi)(\det g)^{1/2}\,\mathrm{d}x\mathrm{d}s \qquad (3.9)\\
&= -\lim_{\epsilon\to 0}\operatorname{Im}\int_0^R\int_M (\partial_i\chi_{\epsilon,s}\bar{\phi})g^{ij}(\partial_j\phi)(\det g)^{1/2}\,\mathrm{d}x\mathrm{d}s\\
&= -2\lim_{\epsilon\to 0}\int_0^R \operatorname{Im}\langle\chi_{\epsilon,s}\phi,(H-\lambda)\phi\rangle_{\mathcal{H}}\,\mathrm{d}s\\
&= -2\int_0^R \operatorname{Im}\langle 1_{B_{R+s}}\phi,\psi\rangle_{\mathcal{H}}\,\mathrm{d}s.
\end{aligned}$$

It follows that

$$\fint_R \operatorname{Im}\langle\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}\,\mathrm{d}r \to 2\operatorname{Im}\langle\psi,\phi\rangle.$$

□

We remark that a small modification of the computation (3.9) yields the more familiar Green's identity

$$\operatorname{Im}\langle\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}=-2\operatorname{Im}\langle 1_{B_r}\phi,\psi\rangle_{\mathcal{H}}\text{ for almost all }r\geq r_0,\tag{3.10a}$$

yielding in particular that $r\to\operatorname{Im}\langle\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}$ is absolutely continuous. A similar computation shows that in fact $r\to\langle\check\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}$ is absolutely continuous for any $\check\phi\in\mathcal{N}$ as it follows from the resulting Green's identity

$$\langle\check\phi,\nabla_\omega\phi\rangle_{\mathcal{G}_r}=\langle 1_{B_r}p_i\check\phi,g^{ij}p_j\phi\rangle+2\big\langle 1_{B_r}\check\phi,(V-\lambda)\phi-\psi\big\rangle\text{ for }r\geq r_0.\tag{3.10b}$$

However, in comparison, we do not know continuity or even local boundedness of the function $r\to\|\nabla_\omega\phi\|_{\mathcal{G}_r}$.

The following technical result will play a major role (see the proofs of Lemmas 3.7–3.9). Recall that the factor $\bar\chi_n$ of Lemma 2.3 was introduce to avoid zeros of $a$ and $b$. This is also the role of the factor $\bar\chi_n$ below.

**Lemma 3.4.** *Let $I\subseteq\mathcal{I}$ be a compact interval. Then we introduce for any large $n$ a function $f_{\check r}(r)$, $r\geq\check r$, depending on any $\check r\geq r_0$ as well as on any $\lambda\in I$ and $\check\phi\in\mathcal{N}$ as follows: Using spherical coordinates we define for $r\geq\check r$*

$$\begin{aligned}\check e&=\exp\biggl(\int_{\check r}^r\pm2\mathrm{i}\tilde b(s,\,\cdot\,)\,\mathrm{d}s\biggr),\\ D\xi(r)&=\exp\biggl(\int_{r_0}^r\Bigl(\mp\mathrm{i}\tilde b+\tfrac12\operatorname{div}\tilde\omega\Bigr)(s,\,\cdot\,)\,\mathrm{d}s\biggr)[\sqrt{b}\mathrm{i}(A\mp a)\phi](r,\cdot)\in\mathcal{G},\\ \check\xi(r)&=\exp\biggl(\int_{r_0}^r\Bigl(\mp\mathrm{i}\tilde b+\tfrac12\operatorname{div}\tilde\omega\Bigr)(s,\,\cdot\,)\,\mathrm{d}s\biggr)[\sqrt{b}\check\phi](r,\cdot)\in\mathcal{G},\\ f_{\check r}(r)&=\langle\check\xi(r),(\check e b^{-1}\bar\chi_n)(r,\cdot)D\xi(r)\rangle_{\mathcal{G}}.\end{aligned}$$

*Then the function $f_{\check r}(\cdot)$ is absolutely continuous on $[\check r,\infty)$ with derivative*

$$\begin{aligned}f'_{\check r}(r)&=T_1+\cdots+T_5;\\ T_1&=\langle(\tilde A\mp\tilde b)\sqrt{b}\check\phi,\check e b^{-1/2}\bar\chi_n(A\mp a)\phi\rangle_{\mathcal{G}_r},\\ T_2&=-2\langle\check\phi,\check e\bar\chi_n\psi\rangle_{\mathcal{G}_r},\\ T_3&=\langle p'\bar{\check e}\check\phi,\bar\chi_n p'\phi\rangle_{\mathcal{G}_r},\\ T_4&=\langle\check\phi,O(r^{-\kappa})(A\mp a)\phi\rangle_{\mathcal{G}_r},\\ T_5&=\langle\check\phi,O(r^{-\kappa})\phi\rangle_{\mathcal{G}_r},\end{aligned}\tag{3.11}$$

*where the bounds of $T_4$ and $T_5$ are uniform in $\lambda\in I$ and $\check r\geq r_0$.*

*Proof.* First we proceed using (2.8b) somewhat unjustified. Compute (formally)

$$f'_{\check r}(r)=\langle(\tilde A\mp\tilde b)\sqrt{b}\check\phi,\check e b^{-1/2}\bar\chi_n(A\mp a)\phi\rangle_{\mathcal{G}_r}-\langle\sqrt{b}\check\phi,(\tilde A\mp\tilde b)\check e b^{-1/2}\bar\chi_n(A\mp a)\phi\rangle_{\mathcal{G}_r},$$

and then substituting for the second term

$$\begin{aligned}\sqrt{b}(\tilde A\mp\tilde b)\check e b^{-1/2}\bar\chi_n(A\mp a)&=\check e\sqrt{b}(\tilde A\pm\tilde b)b^{-1/2}\bar\chi_n(A\mp a)\\&=2\check e\bar\chi_n\bigl(H-\lambda-\tfrac12L\bigr)+O(r^{-\kappa})(A\mp a)+O(r^{-\kappa}).\end{aligned}$$

This yields (3.11). Note that $T_3$ is well-defined since in fact $\bar{\breve{e}}\check{\phi} \in \mathcal{N}$ due to (1.40). By the product rule

$$p'\bar{\breve{e}}\check{\phi} = \mp\Big(p' \int_{\check{r}}^{r} 2\mathrm{i}\tilde{b}(s, \,\cdot\,)\,\mathrm{d}s\Big)\bar{\breve{e}}\check{\phi} + \bar{\breve{e}}p'\check{\phi}, \tag{3.12}$$

yielding that $T_3 \in L^1_{\mathrm{loc}}$ as a function of $r$. Similarly for the other terms showing explicitly that $f'_{\check{r}} \in L^1_{\mathrm{loc}}$. The required (pointwise) uniformity property for $T_4$ and $T_5$ is trivial since the $\check{r}$-dependence is through the oscillatory factor $\breve{e}$ only.

Next we give a rigorous derivation of (3.11) using (2.8b) differently (this argument will not be repeated for the derivation of similar formulas in the proof of Lemmas 3.8 and 3.9). We already argued that all of the above terms make sense and agree with the conclusion of the lemma. We claim that indeed $f_{\check{r}}$ is absolutely continuous. Note that due to (3.10b) and the fact that $\bar{\breve{e}}\check{\phi} \in \mathcal{N}$ we have the representation

$$\begin{aligned} f_{\check{r}}(r) =& \langle 1_{B_r} p_i \bar{\chi}_n \bar{\breve{e}}\check{\phi}, g^{ij} p_j \phi\rangle + 2\big\langle 1_{B_r}\bar{\chi}_n \bar{\breve{e}}\check{\phi}, (V-\lambda)\phi - \psi\big\rangle \\ &+ \big\langle \bar{\chi}_n \bar{\breve{e}}\check{\phi}, (\tfrac12 \Delta r \mp \mathrm{i}a)\phi\big\rangle_{\mathcal{G}_r}. \end{aligned} \tag{3.13}$$

Clearly the first and second terms are absolutely continuous, and by Lemma 3.2 the last one is too.

It is of course doable to compute $f'_{\check{r}}$ using (3.13). However the result is not immediately consistent with the representation from our informal computation. Instead we shall proceed as follows: It remains to show that for $r_1 > \check{r}$

$$f_{\check{r}}(r_1) = \int^{r_1} (T_1 + T_2 + T_3 + T_4 + T_5)\,\mathrm{d}r. \tag{3.14}$$

Let for $r_1 > \check{r}$

$$\chi_\epsilon(r) = \chi((r - r_1)/\epsilon); \quad \epsilon > 0.$$

We compute on one hand

$$\begin{aligned} &\langle (\tilde{A} \pm \tilde{b}) b^{1/2} \chi_\epsilon \bar{\chi}_n \bar{\breve{e}}\check{\phi}, b^{-1/2}(A \mp a)\phi\rangle \\ &\quad= \langle \chi'_\epsilon \bar{\chi}_n \bar{\breve{e}}\check{\phi}, \mathrm{i}(A \mp a)\phi\rangle + \langle \chi_\epsilon (\tilde{A} \pm \tilde{b}) b^{1/2} \bar{\chi}_n \bar{\breve{e}}\check{\phi}, b^{-1/2}(A \mp a)\phi\rangle \\ &\quad= \int \chi'_\epsilon(r) f_{\check{r}}(r)\,\mathrm{d}r + \langle \chi_\epsilon (\tilde{A} \pm \tilde{b}) b^{1/2} \bar{\chi}_n \bar{\breve{e}}\check{\phi}, b^{-1/2}(A \mp a)\phi\rangle, \end{aligned}$$

and on the other hand using (2.8b) (considering $L$ as a form)

$$\begin{aligned} &\langle (\tilde{A} \pm \tilde{b}) b^{1/2} \chi_\epsilon \bar{\chi}_n \bar{\breve{e}}\check{\phi}, b^{-1/2}(A \mp a)\phi\rangle \\ &\quad= \langle \chi_\epsilon \bar{\chi}_n \bar{\breve{e}}\check{\phi}, 2\psi - 2\big(\tfrac12 L + O(r^{-\kappa})(A \mp a) + O(r^{-\kappa})\big)\phi\rangle. \end{aligned}$$

Since $f_{\check{r}}$ is continuous at $r_1$ we obtain using that the right-hand sides are equal and by letting $\epsilon \to 0$ that

$$\begin{aligned} &- f_{\check{r}}(r_1) + \langle 1_{B_{r_1}} (\tilde{A} \pm \tilde{b}) b^{1/2} \bar{\chi}_n \bar{\breve{e}}\check{\phi}, b^{-1/2}(A \mp a)\phi\rangle \\ &= \langle 1_{B_{r_1}} \bar{\chi}_n \bar{\breve{e}}\check{\phi}, 2\psi - 2\big(\tfrac12 L + O(r^{-\kappa})(A \mp a) + O(r^{-\kappa})\big)\phi\rangle \\ &= 2\langle 1_{B_{r_1}} \check{\phi}, \breve{e}\bar{\chi}_n \psi\rangle - \langle 1_{B_{r_1}} p' \bar{\breve{e}}\check{\phi}, \bar{\chi}_n p'\phi\rangle \\ &- \langle 1_{B_{r_1}} \check{\phi}, O(r^{-\kappa})(A \mp a)\phi\rangle - \langle 1_{B_{r_1}} \check{\phi}, O(r^{-\kappa})\phi\rangle. \end{aligned}$$

Moreover

$$\begin{aligned}&\langle 1_{B_{r_1}}(\tilde A \pm \tilde b)b^{1/2}\bar\chi_n\bar{\check e}\check\phi, b^{-1/2}(A\mp a)\phi\rangle\\&=\langle 1_{B_{r_1}}(\tilde A \mp \tilde b)b^{1/2}\bar\chi_n\check\phi, \check e b^{-1/2}(A\mp a)\phi\rangle\\&=\langle 1_{B_{r_1}}(\tilde A \mp \tilde b)b^{1/2}\check\phi, \check e\bar\chi_n b^{-1/2}(A\mp a)\phi\rangle+\langle 1_{B_{r_1}}\check\phi, O(r^{-\kappa})(A\mp a)\phi\rangle.\end{aligned}$$

We conclude (3.14).

□

3.1. **Proof of Theorem 1.14 for easy case.** Suppose in addition to Condition 1.13 that Condition 1.16 (1) holds and consider only $\psi\in\mathcal H_{3/2+}$. We show that (3.4) exists. For convenience we consider only the upper sign. Note that the estimate of Corollary 1.10 holds for some $\beta>1$.

We compute, cf. (2.10) and (2.9a),

$$\begin{aligned}(\tilde A-\tilde b)b^{1/2}&=b^{1/2}(\tilde A-\tilde b-\tfrac{\mathrm i}{2}\tilde\omega^i\nabla_i\ln b)\\&=b^{1/2}(\tilde A-\tilde a+O(r^{-2}))\\&=b^{1/2}\tilde\eta(A-a+O(r^{-2})).\end{aligned}$$

Using then in turn (3.7) and the Cauchy-Schwarz inequality we obtain for $\beta$ slightly bigger than 1

$$\begin{aligned}\int_{r_0}^\infty\|\tfrac{\mathrm d}{\mathrm dr}\xi(r)\|_{\mathcal G}\,\mathrm dr&\le C_\beta\Big(\int_{r_0}^\infty r^{2\beta-1}\big\|[\sqrt b|\mathrm dr|^{-2}\big(A-a+O(r^{-2})\big)\phi]_{|S_r}\big\|^2_{\mathcal G_r}\,\mathrm dr\Big)^{1/2}\\&\le C_1\|\big(A-a\big)\phi\|_{\beta-1/2}+C_2\\&\le C_3<\infty.\end{aligned}$$

Whence the existence of (3.4) follows by integration.

The constant $C_3$ can be chosen locally uniform in $\lambda>\lambda_0$ and arbitrary small if we replace $\int_{r_0}^\infty$ by $\int_R^\infty$, $R>r_0$ big. Whence the limit (3.4) is attained locally uniformly in $(\lambda_0,\infty)$. In addition, since for finite $r$ the map $\lambda\to\xi(r)$ is continuous (cf. (3.5)), we obtain continuity of the map $(\lambda_0,\infty)\ni\lambda\to F^+(\lambda)\psi\in\mathcal G$.

Let us also note that due to Lemma 3.3

$$\|F^+(\lambda)\psi\|=2\operatorname{Im}\langle\psi,\phi\rangle$$

follows from the computation

$$\|F^+(\lambda)\psi\|^2_{\mathcal G}=\lim_{R\to\infty}R^{-1}\int_R^{2R}\|\xi(r)\|^2_{\mathcal G}\,\mathrm dr=\lim_{R\to\infty}⨍_R\|[\sqrt b\phi]_{|S_r}\|^2_{\mathcal G_r}\,\mathrm dr.$$

3.2. **Proof of Theorem 1.14 for general case.** In this subsection we prove the existence of the limit (3.4) for $\psi\in\mathcal H_{1+}$ under Condition 1.13 and then the remaining assertions of Theorem 1.14. We shall consider only the upper sign, since the lower sign can be dealt with in parallel. Throughout the subsection we fix any compact interval $I\subseteq\mathcal I$.

We first investigate properties of spherical derivatives. Let us introduce a "backwards hitting time" by

$$r^{\mathrm{bht}}(x)=\sup\big\{s\le r(x)-r_0\,\big|\,\tilde y(-s,x)\in M\big\};\ x\in E.$$

**Lemma 3.5.** *There exists $C_1 > 0$ such that for all $x \in E$ and $t \in (-r^{\mathrm{bht}}(x), 0]$*

$$\begin{aligned}\ell_*(t,x) &:= \Big(\ell^{ij}(x)[\partial_i \tilde{y}^\alpha(t,x)][\partial_j \tilde{y}^\beta(t,x)]\Big)_{\alpha,\beta} \\ &\le C_1 \big[(r(x)+t)/r(x)\big]^\sigma \ell(\tilde{y}(t,x))\end{aligned} \tag{3.15}$$

*as quadratic forms on the fibers of the cotangent bundle. In particular, for any given $\breve{\sigma} \le \sigma$ with $\breve{\sigma} < \min\{\tau, \rho\}$ there exists $C_2 > 0$ independent of $\lambda \in I$ such that for all $x \in E$*

$$\int_{-r^{\mathrm{bht}}(x)}^0 \big|p'\tilde{b}(\tilde{y}(s,x))\big|\,\mathrm{d}s \le C_2 r(x)^{-\breve{\sigma}/2}. \tag{3.16}$$

*Proof.* We remark that the tensor $\ell_*(t,x)$ is the push-forward of $\ell(x)$ under $\tilde{y}(t,\,\cdot\,)$. To prove the inequality of (3.15) we consider the trace

$$F(t) = g_{\alpha\beta}(\tilde{y}(t,x))\ell^{ij}(x)[\partial_i \tilde{y}^\alpha(t,x)][\partial_j \tilde{y}^\beta(t,x)].$$

Note that we use the Roman and the Greek indices to denote quantities concerning $x$ and $\tilde{y} = \tilde{y}(t,x)$, respectively. The derivative $F'(t)$ is computed as follows: Differentiating the expression

$$(g^*)_{ij}(t,x) := g_{\alpha\beta}(\tilde{y}(t,x))[\partial_i \tilde{y}^\alpha(t,x)][\partial_j \tilde{y}^\beta(t,x)]$$

and using the compatibility condition (2.2) and the flow equation, we compute

$$\begin{aligned}\tfrac{\partial}{\partial t}(g^*)_{ij} &= [\Gamma^\delta_{\gamma\alpha} g_{\delta\beta} + \Gamma^\delta_{\gamma\beta} g_{\alpha\delta}]\tilde{\omega}^\gamma(\partial_i \tilde{y}^\alpha)(\partial_j \tilde{y}^\beta) \\ &\quad + g_{\alpha\beta}(\partial_\gamma \tilde{\omega}^\alpha)(\partial_i \tilde{y}^\gamma)(\partial_j \tilde{y}^\beta) + g_{\alpha\beta}(\partial_\gamma \tilde{\omega}^\beta)(\partial_i \tilde{y}^\alpha)(\partial_j \tilde{y}^\gamma) \\ &= 2(\nabla\tilde{\omega})_{\alpha\beta}(\partial_i \tilde{y}^\alpha)(\partial_j \tilde{y}^\beta),\end{aligned} \tag{3.17}$$

which yields

$$F'(t) = 2\ell^{ij}(x)(\nabla\tilde{\omega}(\tilde{y}(t,x)))_{\alpha\beta}[\partial_i \tilde{y}^\alpha(t,x)][\partial_j \tilde{y}^\beta(t,x)].$$

Next we decompose $\nabla\tilde{\omega} = |\mathrm{d}r|^{-2}\nabla^2 r + (\mathrm{d}|\mathrm{d}r|^{-2}) \otimes \mathrm{d}r$ and substitute in the above formula. The second term does not contribute which follows easily using spherical coordinates. Using then (1.7b) we obtain

$$F'(t) \ge \sigma'(r(x)+t)^{-1}F(t),$$

so that for $t \in (-r^{\mathrm{bht}}(x), 0]$

$$F(t) \le (d-1)\big[(r(x)+t)/r(x)\big]^{\sigma'}.$$

Hence we obtain

$$\Big(\ell^{ij}(x)[\partial_i \tilde{y}^\alpha(t,x)][\partial_j \tilde{y}^\beta(t,x)]\Big)_{\alpha,\beta} \le (d-1)\big[(r(x)+t)/r(x)\big]^{\sigma'} g(\tilde{y}(t,x)).$$

If we write the last inequality in the spherical coordinates again, then there appears no radial component to the left, and hence we can remove the radial component from the right. Thus, using also that $\sigma < \sigma'$, the inequality (3.15) follows.

Now for (3.16) we first use the chain rule. Next by the Cauchy-Schwarz inequality, (1.27), Condition 1.7 and (3.15) with $\sigma$ replaced by $\breve\sigma$ we can estimate

$$\begin{aligned}&\int_{-r^{\mathrm{bht}}(x)}^0 \big|[\partial_\bullet \tilde y^\alpha(s,x)]\big(\partial_\alpha\tilde b\big)(\tilde y(s,x))\big|\,\mathrm{d}s\\ &\le C_4 r(x)^{-\breve\sigma/2}\int_{-r^{\mathrm{bht}}(x)}^0 (r(x)+s)^{-1-\min\{\tau,\rho\}/2+\breve\sigma/2}\,\mathrm{d}s\\ &\le C_5 r(x)^{-\breve\sigma/2},\end{aligned}$$

showing (3.16). □

As we already noted the tensor $\ell_*(t,x)$ is the push-forward of $\ell(x)$ under the map $\tilde y(t,\,\cdot\,)$. Actually Lemma 3.5 is applied only in the spherical coordinates, in which the inequality (3.15) takes a more simplified form, as in the following corollary. However, partly to avoid confusion concerning how we should compare two tensors with different base points, we formulate it using a more convenient geometric terminology.

**Corollary 3.6.** *Let $\breve\sigma\le\sigma$ and $\breve\sigma<\min\{\tau,\rho\}$. Then there exists a constant $C>0$ such that for all $\breve r\ge r_0$ and $u\in C^1_{\mathrm c}(S_{\breve r})$, the function $u(r)=\mathrm{e}^{\mathrm{i}(\breve r-r)\tilde A}u\in C^1_{\mathrm c}(S_r)$ for $r\ge\breve r$ and satisfies*

$$\|u(r)\|_{\mathcal G_r}=\|u\|_{\mathcal G_{r'}},\tag{3.18a}$$

$$\|p'u(r)\|_{\mathcal G_r}\le C\Big((\breve r/r)^{\sigma/2}\|p'u\|_{\mathcal G_{\breve r}}+r^{-\breve\sigma/2}\|u\|_{\mathcal G_{\breve r}}\Big).\tag{3.18b}$$

*More generally the bounds* (3.18a) *and* (3.18b) *are valid for $u\in\mathcal D(L^{1/2}_{\breve r})$ in which case $u(r)\in\mathcal D(L^{1/2}_r)$ for all $r\ge\breve r$.*

*Proof.* Only (3.18b) needs justification (note that the last assertion follows by a density argument). Using the expression in the spherical coordinates

$$u(r)(\sigma)=(\mathrm{e}^{\mathrm{i}(\breve r-r)\tilde A}u)(\sigma)=\exp\bigg(\int_{\breve r}^r -\tfrac12\operatorname{div}\tilde\omega(s,\sigma)\,\mathrm{d}s\bigg)u(\sigma),$$

we compute with $t=\breve r-r$ and $x\in E$ given with coordinates $(r,\sigma)$

$$\begin{aligned}p_iu(r)(\sigma)&=-\mathrm{i}\mathrm{e}^{(\int\cdots)}[\partial_i\tilde y^\alpha(t,x)]\big(\partial_\alpha u\big)(\breve r,\sigma)\\ &\quad+\bigg(\int_t^0\big[\partial_i\tilde y^\alpha(s,x)]\big(\tfrac{\mathrm i}{2}\partial_\alpha\operatorname{div}\tilde\omega\big)(r+s,\sigma)\,\mathrm{d}s\bigg)u(r)(\sigma).\end{aligned}\tag{3.19}$$

The first term is estimated using (3.15), and the second term is estimated as in the proof of (3.16). □

For the remaining part of this section let $\psi\in\mathcal H_{1+}$ and $\phi=R(\lambda+\mathrm{i}0)\psi$, $\lambda\in I\subseteq\mathcal I$.

**Lemma 3.7.** *The following limit exists and is given as*

$$\lim_{r\to\infty}\|\xi(r)\|^2_{\mathcal G}=2\operatorname{Im}\langle\psi,\phi\rangle_{\mathcal H}.\tag{3.20}$$

*Proof.* We use Lemma 3.4 taking there $\check\phi=\phi$. By evaluating at $r=\breve r$ and integrating the derivative on the interval $[\breve r,\infty)$ we then obtain that

$$\lim_{r\to\infty}f_r(r)=0.$$

At this point note that all of the terms $T_1,\dots,T_5$ are integrable due to the Cauchy-Schwarz inequality, Corollaries 1.9 and 1.10, (1.40) and (3.12) (note that if $M^{\mathrm{ex}}=M$ the bound (1.40) follows from (3.16) with $\breve\sigma=1$). Next by taking the imaginary part and using (3.10a) we obtain

$$0=\lim_{r\to\infty}\operatorname{Im}\langle\phi,\nabla_\omega\phi-\mathrm{i}b\phi\rangle_{\mathcal{G}_r}=2\operatorname{Im}\langle\psi,\phi\rangle_{\mathcal{H}}-\lim_{r\to\infty}\|\sqrt{b}\phi(r)\|^2_{\mathcal{G}_r}.$$

Whence indeed

$$\lim_{r\to\infty}\|\xi(r)\|^2_{\mathcal{G}}=\lim_{r\to\infty}\|\sqrt{b}\phi(r)\|^2_{\mathcal{G}_r}=2\operatorname{Im}\langle\psi,\phi\rangle_{\mathcal{H}}.$$

□

**Remark.** It follows from the above proof that the limit (3.20) is attained uniformly in $\lambda\in I$. This property will be used in the proof of Proposition 1.15.

Next decompose

$$\xi=\xi(r)=\exp\biggl(\int_{r_0}^r\Bigl(-\mathrm{i}\tilde b+\tfrac12\operatorname{div}\tilde\omega\Bigr)(s,\,\cdot\,)\,\mathrm{d}s\biggr)[\sqrt{b}\phi](r,\,\cdot\,)$$

as

$$\begin{aligned}\xi&=a^{-1}\xi_++a^{-1}\xi_-;\\ \xi_\pm&=2^{-1}\exp\biggl(\int_{r_0}^r\Bigl(-\mathrm{i}\tilde b+\tfrac12\operatorname{div}\tilde\omega\Bigr)(s,\,\cdot\,)\,\mathrm{d}s\biggr)[\sqrt{b}(a\pm A)\phi](r,\,\cdot\,).\end{aligned}\tag{3.21}$$

At this point the reader is *WARNED* about our use of notation: The quantities $a$ and $\phi$ are considered with the *upper sign only* in this subsection, so for the cases $\pm$ in (3.21) these quantities are the *same* (not to be mixed up with the convention of Lemmas 2.3 and 3.4).

**Lemma 3.8.** *There exists the weak limit*

$$F:=\mathrm{w}\text{–}\mathcal{G}\text{–}\lim_{r\to\infty}\xi(r).$$

*Proof.* Let $g\in C^\infty_{\mathrm c}(S)\subseteq\mathcal{G}$ be given. Due to Lemma 3.7 it suffices to show the existence of

$$C_\pm:=\lim_{r\to\infty}\langle g,a^{-1}(r)\xi_\pm(r)\rangle_{\mathcal{G}}.$$

*Step I. $C_-=0$.* Writing

$$g=\exp\biggl(\int_{r_0}^r\tfrac12\operatorname{div}\tilde\omega(s,\,\cdot\,)\,\mathrm{d}s\biggr)u(r)$$

we note that $u(r')\in C^\infty_{\mathrm c}(S_{r'})$ for $r'\geq R_n$ with $n$ large enough. We can write $u(r)=\mathrm{e}^{\mathrm{i}(r'-r)\tilde A}u(r')\in C^\infty_{\mathrm c}(S_r)$ for $r\geq r'$. Let $\bar\chi_n=1-\chi(r/R_n)$ so that $\bar\chi_nu\in\mathcal{N}$ (and such that all zeros of $a$ and $b$ are in $B_{R_n}$). Introduce then for $r\geq r_0$

$$\begin{aligned}\check\phi&=\exp\biggl(\int_{r_0}^r\mathrm{i}\tilde b(s,\,\cdot\,)\,\mathrm{d}s\biggr)b^{-1/2}\bar\chi_nu(r),\\ \check\xi&=\exp\biggl(\int_{r_0}^r\Bigl(-\mathrm{i}\tilde b+\tfrac12\operatorname{div}\tilde\omega\Bigr)(s,\,\cdot\,)\,\mathrm{d}s\biggr)[\sqrt{b}\check\phi](r,\cdot)\in\mathcal{G}.\end{aligned}$$

Note that we can consider $\check\phi$ as an element of $\mathcal{N}$ and that $\check\xi(r) = g$. Also note that $\xi_- = \mathrm{i}2^{-1}D\xi$ in terms of notation of Lemma 3.4. We introduce as in Lemma 3.4

$$\check e = \exp\Big(\int_{\check r}^{r} 2\mathrm{i}\tilde b(s,\ \cdot\ )\,\mathrm{d}s\Big);\ r \geq \check r \geq r_0.$$

By the proof of Lemma 3.4 with this choice of $\check\phi$ (and by using (2.8c) rather than (2.8b)) it follows by evaluating at $r = \check r \geq 2R_n$ and integrating the derivative

$$\begin{aligned}\tfrac{2}{\mathrm{i}}\tfrac{\mathrm{d}}{\mathrm{d}r}\langle\check\xi(r), \check e a^{-1}\xi_-(r)\rangle_{\mathcal{G}} &= \langle(\tilde A - \tilde b)\sqrt{b}\check\phi, \check e a^{-1}\sqrt{b}(A-a)\phi\rangle_{\mathcal{G}_r}\\ &\quad - \langle\sqrt{b}\check\phi, (\tilde A - \tilde b)\check e a^{-1}\sqrt{b}(A-a)\phi\rangle_{\mathcal{G}_r}\\ &= -\langle\sqrt{b}\check\phi, \check e(\tilde A + \tilde b)a^{-1}\sqrt{b}(A-a)\phi\rangle_{\mathcal{G}_r}\end{aligned}$$

on the interval $[\check r, \infty)$ that $C_- = \lim_{r\to\infty}\langle g, a^{-1}\xi_-(r)\rangle_{\mathcal{G}}$ exists and in fact is given by $C_- = 0$. Note that the analogue of the expression $T_1$ of Lemma 3.4 of the derivative vanishes, and that the corresponding terms $T_2, \dots, T_5$ are integrable (uniformly in $\check r$). For example it follows from Lemma 3.5 and Corollary 3.6 that

$$\|p'\bar{\check e}\tfrac{b}{\bar a}\check\phi\|_{\mathcal{G}_r} \leq C\big((r')^{1/2}\|p'u(r')\|_{\mathcal{G}_{r'}} + \|u(r')\|_{\mathcal{G}_{r'}}\big)r^{-1/2};\ r \geq r'. \tag{3.22}$$

(Here $C$ may depend on the support of $g$.) This estimate can be applied with $r' = 2R_n$ (for example) to treat the analogue of the expression $T_3$ of Lemma 3.4.

*Step II.* $C_+$ exists. Similarly we have

$$2\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}r}\langle\check\xi(r), a^{-1}\xi_+(r)\rangle_{\mathcal{G}} = -\langle\sqrt{b}\check\phi, (\tilde A - \tilde b)a^{-1}\sqrt{b}(A+a)\phi\rangle_{\mathcal{G}_r}$$

To show that $C_+$ exists it suffices to argue that the derivative is integrable (since now there is no factor $\check e$ and whence no $\check r$-dependence to control). As in Step I there are four terms to consider, say $T_2, \dots, T_5$. More precisely these terms are the contributions from four terms arising by the following computation. We compute using (2.9a), (2.9c) and in the last step (2.8a)

$$\begin{aligned}&\tfrac{a}{\sqrt{b}}(\tilde A - \tilde b)\tfrac{\sqrt{b}}{a}(A+a)\\ &= (A - \bar a)\tilde\eta(A+a) + O(r^{-\kappa})(A-a) + O(r^{-\kappa})\\ &= (A-a)\tilde\eta(A+a) + \tilde\eta 4\mathrm{i}(\operatorname{Im} a)a + \big(\tilde\eta 2\mathrm{i}(\operatorname{Im} a) + O(r^{-\kappa})\big)(A-a) + O(r^{-\kappa})\\ &= (A+a)\tilde\eta(A-a) + \tilde\eta\big(2(p^r a) + 4\mathrm{i}(\operatorname{Im} a)a\big) + O(r^{-\kappa/2})(A-a) + O(r^{-\kappa})\\ &= (A+a)\tilde\eta(A-a) + O(r^{-\kappa/2})(A-a) + O(r^{-\kappa})\\ &= 2(H-\lambda) - L + O(r^{-\kappa/2})(A-a) + O(r^{-\kappa}).\end{aligned}$$

We can now proceed as in Step I. In particular we can use (3.22) with $\check e = 1$ to treat the term $T_3$ which is the analogue of $T_3$ of Lemma 3.4.

□

**Lemma 3.9.** *The quantity $F$ is the strong limit*

$$F = \mathcal{G}\text{–}\lim_{r\to\infty}\xi(r).$$

*Proof.* Due to Corollary 1.10 there exists $C_1 > 0$ and a sequence $r_n \to \infty$ such that

$$\|(A-a)\phi\|^2_{\mathcal{G}_{r_n}} + \|p'\phi\|^2_{\mathcal{G}_{r_n}} \leq C_1/r_n. \tag{3.23}$$

To show the existence of the strong limit it suffices to show that

$$\lim_{n\to\infty}\langle\xi(r_n),F-\xi(r_n)\rangle_{\mathcal G}=0. \tag{3.24}$$

Due to (3.23) it suffices in turn to show that

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle\xi(r_n),(a^{-1}\xi_+)(r_m)-(a^{-1}\xi_+)(r_n)\rangle_{\mathcal G}=0. \tag{3.25}$$

Now we claim that proceeding as in Step II of the proof of Lemma 3.8 (replacing $g\to\xi(r_n)$ and now integrating from $r_n$) indeed (3.25) follows. Note for the analogue term $T_3$ that Lemma 3.5 and Corollary 3.6 yield

$$\|p'\tfrac{b}{\bar a}\check\phi_n\|_{\mathcal G_r}\le C_2\big(r_n^{1/2}\|p'\phi\|_{\mathcal G_{r_n}}+\|\phi\|_{\mathcal G_{r_n}}\big)r^{-1/2};$$
$$r\ge r_n,\quad \check\phi_n=\mathrm e^{\mathrm i(r_n-r)(\tilde A-\tilde b)}[\phi]_{S_{r_n}}.$$

In combination with (3.23) we then obtain

$$\|p'\tfrac{b}{\bar a}\check\phi_n\|_{\mathcal G_r}\le C_3r^{-1/2},$$

which suffices for integrability and smallness $o(n^0)$ of the integral (due to the Cauchy-Schwarz inequality and Corollary 1.10) essentially showing (3.25). □

*Proof of Theorem 1.14.* The definition (1.43) is justified by Lemma 3.9. Clearly (1.44) follows from Lemma 3.7.

It remains to show the continuity statement. We shall basically follow the scheme of [Co]. Due to (1.44), Corollary 1.9 and the density of $C^\infty_{\mathrm c}(S)\subseteq\mathcal G$ the continuity of the map $I\ni\lambda\to F^+(\lambda)\psi\in\mathcal G$ follows if we can show continuity of $\langle F^+(\cdot)\psi,g\rangle_{\mathcal G}$ for any $\psi\in\mathcal H_{1+}$ and $g\in C^\infty_{\mathrm c}(S)$. Let us for any such $g$ introduce approximate generalized eigenfunctions $\phi^\pm[g]\in\mathcal N\cap B^*$ by specifying in the spherical coordinates

$$\phi^\pm[g](r,\sigma)=\bar\chi_n(r)b^{-1/2}(r,\sigma)\exp\bigg(\int_{r_0}^r\Big(\pm\mathrm i\tilde b-\tfrac12\operatorname{div}\tilde\omega\Big)(s,\sigma)\,\mathrm ds\bigg)g(\sigma). \tag{3.26}$$

The factor $\bar\chi_n$ is chosen as a cut-off function, possibly depending on $I$ and the support of $g$, to assure the property $\phi^\pm[g]\in\mathcal N$ (as in the proof of Lemma 3.8). Note that these vectors are essentially the same as those introduced at (1.48) (this is why we are using the same notation). We shall use the previous notation $\xi,\xi_+$ and $\phi$. First calculate (for $m$ sufficiently large)

$$\begin{aligned}&2\langle\psi,\chi_m\phi^+[g]\rangle=\\&-\mathrm i\langle(A+\bar a)\phi,\chi_m'\phi^+[g]\rangle+\langle(A+\bar a)\phi,\chi_m\tilde\eta(A-a)\phi^+[g]\rangle\\&\quad+\int_{r_0}^\infty\chi_m(r)\big(\langle p'\phi,p'\phi^+[g]\rangle_{\mathcal G_r}+\langle\phi,O(r^{-\kappa})\phi^+[g]\rangle_{\mathcal G_r}\big)\,\mathrm dr,\end{aligned}$$

cf. (2.8a). Note for the first term to the right that

$$\langle(A+\bar a)\phi,\chi_m'\phi^+[g]\rangle=\int_{r_0}^\infty\chi_m'(r)\langle(A+\bar a)\phi,\phi^+[g]\rangle_{\mathcal G_r}\,\mathrm dr,$$

and

$$\begin{aligned}\langle(A+\bar a)\phi,\phi^+[g]\rangle_{\mathcal G_r}&=\langle(\bar a-a)\phi,\phi^+[g]\rangle_{\mathcal G_r}+2\langle(b^{-1}\xi_+)(r),g\rangle_{\mathcal G}\\&=2\langle(a^{-1}\xi_+)(r),g\rangle_{\mathcal G}+2\langle((b^{-1}-a^{-1})\xi_+)(r),g\rangle_{\mathcal G}+2\mathrm i\langle(\operatorname{Im}a)\phi,\phi^+[g]\rangle_{\mathcal G_r}.\end{aligned}$$

Note for the second term to the right that

$$\begin{aligned}\big(A-a+O(r^{-\kappa})\big)\phi^+[g] &= \big(b^{-1/2}(A-b)b^{1/2}+\tfrac{\mathrm{i}}{2}\nabla_\omega \ln|\mathrm{d}r|^2\big)\phi^+[g] \\ &= O(r^{-\infty}),\end{aligned} \tag{3.27}$$

in fact here the last term vanishes for $r$ large. These considerations allow us to take $m\to\infty$ and we obtain (using that $\langle (a^{-1}\xi_+)(r), g\rangle_{\mathcal{G}} \to \langle F^+(\lambda)\psi, g\rangle_{\mathcal{G}}$ for $r\to\infty$) that

$$\begin{aligned}2\mathrm{i}\langle F^+(\lambda)\psi, g\rangle_{\mathcal{G}} &= 2\langle \psi, \phi^+[g]\rangle - \langle (A+\bar a)\phi, \tilde\eta(A-a)\phi^+[g]\rangle \\ &\quad - \int_{r_0}^\infty \big(\langle p'\phi, p'\phi^+[g]\rangle_{\mathcal{G}_r} + \langle \phi, O(r^{-\kappa})\phi^+[g]\rangle_{\mathcal{G}_r}\big)\,\mathrm{d}r.\end{aligned}$$

By tracing the $\lambda$-dependence we conclude from this representation that indeed $\langle F^+(\cdot)\psi, g\rangle_{\mathcal{G}}$ is continuous. □

The last formula reads more compactly (although less precisely)

$$\mathrm{i}\langle F^+(\lambda)\psi, g\rangle_{\mathcal{G}} - \langle \psi, \phi^+[g]\rangle = -\langle \phi, (H-\lambda)\phi^+[g]\rangle, \tag{3.28}$$

where the right-hand side is given an interpretation very similar to (2.8a).

3.3. **Properties of distorted Fourier transform.** We first prove Proposition 1.15. Throughout this subsection we continue to consider only the upper sign.

*Proof of Proposition 1.15.* We first prove (1.45) for $\psi\in B$. It is a direct consequence of Theorem 1.14 that (1.45) holds for $\psi\in\mathcal{H}_{1+}$, and we have already seen that the left-hand side extends continuously in $\psi\in B$. Hence it suffices to show the existence and continuity of the right-hand side in $B$. By Theorem 1.6 these matters reduce to the following estimate, a version of which appears in a similar context in [Sk]: For any $\psi\in B$

$$\sup_{R>r_0}\Big\| \fint_R \xi(r)\,\mathrm{d}r\Big\|_{\mathcal{G}} \le C\|\phi\|_{B^*}. \tag{3.29}$$

To show (3.29) we write $\xi(r)=\mathrm{e}^{\mathrm{i}(r-r_0)(\tilde A^{\mathrm{ex}}-\tilde b^{\mathrm{ex}})}u(r)$ and note that

$$\Big\| \fint_R \xi(r)\,\mathrm{d}r\Big\|_{\mathcal{G}} \le \fint_R \|u(r)\|_{\mathcal{G}_r}\,\mathrm{d}r.$$

Next for any $R>r_0$ we choose $n\ge 0$ such that $R_n\le 2R<R_{n+1}$ and use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}\Big\| \fint_R \xi(r)\,\mathrm{d}r\Big\|_{\mathcal{G}}^2 &\le \fint_R \|u(r)\|^2_{\mathcal{G}_r}\,\mathrm{d}r \\ &\le 2\sum_{\nu=0}^n (R_\nu/R_n)R_\nu^{-1}\|F_\nu\sqrt{b}\phi\|^2 \\ &\le 4\|\sqrt{b}\phi\|^2_{B*}.\end{aligned}$$

Hence we have shown (3.29) and therefore that (1.45) holds for any $\psi\in B$.

To show that $\fint_R \xi(r)\,\mathrm{d}r\to F^\pm(\lambda)\psi$ locally uniformly in $\lambda$ we can assume that $\psi\in\mathcal{H}_{1+}$. Next note that

$$\Big\| F^+(\lambda)\psi - \fint_{R_1}\xi(r_1)\,\mathrm{d}r_1\Big\|_{\mathcal{G}}^2 \le \lim_{R_2\to\infty}\fint_{R_1}\fint_{R_2}\|\xi(r_2)-\xi(r_1)\|^2_{\mathcal{G}}\,\mathrm{d}r_2\mathrm{d}r_1.$$

We look at $R_2 > 2R_1$ and write

$$\begin{aligned}\|\xi(r_2)-\xi(r_1)\|^2_{\mathcal G} &= \|\xi(r_2)\|^2_{\mathcal G} - \|\xi(r_1)\|^2_{\mathcal G} - 2\mathrm{Re}\,\langle \xi(r_1), \xi(r_2)-\xi(r_1)\rangle_{\mathcal G}\\ &= \|\xi(r_2)\|^2_{\mathcal G} - \|\xi(r_1)\|^2_{\mathcal G}\\ &\quad - 2\mathrm{Re}\,\langle \xi(r_1), (a^{-1}\xi_-)(r_2)\rangle_{\mathcal G} + 2\mathrm{Re}\,\langle \xi(r_1), (a^{-1}\xi_-)(r_1)\rangle_{\mathcal G}\\ &\quad - 2\mathrm{Re}\,\langle \xi(r_1), (a^{-1}\xi_+)(r_2) - (a^{-1}\xi_+)(r_1)\rangle_{\mathcal G}.\end{aligned}$$

The first term contributes to the $R_2$-limit by $2\,\mathrm{Im}\langle\psi,\phi\rangle$ due to Lemma 3.3, the second term by $-\fint_{R_1}\|\xi(r_1)\|^2_{\mathcal G}\,\mathrm{d}r_1$, the third term by 0 (cf. Corollary 1.10), the fourth term by $\fint_{R_1} 2\mathrm{Re}\,\langle \xi(r_1), (a^{-1}\xi_-)(r_1)\rangle_{\mathcal G}\,\mathrm{d}r_1$ and the last term by $o(R_1^0)$ (by the proof of Lemma 3.9). We readily check that $\fint_{R_1} 2\mathrm{Re}\,\langle \xi(r_1), (a^{-1}\xi_-)(r_1)\rangle_{\mathcal G}\,\mathrm{d}r_1 \to 0$ locally uniformly in $\lambda$, and similarly that $\fint_{R_1}\|\xi(r_1)\|^2_{\mathcal G}\,\mathrm{d}r_1 \to 2\,\mathrm{Im}\langle\psi,\phi\rangle$ and the quantity $o(R_1^0)\to 0$ locally uniformly in $\lambda$. This is by Corollary 1.10 and the proofs of Lemmas 3.3 and 3.9, respectively.

Next, we note that $B\cap\mathcal H_{\mathcal I}$ is dense in $\mathcal H_{\mathcal I}$. In fact for any $\psi\in B$ and any $f\in C^\infty_{\mathrm c}(\mathcal I)$ the vector $f(H)\psi\in B$, cf. [Hö, Theorem 14.1.4]. Due to Stone's formula and (1.44) we have

$$\|F^+\psi\|_{\widetilde{\mathcal H}_{\mathcal I}} = \|\psi\|_{\mathcal H_{\mathcal I}};\quad \psi\in B\cap\mathcal H_{\mathcal I},$$

so the operator $F^+$ extends as an isometry from $B\cap\mathcal H_{\mathcal I}\subseteq\mathcal H_{\mathcal I}$ to an isometry $\mathcal H_{\mathcal I}\to\widetilde{\mathcal H}_{\mathcal I}$ (denoted also by $F^+$). It remains to show that $F^+H_{\mathcal I}\subseteq M_\lambda F^+$ or equivalently that $F^+(H_{\mathcal I}-\mathrm i)^{-1} = (M_\lambda-\mathrm i)^{-1}F^+$. Whence it suffices to show that

$$F^+(H-\mathrm i)^{-1}\psi = (M_\lambda-\mathrm i)^{-1}F^+\psi \text{ for any } \psi\in B\cap\mathcal H_{\mathcal I}.$$

Using the resolvent equations

$$R(\lambda\pm\mathrm i0)R(\mathrm i) = (\lambda-\mathrm i)^{-1}R(\lambda\pm\mathrm i0) - (\lambda-\mathrm i)^{-1}R(\mathrm i), \tag{3.30}$$

we obtain

$$F^+(\lambda)R(\mathrm i)\psi = \lim_{R\to\infty}\fint_R(\lambda-\mathrm i)^{-1}\xi(r)\,\mathrm{d}r = (\lambda-\mathrm i)^{-1}F^+(\lambda)\psi.$$

Note that due to the Cauchy-Schwarz inequality the second term of (3.30) does not contribute to the limit. □

Now we embark on the proof of Theorem 1.17. Note that under the conditions of the lemma below a priori we can write

$$R(\mathrm i)L\phi^+[g] = \mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{m\to\infty} R(\mathrm i)\chi_m L\phi^+[g]\in B^*,$$

meaning that for any $\check\psi\in B$

$$\begin{aligned}\langle R(\mathrm i)L\phi^+[g],\check\psi\rangle &= \langle p'\phi^+[g], p'R(-\mathrm i)\check\psi\rangle\\ &= \lim_{m\to\infty}\int\chi_m(r)\langle p'\phi^+[g], p'R(-\mathrm i)\check\psi\rangle_{\mathcal G_r}\,\mathrm{d}r.\end{aligned}$$

**Lemma 3.10.** *Suppose Condition 1.16. For any $g\in C^\infty_{\mathrm c}(S)$ let $\phi^+[g]\in\mathcal N\cap B^*$ be given by* (3.26) *(where $n$ is large, but locally independent of $\lambda>\lambda_0$). Then*

$$R(\mathrm i)L\phi^+[g] = \mathrm{w}\text{–}\mathrm{B}\text{–}\lim_{m\to\infty} R(\mathrm i)\chi_m L\phi^+[g]\in B,$$

*meaning that the vector* $\psi = R(\mathrm{i})L\phi^+[g]$*, a priori in* $B^*$*, actually is in* $B$ *and that for all* $\check\phi \in B^*$

$$\begin{aligned}\langle \psi, \check\phi\rangle &= \langle p'\phi^+[g], p'R(-\mathrm{i})\check\phi\rangle\\ &= \lim_{m\to\infty}\int \chi_m(r)\langle p'\phi^+[g], p'R(-\mathrm{i})\check\phi\rangle_{\mathcal{G}_r}\,\mathrm{d}r.\end{aligned}$$

*In fact*

$$R(\mathrm{i})L\phi^+[g] \in B\cap\mathcal{H}^1. \tag{3.31}$$

When Condition 1.16 (1) holds, the proof of Lemma 3.10 is rather simple since an essential estimate is already done in Corollary 3.6. However, when Condition 1.16 (2) holds, we need to estimate one more derivative than in Corollary 3.6 and this requires technical computations, see Lemma 3.13. Hence for the moment we postpone the proof of Lemma 3.10 and first prove Theorem 1.17 which in turn clearly is a direct consequence of Lemma 3.10 and the following lemma.

**Lemma 3.11.** *If for all* $g\in C^\infty_{\mathrm{c}}(S)$ *the vector* $\phi^+[g]$ *of* (3.26) *(depending on* $\lambda>\lambda_0$*) satisfies* (3.31)*, then* $F^+\colon \mathcal{H}_{\mathcal{I}} \to \widetilde{\mathcal{H}}_{\mathcal{I}}$ *is a unitary diagonalizing transform.*

*Proof.* We consider for $\lambda\in\mathcal{I}$ the operator $F^+(\lambda)\colon B\to\mathcal{G}$ of Proposition 1.15. From the same result we know that $F^+$ is an isometry.

*Step I.* First we show that $F^+(\lambda)$ has dense range. This is equivalent to showing that $F^+(\lambda)^*:\mathcal{G}\to B^*$ is injective, and for that we will use the representation (3.28) of $F^+(\lambda)^*g$ for $g\in C^\infty_{\mathrm{c}}(S)$. For the term on the right-hand side for such $g$ we claim the bound

$$(A+a)R(\lambda-\mathrm{i}0)(H-\lambda)\phi^+[g]\in B^*_0. \tag{3.32}$$

To obtain (3.32) we use (3.30), reducing the problem to show that

$$(A+a)\psi_+,\ (A+a)R(\lambda-\mathrm{i}0)\psi_+\in B^*_0;\quad \psi_+ = R(\mathrm{i})(H-\lambda)\phi^+[g].$$

The interpretation of $\psi_+$ is given as in the proof of Theorem 1.14 which amounts to expanding $(H-\lambda)\phi^+[g]$ into a sum of three terms, cf. (2.8a). Each term is in $B\cap\mathcal{H}^1$, so consequently $\psi_+\in B\cap\mathcal{H}^1$. Note that at this point we use (3.27) and (3.31). Using that $\psi_+\in B\cap\mathcal{H}^1$ we deduce (3.32) by Corollary 1.10.

Now using (3.27), (3.28) and (3.32) we obtain

$$g = \mathcal{G}\text{–}\lim_{R\to\infty} ⨍_R \mathrm{e}^{\mathrm{i}(r-r_0)(\tilde A^{\mathrm{ex}}-\tilde b^{\mathrm{ex}})}\big[\tfrac{b^{1/2}\mathrm{i}}{2a}(a+A)F^+(\lambda)^*g\big]_{|S_r}\,\mathrm{d}r, \tag{3.33}$$

and since

$$(a+A)F^+(\lambda)^* = (\lambda-\mathrm{i})\big\{(a+A)R(\mathrm{i})\big\}F^+(\lambda)^*\in\mathcal{B}(\mathcal{G},B^*),$$

we conclude (3.33) for all $g\in\mathcal{G}$ by a continuity argument essentially identical with the one given in the first part of the proof of Proposition 1.15. In particular indeed $F^+(\lambda)^*:\mathcal{G}\to B^*$ is injective.

*Step II.* We prove the unitarity of $F^+\colon\mathcal{H}_{\mathcal{I}}\to\widetilde{\mathcal{H}}_{\mathcal{I}}$. Since we know $F^+H_{\mathcal{I}}\subseteq M_\lambda F^+$ from Proposition 1.15 it then follows that $F^+H_{\mathcal{I}} = M_\lambda F^+$, and the proof is done.

By using Proposition 1.15 (possibly in combination with (3.49)) we obtain that

$$F^+(\lambda)f(H)\psi = f(\lambda)F^+(\lambda)\psi \text{ for all } f\in C^\infty_{\mathrm{c}}(\mathcal{I}) \text{ and } \psi\in B. \tag{3.34}$$

Assuming $g(\cdot) \in \ker(F^+)^* \subseteq \widetilde{\mathcal{H}}_{\mathcal{I}}$ it suffices to show that $g(\lambda) = 0$ for a.e. $\lambda \in \mathcal{I}$. We shall mimic the proof of [ACH, Theorem 1.1]. For any $f \in C_c^\infty(\mathcal{I})$ and $\psi \in B$

$$\int_{\mathcal{I}} f(\lambda)\langle g(\lambda), F^+(\lambda)\psi\rangle_{\mathcal{G}}\, \mathrm{d}\lambda = \langle (F^+)^* g(\cdot), f(H)\psi\rangle_{\mathcal{H}_{\mathcal{I}}} = 0.$$

We apply this to the elements of a countable and dense subset, say $\{\psi_k\}_{k=1}^\infty \subseteq B$, and conclude that there exists a set $N \subseteq \mathcal{I}$ of measure 0 such that

$$\langle g(\lambda), F^+(\lambda)\psi_k\rangle_{\mathcal{G}} = 0 \text{ for all } k \in \mathbb{N} \text{ and } \lambda \in \mathcal{I} \setminus N.$$

Since $\{F^+(\lambda)\psi_k\}_{k=1}^\infty \subseteq \mathcal{G}$ is dense (by Step I) we conclude that $g(\cdot) = 0$. Hence $F^+ \colon \mathcal{H}_{\mathcal{I}} \to \widetilde{\mathcal{H}}_{\mathcal{I}}$ is surjective and therefore unitary. □

**Remarks 3.12.** We used above the representation in terms of the vectors $\phi^\pm[g] \in \mathcal{N}$ of (3.26) (here stated for both signs)

$$\begin{gathered}\pm\mathrm{i}F^\pm(\lambda)^* g = \phi^\pm[g] - \psi_\pm - (\lambda - \mathrm{i})R(\lambda \mp \mathrm{i}0)\psi_\pm;\\ g \in C_c^\infty(S), \quad \psi_\pm = R(\mathrm{i})(H-\lambda)\phi^\pm[g] \in B \cap \mathcal{H}^1.\end{gathered} \tag{3.35a}$$

For comparison we obtain using Corollary 1.11

$$0 = \phi^\pm[g] - \psi_\pm - (\lambda - \mathrm{i})R(\lambda \pm \mathrm{i}0)\psi_\pm.$$

In particular

$$\phi^\pm[g] - (\lambda - \mathrm{i})R(\lambda \pm \mathrm{i}0)\psi_\pm \in B_0^*,$$

which leads to

$$g = (\lambda - \mathrm{i})F^\pm(\lambda)\psi_\pm; \quad g \in C_c^\infty(S), \quad \psi_\pm = R(\mathrm{i})(H-\lambda)\phi^\pm[g]. \tag{3.35b}$$

The formulas (3.35a) and (3.35b) will be used in Section 3.4, however we stress that the vectors of (3.26) are given with a cutoff to make them elements of $\mathcal{N}$. The vectors $\phi^\pm[g]$ given by (1.48) do not necessarily enjoy this property, and in fact the above formulas might not be valid in general when $\phi^\pm[g]$ (with $g \in C_c^\infty(S)$) are given by (1.48).

*Proof of Theorem 1.17.* The statement is obvious from Proposition 1.15 and Lemmas 3.10 and 3.11. □

Finally we prove Lemma 3.10. We begin with a technical estimate required for the case of Condition 1.16 (2).

**Lemma 3.13.** *Suppose Condition 1.16 (2) and let $\check\sigma < \min\{\sigma, \tau, \rho\}$. Then for any compact interval $I \subseteq \mathcal{I}$ there exists a constant $C > 0$ such that for all $\lambda \in I$, $\check r \ge r_0$ and $u \in C_c^2(S_{\check r})$, the function $u(r) = \mathrm{e}^{\mathrm{i}(\check r - r)(\tilde A - \tilde b)}u \in C_c^2(S_r)$ for $r \ge \check r$ and satisfies*

$$\|L_r u(r)\|_{\mathcal{G}_r} \le C\Big((1/r)^{\check\sigma}\|u\|_{\mathcal{G}_{\check r}} + (\check r^{1/2}/r)^{\check\sigma}\|\nabla' u\|_{\mathcal{G}_{\check r}} + (\check r/r)^{\check\sigma}\|\nabla'^2 u\|_{\mathcal{G}_{\check r}}\Big), \tag{3.36}$$

*where $\nabla'$ is the Levi–Civita connection associated with the induced Riemannian metric on the $r$-sphere $S_{\check r}$, i.e. $g_{\check r} = \iota_{\check r}^* g$.*

*Proof. Step I.* In this proof we make use of the geometric derivatives of a mapping presented in Subsection 2.1. For any $t \in [\check r - r, 0]$ and $x \in \tilde y(-t, S_{r+t}) \subseteq S_r$ we define the quantity $\big(\Delta_r \tilde y^\alpha(t,x)\big)_{\alpha=2,\dots,d}$ as a tangent of $S_{r+t}$ at $\tilde y(t,x)$ by

$$\Delta_r \tilde y^\alpha(t,x) = \ell^{ij}(\nabla^2 \tilde y)^\alpha{}_{ij}(t,x).$$

Note that here again we adopt the same convention for the Roman and Greek indices as in the proof of Lemma 3.5. Let us abbreviate simply $\tilde{y} = \tilde{y}(t,x)$ and use the formula (2.7). Then we can write the above quantity more explicitly:

$$\Delta_r \tilde{y}^\alpha(t,x) = \Delta_r \tilde{y}^\alpha = \ell^{ij}\partial_i\partial_j\tilde{y}^\alpha - \ell^{ij}\Gamma^k_{ij}\partial_k\tilde{y}^\alpha + \ell^{ij}\Gamma^\alpha_{\beta\gamma}(\partial_i\tilde{y}^\beta)(\partial_j\tilde{y}^\gamma). \tag{3.37}$$

Hereinafter in this proof all the indices run only over the angular components in the spherical coordinates. Whence the distinguished radial index $\alpha,\beta,\gamma,\delta,i,j,k = r$ does not enter in the summations there. At this point it should be noted that it does not cause a confusion, because $\Gamma^\alpha_{\beta\gamma} = \Gamma'^\alpha_{\beta\gamma}$ for $\alpha,\beta,\gamma \neq r$ where $\Gamma'$ is the Christoffel symbol for the submanifold $S_{r+t}$ (and similarly for $\Gamma^k_{ij}$).

Similarly, the quantity $\big(L\tilde{y}^\alpha(t,x)\big)_{\alpha=2,\dots,d}$ is defined as a tangent of $S_{r+t}$ at $\tilde{y}(t,x)$ by

$$L\tilde{y}^\alpha(t,x) = L\tilde{y}^\alpha = -\Delta_r\tilde{y}^\alpha + \tfrac12(\partial_i \ln|\mathrm{d}r|^2)\ell^{ij}\partial_j\tilde{y}^\alpha, \tag{3.38}$$

cf. (2.11). We note that the quantity (3.38) is defined to satisfy that for any function $v \in C^2(S_{r+t})$

$$\begin{aligned} L\big[v(\tilde{y}(t,x))\big] &= \big[L\tilde{y}^\alpha(t,x)\big]\big(\partial_\alpha v\big)(\tilde{y}(t,x)) \\ &\quad - \ell^{ij}(x)\big[\partial_i\tilde{y}^\alpha(t,x)\big]\big[\partial_j\tilde{y}^\beta(t,x)\big]\big(\nabla'^2 v\big)_{\alpha\beta}(\tilde{y}(t,x)), \end{aligned} \tag{3.39}$$

where $\nabla'$ is the covariant derivative on the submanifold $S_{r+t}$,

*Step II.* Next let us estimate the tensor (3.37). We claim that for any $\breve{\sigma} < \min\{\sigma,\tau\}$ there exists $C_1 > 0$ such that uniformly in $r \geq \check{r} \geq r_0$, $t \in [\check{r} - r, 0]$ and $x \in \tilde{y}(r-\check{r}, S_{\check{r}}) \subseteq S_r$

$$F(t) := \ell_{\alpha\beta}(\Delta_r\tilde{y}^\alpha)(\Delta_r\tilde{y}^\beta) = |\Delta_r\tilde{y}|^2 \leq C_1(r+t)^{\breve{\sigma}}/r^{2\breve{\sigma}}. \tag{3.40}$$

To prove (3.40) we shall establish a differential inequality for $F(t)$, cf. the proof of Lemma 3.5. Obviously, using the shorthand notation $v^\alpha = \Delta_r\tilde{y}^\alpha(t,x)$ for $\alpha \neq r$,

$$F'(t) = \Big((\partial_t\ell_{\alpha\beta})v^\alpha + 2\ell_{\alpha\beta}\big[\partial_t\Delta_r\tilde{y}^\alpha\big]\Big)v^\beta.$$

Next recall

$$\Gamma'^\alpha_{\delta\gamma} = \tfrac12\ell^{\alpha\beta}\big(\partial_\delta\ell_{\beta\gamma} + \partial_\gamma\ell_{\beta\delta} - \partial_\beta\ell_{\delta\gamma}\big), \tag{3.41a}$$

and similarly for the Christoffel symbol on $M$. The latter yields

$$|\mathrm{d}r|^2\partial_r\ell_{\alpha\beta} = -2\Gamma^r_{\alpha\beta} = 2(\nabla^2 r)_{\alpha\beta};\ \alpha,\beta \neq r. \tag{3.41b}$$

Then we calculate using (3.41b) and that $\tilde{\omega}^\eta = \delta_{\eta r}$ (here and below possibly $\eta = r$)

$$\begin{aligned} &(\partial_t\ell_{\alpha\beta})v^\alpha + 2\ell_{\alpha\beta}[\partial_t\Delta_r\tilde{y}^\alpha] \\ &= (\partial_\eta\ell_{\alpha\beta})\tilde{\omega}^\eta v^\alpha + 2\ell_{\alpha\beta}\Big(\ell^{ij}\partial_i\partial_j\tilde{\omega}^\alpha - \big[\ell^{ij}\Gamma^k_{ij}\big]\partial_k\tilde{\omega}^\alpha \\ &\qquad + (\partial_\eta\Gamma^\alpha_{\delta\gamma})\tilde{\omega}^\eta\ell^{ij}[\partial_i\tilde{y}^\delta][\partial_j\tilde{y}^\gamma] + 2\Gamma^\alpha_{\delta\gamma}\ell^{ij}(\partial_i\tilde{\omega}^\delta)[\partial_j\tilde{y}^\gamma]\Big) \\ &= (\partial_r\ell_{\alpha\beta})v^\alpha + 2\ell_{\alpha\beta}(\partial_r\Gamma^\alpha_{\delta\gamma})\ell^{ij}[\partial_i\tilde{y}^\delta][\partial_j\tilde{y}^\gamma] \\ &= 2|\mathrm{d}r|^{-2}(\nabla^2 r)_{\alpha\beta}v^\alpha + 2\ell_{\alpha\beta}(\partial_r\Gamma^\alpha_{\delta\gamma})\ell^{ij}[\partial_i\tilde{y}^\delta][\partial_j\tilde{y}^\gamma]. \end{aligned} \tag{3.42}$$

A computation using (3.41a), (3.41b) and (1.8) (details skipped) yields

$$\partial_r\Gamma'^\alpha_{\delta\gamma} = (\nabla(|\mathrm{d}r|^{-2}\nabla^2 r))_\delta{}^\alpha{}_\gamma + (\nabla(|\mathrm{d}r|^{-2}\nabla^2 r))_\gamma{}^\alpha{}_\delta - |\mathrm{d}r|^{-2}(\nabla^3 r)^\alpha{}_{\delta\gamma},$$

which we can insert on the right-hand side of (3.42). Using the Cauchy-Schwarz inequality, Lemma 3.5 and Condition 1.16 (2) the contribution from the last term of (3.42) can be estimated from below as follows: For any $\epsilon > 0$

$$\cdots \geq -\epsilon(r+t)^{-1}F(t) - \epsilon^{-1}C_2(r+t)^{2\sigma-\tau-1}/r^{2\sigma}.$$

With (1.7b) this leads to the final estimate (seen by taking $\epsilon \leq \sigma - \check{\sigma}$)

$$F'(t) \geq \check{\sigma}(r+t)^{-1}F(t) - C_3(r+t)^{2\sigma-\tau-1}/r^{2\sigma}. \tag{3.43}$$

On the other hand we can compute $F(0) = 0$ by using (3.37), and then the Gronwall's inequality yields the claimed estimate (3.40).

*Step III.* Here we compute and estimate $Lu(r)$. For simplicity we would like to use the spherical coordinates as in (3.19). To avoid any confusion concerning base points let us write $\sigma \in \tilde{y}(r-\check{r}, \operatorname{supp} u) \subseteq S_r$ and $\sigma(s) = \tilde{y}(s-r,\sigma) \in S_s$, $s \in [\check{r}, r]$. Then we can write

$$u(r)(\sigma) = (\mathrm{e}^{\mathrm{i}(\check{r}-r)(\tilde{A}-\tilde{b})}u)(\sigma) = \exp\biggl(\int_{\check{r}}^{r}\Bigl(\mathrm{i}\tilde{b} - \tfrac{1}{2}\operatorname{div}\tilde{\omega}\Bigr)(s,\sigma(s))\,\mathrm{d}s\biggr)u(\sigma(\check{r})). \tag{3.44}$$

Using (3.44) and (3.39) we obtain the following expression:

$$\begin{aligned}
&Lu(r)(\sigma)\\
&= [L\sigma^\alpha(\check{r})](\mathrm{e}^{\mathrm{i}(\check{r}-r)(\tilde{A}-\tilde{b})}\partial_\alpha u(\check{r}))(\sigma)\\
&\quad - \ell^{ij}[\partial_i\sigma^\alpha(\check{r})][\partial_j\sigma^\beta(\check{r})]\bigl(\mathrm{e}^{\mathrm{i}(\check{r}-r)(\tilde{A}-\tilde{b})}(\nabla'^2 u(\check{r}))_{\alpha\beta}\bigr)(\sigma)\\
&\quad + \biggl(\int_{\check{r}}^{r}[L\sigma^\alpha(s)]\bigl[\partial_\alpha\bigl(\mathrm{i}\tilde{b} - \tfrac{1}{2}\operatorname{div}\tilde{\omega}\bigr)\bigr](s,\sigma(s))\,\mathrm{d}s\biggr)u(r)(\sigma)\\
&\quad - \biggl(\int_{\check{r}}^{r}\ell^{ij}[\partial_i\sigma^\alpha(s)][\partial_j\sigma^\beta(s)]\bigl(\nabla'^2\bigl(\mathrm{i}\tilde{b} - \tfrac{1}{2}\operatorname{div}\tilde{\omega}\bigr)\bigr)_{\alpha\beta}(s,\sigma(s))\,\mathrm{d}s\biggr)u(r)(\sigma)\\
&\quad - 2\ell^{ij}\biggl(\int_{\check{r}}^{r}[\partial_i\sigma^\alpha(s)]\bigl[\partial_\alpha\bigl(\mathrm{i}\tilde{b} - \tfrac{1}{2}\operatorname{div}\tilde{\omega}\bigr)\bigr](s,\sigma(s))\,\mathrm{d}s\biggr)\\
&\qquad \cdot [\partial_j\sigma^\beta(\check{r})]\bigl(\mathrm{e}^{\mathrm{i}(\check{r}-r)(\tilde{A}-\tilde{b})}\partial_\beta u(\check{r})\bigr)(\sigma)\\
&\quad - \ell^{ij}\biggl(\int_{\check{r}}^{r}[\partial_i\sigma^\alpha(s)]\bigl[\partial_\alpha\bigl(\mathrm{i}\tilde{b} - \tfrac{1}{2}\operatorname{div}\tilde{\omega}\bigr)\bigr](s,\sigma(s))\,\mathrm{d}s\biggr)\\
&\qquad \cdot \biggl(\int_{\check{r}}^{r}[\partial_j\sigma^\alpha(s)]\bigl[\partial_\alpha\bigl(\mathrm{i}\tilde{b} - \tfrac{1}{2}\operatorname{div}\tilde{\omega}\bigr)\bigr](s,\sigma(s))\,\mathrm{d}s\biggr)u(r)(\sigma).
\end{aligned} \tag{3.45}$$

Then by (3.15), (3.38) and (3.40) we can verify (3.36). In fact, it is clear that the first and the second terms to the right of (3.45) satisfy the desired estimate. The other terms can be treated more or less in a similar manner, and hence we consider only the fifth term. By the Cauchy–Schwarz inequality, (3.15)

$$\begin{aligned}
&\biggl|\ell^{ij}\biggl(\int_{\check{r}}^{r}[\partial_i\sigma^\alpha(s)]\bigl[\partial_\alpha\bigl(\mathrm{i}\tilde{b} - \tfrac{1}{2}\operatorname{div}\tilde{\omega}\bigr)\bigr](s,\sigma(s))\,\mathrm{d}s\biggr)[\partial_j\sigma^\beta(\check{r})]\biggr|\\
&\leq C_4(\check{r}^{1/2}/r)^{\check{\sigma}}\int_{\check{r}}^{r}s^{-1-\min\{\tau,\rho\}/2+\check{\sigma}/2}\,\mathrm{d}s\\
&\leq C_5(\check{r}^{1/2}/r)^{\check{\sigma}}.
\end{aligned}$$

This is the desired estimate. We omit the rest of the argument. □

*Proof of Lemma 3.10.* Write with $\check r = R_n$ (recall that possibly $n$ depends on the support of $g$)

$$\begin{aligned}\phi^+ &= \bar\chi_n(r) b^{-1/2} \mathrm{e}^{\mathrm{i}(\check r - r)(\tilde A - \tilde b)} u;\\ u &= \mathrm{e}^{\mathrm{i}(r_0 - \check r)(\tilde A - \tilde b)} g \in C^1_{\mathrm{c}}(S_{\check r}).\end{aligned}$$

First let us assume Condition 1.16 (1). We decompose

$$R(\mathrm{i})L\phi^+ = (R(\mathrm{i})r^{-s}p')(p' r^s \phi^+);\quad s > 1/2. \tag{3.46}$$

Since the first factor is bounded as $\mathcal H \to \mathcal H_s \cap \mathcal H^1 \subseteq B \cap \mathcal H^1$, it suffices to show that the second factor belongs to $\mathcal H$ for some $s > 1/2$. We combine Condition 1.16 (1) with Lemma 3.5 and Corollary 3.6 (these results applied with any $\breve\sigma \in (2, \min\{\sigma, \tau, \rho\})$ and conclude that indeed $p' r^s \phi^+ \in \mathcal H$ for some $s > 1/2$. Of course the conclusion $R(\mathrm{i})L\phi^+ = \text{w–B–}\lim_{m\to\infty}(R(\mathrm{i})r^{-s}p')\chi_m(p' r^s \phi^+)$ follows from this argument.

Next, assuming Condition 1.16 (2), we note that $u \in C^2_{\mathrm{c}}(S_{\check r})$ and decompose

$$R(\mathrm{i})L\phi^+ = \big(R(\mathrm{i})r^{-s}\big)\Big(\int \oplus L_r r^s \phi^+ \,\mathrm{d}r\Big);\quad s > 1/2,$$

and proceed using Lemma 3.5 and Corollary 3.6 to bound $L_r\phi^+ - b^{-1/2}L_r b^{1/2}\phi^+$ and Lemma 3.13 to bound $b^{-1/2}L_r b^{1/2}\phi^+$, respectively. Next we introduce a factor $\chi_m$ as above and conclude similarly. □

### 3.4. **Scattering matrix and characterization of generalized eigenfunctions.**

In this subsection we prove Theorem 1.19. Throughout the subsection we assume Condition 1.16, and we fix $\lambda \in \mathcal I$.

We begin with a partial uniqueness result.

**Lemma 3.14.** *Suppose $\phi \in \mathcal E_\lambda$ and $\xi_\pm \in \mathcal G$ satisfy*

$$\phi - \phi^+[\xi_+] + \phi^-[\xi_-] \in B_0^*. \tag{3.47}$$

*Then $\xi_\pm$ are uniquely determined by $\phi$. Moreover*

$$\|\xi_+\|^2_{\mathcal G} + \|\xi_-\|^2_{\mathcal G} = \lim_{R\to\infty} R^{-1}\int_{B_{2R}\setminus B_R} b|\phi|^2 (\det g)^{1/2}\mathrm{d}x, \tag{3.48a}$$

$$\|\xi_+\|_{\mathcal G} = \|\xi_-\|_{\mathcal G}. \tag{3.48b}$$

*Proof.* The uniqueness statement follows from (3.48a), which in turn is proved as follows:

$$\begin{aligned}&\lim_{R\to\infty} R^{-1}\int_{B_{2R}\setminus B_R} b|\phi|^2 (\det g)^{1/2}\mathrm{d}x\\ &= \lim_{R\to\infty} R^{-1}\int_{B_{2R}\setminus B_R} b|\phi^+[\xi_+] - \phi^-[\xi_-]|^2 (\det g)^{1/2}\mathrm{d}x\\ &= \|\xi_+\|^2_{\mathcal G} + \|\xi_-\|^2_{\mathcal G} - 2\mathrm{Re}\lim_{R\to\infty} \fint_R \langle \xi_+, \exp\big(-2\mathrm{i}\int_{r_0}^r \tilde b(s,\,\cdot\,)\,\mathrm{d}s\big)\xi_-\rangle_{\mathcal G}\,\mathrm{d}r\end{aligned}$$

The last term vanishes as may be seen by first writing

$$\exp\Big(-2\mathrm{i}\int_{r_0}^r \tilde b(s,\,\cdot\,)\,\mathrm{d}s\Big) = (-2\mathrm{i}\tilde b)^{-1}\tfrac{\mathrm{d}}{\mathrm{d}r}\exp\Big(-2\mathrm{i}\int_{r_0}^r \tilde b(s,\,\cdot\,)\,\mathrm{d}s\Big)$$

and then integrate by parts picking up a sum of decaying factors. Note that indeed $\frac{\mathrm{d}}{\mathrm{d}r}\tilde b(r,\cdot) = o(R^0)$ uniformly in the angle variable (so that the Cauchy-Schwarz inequality applies).

As for (3.48b) first note that $A\phi \in B^*$, which comes from the representation $A\phi = (\lambda - \mathrm{i})AR(\mathrm{i})\phi$ and the fact that $AR(\mathrm{i}) \in \mathcal{B}(B^*)$. Then we compute

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle \mathrm{i}[H,\chi_n]\rangle_\phi \\
&= \lim_{n\to\infty} \langle A\chi_n'\rangle_\phi \\
&= \lim_{n\to\infty} \langle A\phi, \chi_n'(\phi^+[\xi_+] - \phi^-[\xi_-])\rangle \\
&= \lim_{n\to\infty} \langle \phi, \chi_n'(A\phi^+[\xi_+] - A\phi^-[\xi_-])\rangle \\
&= \lim_{n\to\infty} \langle \phi, \chi_n'(b\phi^+[\xi_+] + b\phi^-[\xi_-])\rangle \\
&= \lim_{n\to\infty} \langle \phi^+[\xi_+] - \phi^-[\xi_-], \chi_n' b(\phi^+[\xi_+] + \phi^-[\xi_-])\rangle \\
&= \|\xi_+\|^2_{\mathcal{G}} - \|\xi_-\|^2_{\mathcal{G}},
\end{aligned}$$

where in the last step we integrated by parts as in the proof of (3.48a). □

Next, we construct $\xi_\pm \in \mathcal{G}$ from $\phi \in \mathcal{E}_\lambda$. Note for comparison that $F^\pm(\lambda)^*\xi \in \mathcal{E}_\lambda$ for any $\xi \in \mathcal{G}$ (readily proven by using $F^\pm(\lambda)^* = (\lambda - \mathrm{i})R(\mathrm{i})F^\pm(\lambda)^*$, cf. the proof of Lemma 3.11).

**Lemma 3.15.** *For any* $\phi \in \mathcal{E}_\lambda$ *there exist* $\xi_\pm \in \mathcal{G}$ *such that* (1.50b) *hold.*

*Proof.* We use the scheme of proof of [Sk, Proposition 6.2]. By the definition of $S(\lambda)$ it suffices to show that for any $\phi \in \mathcal{E}_\lambda$ the representation $\phi = \mathrm{i}F^+(\lambda)^*\xi$ for some $\xi \in \mathcal{G}$ holds.

Pick $f \in C^\infty_{\mathrm{c}}(\mathbb{R})$ with $f(t) = t$ in neighbourhood of $t = \lambda$. Whence $f(H)\phi = \lambda\phi$. We introduce (for a fixed large $m$)

$$\phi_\pm = \tfrac{1}{2b}\bar\chi_m(A \pm b)\phi \in B^*, \quad \xi_n = F^+(\lambda)\chi_n\big(f(H) - \lambda\big)\phi_+;\ n \in \mathbb{N}.$$

The sequence $(\xi_n) \subseteq \mathcal{G}$ is bounded. Indeed since $F^+(\lambda)\big(f(H) - \lambda\big) = 0$ (cf. (3.34)) we compute using (3.49) (stated below) and estimate uniformly in $n \in \mathbb{N}$ and in $g \in C^\infty_{\mathrm{c}}(S)$, $\|g\|_{\mathcal{G}} = 1$,

$$\begin{aligned}
\langle g, \xi_n\rangle_{\mathcal{G}} &= \mathrm{i}\langle F^+(\lambda)^* g, \big(A\chi_n' + \mathrm{i}|\mathrm{d}r|^2\chi_n''/2\big) f'(H)\phi_+\rangle_{B^*\times B}, \\
|\langle g, \xi_n\rangle_{\mathcal{G}}| &\le C_1\big|\big(\|AF^+(\lambda)^* g\|_{B^*} + \|F^+(\lambda)^* g\|_{B^*}\big) \le C_2.
\end{aligned}$$

Next we choose a weakly convergent subsequence of $(\xi_n)$, cf. [Yo, Theorem 1 p. 126]. Whence, possibly upon changing notation, $\text{w–}\lim_{n\to\infty}\xi_n =: \xi \in \mathcal{G}$. For this $\xi$

and with $\check f(t) := (f(t)-\lambda)(t-\lambda)^{-1}$ we compute

$$\begin{aligned}
&\mathrm{i}F^+(\lambda)^*\xi\\
&= \mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} \mathrm{i}F^+(\lambda)^*F^+(\lambda)\chi_n\big(f(H)-\lambda\big)\phi_+\\
&= \mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} \big(R(\lambda+\mathrm{i}0)-R(\lambda-\mathrm{i}0)\big)\chi_n\big(f(H)-\lambda\big)\phi_+\\
&= \mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} \Big(R(\lambda+\mathrm{i}0)\chi_n\big(f(H)-\lambda\big)\phi_+ + R(\lambda-\mathrm{i}0)\chi_n\big(f(H)-\lambda\big)(\phi-\phi_+)\Big)\\
&= \mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} \Big(\check f(H)\chi_n\phi + R(\lambda+\mathrm{i}0)[\chi_n,f(H)]\phi_+ + R(\lambda-\mathrm{i}0)[\chi_n,f(H)](\phi-\phi_+)\Big)\\
&= \check f(H)\phi + \mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} \Big(R(\lambda+\mathrm{i}0)[\chi_n,f(H)]\phi_+ + R(\lambda-\mathrm{i}0)[\chi_n,f(H)](\phi-\phi_+)\Big).
\end{aligned}$$

The first term simplifies as $\check f(H)\phi=\phi$. To compute the last term we represents in a standard fashion (in terms of an almost analytic extension $\tilde f$)

$$f(H)=\int_{\mathbb C} R(z)\mathrm{d}\mu(z);\ \mathrm{d}\mu(z)=-(2\pi\mathrm{i})^{-1}\bar\partial\tilde f(z)\mathrm{d}z\mathrm{d}\bar z, \tag{3.49}$$

allowing us to compute

$$[\chi_n,f(H)]=-\mathrm{i}\int_{\mathbb C} R(z)\big(A\chi_n'+\mathrm{i}|\mathrm{d}r|^2\chi_n''/2\big)R(z)\mathrm{d}\mu(z).$$

Whence due to Corollary 1.11 (note also that the second term with the factor $\chi_n''$ does not contribute to the limit)

$$\begin{aligned}
&\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} R(\lambda+\mathrm{i}0)[\chi_n,f(H)]\phi_+\\
&= -\mathrm{i}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty}\int_{\mathbb C} R(z)R(\lambda+\mathrm{i}0)A\chi_n'R(z)\mathrm{d}\mu(z)\phi_+\\
&= \mathrm{i}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty}\int_{\mathbb C} R(z)R(\lambda+\mathrm{i}0)b\chi_n'R(z)\mathrm{d}\mu(z)\phi_+\\
&= \mathrm{i}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty}\int_{\mathbb C} R(z)R(\lambda+\mathrm{i}0)R(z)\mathrm{d}\mu(z)b\chi_n'\phi_+\\
&= -\tfrac{\mathrm{i}}{2}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} f'(H)R(\lambda+\mathrm{i}0)\chi_n'\bar\chi_m(A+b)\phi\\
&= -\tfrac{\mathrm{i}}{2}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} f'(H)R(\lambda+\mathrm{i}0)(A+b)\chi_n'\phi\\
&= 0.
\end{aligned}$$

Similarly, using that $\phi-\phi_+=\chi_m\phi-\phi_-$,

$$\begin{aligned}
&\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} R(\lambda-\mathrm{i}0)[\chi_n,f(H)](\phi-\phi_+)\\
&= -\tfrac{\mathrm{i}}{2}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} f'(H)R(\lambda-\mathrm{i}0)\chi_n'\bar\chi_m\big((A-b)\phi\big)\\
&= -\tfrac{\mathrm{i}}{2}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} f'(H)R(\lambda-\mathrm{i}0)(A-b)\chi_n'\phi\\
&= 0.
\end{aligned}$$

Whence we have shown that $\phi=\mathrm{i}F^+(\lambda)^*\xi$ for the constructed $\xi$. □

**Lemma 3.16.** *For all $\psi\in B$ and all $\lambda\in\mathcal I$*

$$\sqrt{b}R(\lambda\pm\mathrm{i}0)\psi-1_M\mathrm{e}^{\mathrm{i}(r_0-r)(\tilde A^{\mathrm{ex}}\mp\tilde b^{\mathrm{ex}})}F^\pm(\lambda)\psi\in B_0^*. \tag{3.50}$$

*Proof.* This is obvious from Lemma 3.9 for $\psi \in \mathcal{H}_{1+}$. The general case is treated by an approximation argument (as in the proof of [Sk, Corollary 5.5]). □

A construction of $\phi \in \mathcal{E}_\lambda$ from $\xi_\pm \in \mathcal{G}$ may intuitively seem most feasible when $\xi_\pm$ satisfies the Dirichlet boundary condition. We first give such construction for $\xi_\pm \in C_{\mathrm{c}}^\infty(S)$ and shortly extend it allowing any $\xi_\pm \in \mathcal{G}$.

**Lemma 3.17.** *For any $\xi_- \in C_{\mathrm{c}}^\infty(S)$ introduce $\phi^-[\xi_-] \in \mathcal{N} \cap B^*$ by* (3.26) *(rather than by* (1.48)*) and define then $\phi \in \mathcal{E}_\lambda$ and $\xi_+ \in \mathcal{G}$ by*

$$\begin{aligned} \phi &= \psi_- + (\lambda - \mathrm{i})R(\lambda + \mathrm{i}0)\psi_- - \phi^-[\xi_-], \\ \xi_+ &= (\lambda - \mathrm{i})F^+(\lambda)\psi_-; \quad \psi_- = R(\mathrm{i})(H - \lambda)\phi^-[\xi_-]. \end{aligned} \tag{3.51}$$

*Then* (1.50a) *and* (1.50b) *hold for $\{\xi_-, \xi_+, \phi\}$.*

*Proof.* Note that $\psi_- \in B$, cf. the proof of Lemma 3.11, and that (1.50a) holds with the approximate eigenfunctions of (1.48) if the estimate is valid for those defined by (3.26) (obviously the difference is in $B_0^*$). We combine (3.35a) and (3.35b) (with the lower sign only) and Lemma 3.16 (with the upper sign). □

Similarly we can first specify $\xi_+ \in C_{\mathrm{c}}^\infty(S)$ (the proof is similar).

**Lemma 3.18.** *For any $\xi_+ \in C_{\mathrm{c}}^\infty(S)$ introduce $\phi^+[\xi_+] \in \mathcal{N} \cap B^*$ by* (3.26) *and define $\phi \in \mathcal{E}_\lambda$ and $\xi_- \in \mathcal{G}$ by*

$$\begin{aligned} \phi &= \phi^+[\xi_+] - \psi_+ - (\lambda - \mathrm{i})R(\lambda - \mathrm{i}0)\psi_+, \\ \xi_- &= (\lambda - \mathrm{i})F^-(\lambda)\psi_+; \quad \psi_+ = R(\mathrm{i})(H - \lambda)\phi^+[\xi_+]. \end{aligned} \tag{3.52}$$

*Then* (1.50a) *and* (1.50b) *hold for $\{\xi_-, \xi_+, \phi\}$.*

*Proof of Theorem 1.19.* Let any $\xi_- \in \mathcal{G}$ be given, and choose a sequence $\xi_{-,n} \in C_{\mathrm{c}}^\infty(S)$ such that $\xi_{-,n} \to \xi_-$ in $\mathcal{G}$ as $n \to \infty$. By Lemma 3.17 we have

$$\mathrm{i}F^-(\lambda)^*\xi_{-,n} - \phi^+[S(\lambda)\xi_{-,n}] + \phi^-[\xi_{-,n}] \in B_0^*$$

(with the approximate eigenfunctions of (1.48)). By the continuity of $F^-(\lambda)^*$, $S(\lambda)$ and $\phi^\pm[\,\cdot\,]$ we obtain, letting $n \to \infty$,

$$\mathrm{i}F^-(\lambda)^*\xi_- - \phi^+[S(\lambda)\xi_-] + \phi^-[\xi_-] \in B_0^*. \tag{3.53}$$

Whence (1.50a) and (1.50b) hold for $\{\xi_-, S(\lambda)\xi_-, \mathrm{i}F^-(\lambda)^*\xi_-\}$, and the existence part of (i) follows when $\xi_- \in \mathcal{G}$ is given first. We can proceed similarly using Lemma 3.18 when $\xi_+ \in \mathcal{G}$ is given first, and whence, with Lemma 3.15, the existence part of (i) is completed. In addition, the correspondences for either $\xi_- \in \mathcal{G}$ or $\xi_+ \in \mathcal{G}$ given first are given by (1.50b).

To complete (i) it remains to prove the uniqueness part. Note that we already have a partial result in Lemma 3.14 (for $\phi$ given first). Let $\xi_- \in \mathcal{G}$ be given and suppose that $\phi - \phi^+[\xi_+] + \phi^-[\xi_-] \in B_0^*$ for some $\phi \in \mathcal{E}_\lambda$ and $\xi_+ \in \mathcal{G}$. By linearity we may assume that $\xi_- = 0$, and it suffices to show that $\xi_+ = 0$ and $\phi = 0$. Clearly by Lemma 3.14 the vector $\xi_+ = 0$ and whence $\phi \in B_0^*$. By Theorem 1.4 it then follows that $\phi = 0$. We can argue similarly if $\xi_+ \in \mathcal{G}$ is given. We have shown (i) and the formulas (1.50b). The assertion (1.50c) for the upper sign follows from (3.33). We can argue similarly for the lower sign. Whence (ii) is shown.

The formulas (1.50d) are immediate consequences of (3.48a) and (3.48b), and in combination with (i) and (ii) we conclude that indeed $F^\pm(\lambda)^*\colon \mathcal{G} \to \mathcal{E}_\lambda\,(\subseteq B^*)$ are bi-continuous. We have shown (iii).

Finally, since $F^{\pm}(\lambda)^*$ are injective and have closed range in $B^*$ (by (iii)), Banach's closed range theorem [Yo, Theorem p. 205] implies that the range of $F^{\pm}(\lambda)$ for both signs coincides with $\mathcal{G}$. We conclude that the range of $\delta(H-\lambda) = (2\pi)^{-1}F^{\pm}(\lambda)^*F^{\pm}(\lambda)$ coincides with $\mathcal{E}_\lambda$. Hence (iv) is shown. □

3.5. **Counter examples, open problems.** We consider modifications of the model of Example 1.18 and show that the asymptotics of the generalized eigenfunctions in $B^*$ for these models are *not* given by (1.50a). Fix $\kappa \in (0,1)$, let $\theta := xy^{-\kappa}$ for $y>0$ and let $r^2 := \kappa x^2 + y^2$. Consider $M \subset \mathbb{R}^2$ with an end described as

$$E = \{(x,y) \in \mathbb{R}\times\mathbb{R}_+ | \quad r > r_0, \quad -1 < \theta < 1\},$$

which is a cylinder in the variables $r$ and $\theta$. The (inverse) metric in these coordinates are

$$g^{rr} = N_r := |\mathrm{d}r|^2, \quad g^{\theta\theta} = N_\theta := |\mathrm{d}\theta|^2, \quad g^{r\theta} = 0.$$

Using the short-hand notation $|g| = \det g = N_r^{-1}N_\theta^{-1}$ we compute

$$\begin{aligned} |g|^{1/4}\Delta|g|^{-1/4} &= \partial_r N_r \partial_r + \partial_\theta N_\theta \partial_\theta + W_r + W_\theta; \\ W_r &= -N_r(\partial_r \ln|g|)^2/16 - (\partial_r N_r \partial_r \ln|g|)/4, \\ W_\theta &= -N_\theta(\partial_\theta \ln|g|)^2/16 - (\partial_\theta N_\theta \partial_\theta \ln|g|)/4. \end{aligned}$$

We also compute

$$\begin{aligned} &\partial_r x = \tfrac{\kappa x}{rN_r}, \quad \partial_\theta x = \tfrac{1}{y^\kappa N_\theta}, \quad \partial_r y = \tfrac{y}{rN_r}, \quad \partial_\theta y = -\tfrac{\kappa\theta}{yN_\theta}, \\ &N_r = 1 - (\kappa-\kappa^2)\theta^2 y^{2\kappa} r^{-2}, \quad N_\theta = y^{-2\kappa} + \tfrac{\kappa^2\theta^2}{y^2}, \\ &\partial_r N_r = 2(1-\kappa N_r^{-1})(\kappa-\kappa^2)\theta^2 y^{2\kappa} r^{-3} = O(r^{2\kappa-3}), \quad \partial_r^2 N_r = O(r^{2\kappa-4}), \\ &\quad \partial_\theta N_r = -2(1-\kappa^2 N_\theta^{-1}\theta^2 y^{-2})(\kappa-\kappa^2)\theta y^{2\kappa} r^{-2} = O(r^{2\kappa-2}), \quad \partial_\theta^2 N_r = O(r^{2\kappa-2}), \\ &\partial_r N_\theta = -\tfrac{2\kappa}{rN_r}\big(y^{-2\kappa} + \kappa\theta^2 y^{-2}\big) = O(r^{-1-2\kappa}), \quad \partial_r^2 N_\theta = O(r^{-2-2\kappa}), \\ &\partial_\theta N_\theta = \tfrac{2\kappa^2\theta}{y^2 N_\theta}\big(y^{-2\kappa} + \kappa\theta^2 y^{-2}\big) + \tfrac{2\kappa^2\theta}{y^2} = \tfrac{4\kappa^2\theta}{r^2}\big(1 + O\big(r^{2\kappa-2}\big)\big), \\ &\quad \partial_\theta^2 N_\theta = \tfrac{4\kappa^2}{r^2}\big(1 + O\big(r^{2\kappa-2}\big)\big). \end{aligned}$$

Using these formulas and $\partial_* \ln|g| = -(\partial_* N_r)/N_r - (\partial_* N_\theta)/N_\theta$ we get

$$W_r = O(r^{-2}), \quad W_\theta = O(r^{-2}).$$

We consider for $\kappa \in (0,1/2]$ the approximate outgoing eigenfunction (corresponding to any $\lambda > 0$ and here with $r_0 = r_0(\lambda)$ chosen big enough)

$$\begin{aligned} \phi^+ &:= \bar\chi_n |g|^{-1/4} b^{-1/2} \mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} u(\theta) \\ &\approx C(\lambda) r^{-\kappa/2} \mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} u(\theta). \end{aligned} \tag{3.54}$$

Here

$$b = \sqrt{2(\lambda - \tfrac{\mu(\lambda)}{r^{2\kappa}})} \approx \sqrt{2\lambda} - \tfrac{\mu(\lambda)}{\sqrt{2\lambda}} r^{-2\kappa},$$

$u = u(\theta)$ is any Dirichlet eigenstate of the operator on $L^2((-1,1),\mathrm{d}\theta)$ given by

$$\begin{aligned} H_D &:= -\tfrac{1}{2}\partial_\theta^2 \text{ for } \kappa < 1/2, \\ H_D &:= -\tfrac{1}{2}\partial_\theta^2 - \tfrac{\lambda\theta^2}{4} \text{ for } \kappa = 1/2, \end{aligned}$$

and $\mu(\lambda)$ is the corresponding eigenvalue. To see why this is an approximate eigenfunction we first note that $\phi^+ \in \mathcal{N} \cap B^*$. We claim that in fact

$$(H-\lambda)\phi^+ \in r^{2\kappa-2}B^* \subseteq B \text{ for } \kappa < 1/2,$$
$$(H-\lambda)\phi^+ \in r^{-2}B^* \subseteq B \text{ for } \kappa = 1/2.$$

We compute for $\kappa = 1/2$ (skipping the details for $\kappa < 1/2$)

$$\partial_r N_r \partial_r \big(b^{-1/2} \mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} u\big) = \big(-b^2 + \tfrac{\lambda\theta^2}{2r} + O(r^{-2})\big) b^{-1/2}\mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} u,$$
$$\partial_\theta N_\theta \partial_\theta \big(b^{-1/2} \mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} u\big) = b^{-1/2}\mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} \big(\tfrac{1}{r}\partial_\theta^2 u + O(r^{-2})\big).$$

In the first identity we substitute $b^2 = 2(\lambda - \frac{\mu(\lambda)}{r})$. Then we collect our computations and indeed obtain

$$(H-\lambda)\phi^+ = (H-\lambda)\phi^+ - |g|^{-1/4} b^{-1/2} \mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} r^{-1}\big(H_D - \mu(\lambda)\big)u \in r^{-2}B^*.$$

Next we define

$$\phi_u = \phi^+ - R(\lambda - \mathrm{i}0)(H-\lambda)\phi^+.$$

This $\phi_u$ is in $\mathcal{E}_\lambda$ with non-trivial prescribed outgoing asymptotics. If we look at all eigenstates of $H_D$, say numbered by $k \in \mathbb{N}$, we obtain several generalized eigenfunctions this way. Note that for $\kappa = 1/2$

$$\mathrm{e}^{\mathrm{i}\int_{r_0}^r b\,\mathrm{d}r} \approx \mathrm{e}^{\mathrm{i}\sqrt{2\lambda}r} \exp\big(-\mathrm{i}\tfrac{\mu(\lambda)}{\sqrt{2\lambda}} \ln r\big).$$

Due to the non-trivial factor

$$\exp\big(-\mathrm{i}\tfrac{\mu(\lambda,k)}{\sqrt{2\lambda}} \ln r\big)$$

the asymptotics (1.50a) is readily seen to be *incorrect* (seen by using just two of the constructed generalized eigenfunctions). By a similar reasoning this conclusion is also valid for $\kappa < 1/2$.

The methods of this paper (in combination with other ingredients) should yield a modification of Theorem 1.19 where the asymptotics of *any* $\phi \in \mathcal{E}_\lambda$ should be provided by functions of the form (3.54) and their incoming counterparts, say in combination denoted by $\{\phi_k^\pm | k \in \mathbb{N}\}$. This would intuitively yield the identification of the limiting space as $\mathcal{G} = l^2(\mathbb{N})$, but we shall not elaborate at this point.

For $\kappa \in (1/2, 1)$ we do not know how to construct approximate outgoing eigenfunctions in $\mathcal{N} \cap B^*$. If for example we take $b = \sqrt{2\lambda}$ and $u$ any nonzero function in the domain of the Dirichlet Laplacian on $(-1, 1)$ in (3.54) we obtain

$$(H - \lambda N_r)\phi^+ \in r^{-2\kappa}B^* \subseteq B,$$

which shows that

$$(H-\lambda)\phi^+ \notin B,$$

since $1 - N_r \approx (\kappa - \kappa^2)\theta^2 r^{2\kappa-2}$ is long-range for $\kappa \in (1/2, 1)$. The reader might think that a better approximation to the eikonal equation than $\sqrt{2\lambda}r$ could be given to construct concrete approximate outgoing eigenfunctions in $\mathcal{N} \cap B^*$ to cure this deficiency, however a closer examination indicates that this is not feasible (note that the forward flow property is a severe restriction). The ansatz (1.48) has a similar deficiency. Whence the asymptotics of the generalized eigenfunctions in $\mathcal{E}_\lambda$ is not known to us for $\kappa \in (1/2, 1)$.

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