Čech cohomology

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$$(d\alpha)(U_0, \ldots, U_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \alpha(U_0 \cap \cdots \cap \widehat{U}_i \cap \cdots \cap U_{k+1})$$

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We have $d^2 = 0$ and can hence define $\check{Z}^k(\mathcal{U}, \mathcal{F})$, $\check{B}^k(\mathcal{U}, \mathcal{F})$, and $\check{H}^k(\mathcal{U}, \mathcal{F}) = \check{Z}^k(\mathcal{U}, \mathcal{F}) / \check{B}^k(\mathcal{U}, \mathcal{F})$, the **$k$th Čech cohomology of the cover $\mathcal{U}$**.

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Goal: Free this from dependence on $\mathcal{U}$.

Let $\mathcal{V}$ be a (totally ordered) refinement of $\mathcal{U}$ with **refinement map** $\tau \colon \mathcal{V} \to \mathcal{U}$, meaning that $V \subset \tau(V)$ for all $V \in \mathcal{V}$.

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Proof. If τ' is another, we define a chain homotopy $h\colon \check{C}^{k+1}(\mathcal{U}, \mathcal{F}) \to \check{C}^k(\mathcal{V}, \mathcal{F})$ such that $\tau^{\#} - (\tau')^{\#} = hd + dh$.

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Proceed as in the proof of Theorem 2.10 in Hatcher: *Algebraic Topology*. □

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Then $\{\check{H}^k(\mathcal{U}, \mathcal{F}) \mid \mathcal{U} \text{ cover}\}$ is a directed system. Hence we may define $\check{H}^k(M, \mathcal{F}) = \varinjlim \check{H}^k(\mathcal{U}, M)$, the **$k$th Čech cohomology of M with respect to $\mathcal{F}$**.

Lemma 2

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Theorem 3

For any fine resolution $0 \to \mathcal{F} \to \mathcal{F}^*$ and any corresponding FR-trivial cover $\mathcal{U}$, there is a canonical isomorphism $\check{H}^k(\mathcal{U}, \mathcal{F}) \cong H^k(M, \mathcal{F})$ to the sheaf cohomology of M.

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Corollary 4

1. $\check{H}^k(M, \mathcal{F}) \cong H^k(M, \mathcal{F})$*, because FR-trivial covers are cofinal among covers (very non-trivial!).*

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Corollary 4

1. *$\check{H}^k(M, \mathcal{F}) \cong H^k(M, \mathcal{F})$, because FR-trivial covers are cofinal among covers (very non-trivial!).*
2. *If $\mathcal{U}$ satisfies $\check{H}^k(U_0 \cap \ldots \cap U_k, \mathcal{F}) = 0$, for all finite intersections and all $k > 0$, then $\check{H}^k(\mathcal{U}, \mathcal{F}) \cong \check{H}^k(M, \mathcal{F})$ (proved using spectral sequences).*

Consider the following bicomplex (meaning that all squares anticommute), with all rows and columns exact but the first ones.

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 & & \check{C}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}) & \xrightarrow[\alpha_0]{} & \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
\mathcal{F}^0(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\alpha_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\beta_0]{} & \cdots & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\mathcal{F}^1(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^1) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^1) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}^1) & \longrightarrow & \cdots & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\mathcal{F}^2(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \cdots & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & &
\end{array}
$$

Consider the following bicomplex (meaning that all squares anticommute), with all rows and columns exact but the first ones.

$$
\begin{array}{ccccccccccc}
 & & \check{C}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}) & \underset{\alpha_0}{\longrightarrow} & \cdots & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
\mathcal{F}^0(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^0) & \underset{\alpha_1}{\longrightarrow} & \check{C}^2(\mathcal{U},\mathcal{F}^0) & \underset{\beta_0}{\longrightarrow} & \cdots & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
\mathcal{F}^1(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^1) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^1) & \underset{\beta_1}{\longrightarrow} & \check{C}^2(\mathcal{U},\mathcal{F}^1) & \longrightarrow & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
\mathcal{F}^2(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

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 & & \downarrow & & \downarrow & & \downarrow & & & \\
\mathcal{F}^0(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\alpha_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\beta_0]{} & \cdots & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\mathcal{F}^1(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^1) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\beta_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^1) & \xrightarrow[0]{} & \cdots & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\mathcal{F}^2(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \cdots & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & \check{C}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}) & \xrightarrow[\alpha_0]{} & \cdots \quad 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^0(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\alpha_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\beta_0]{} & \cdots \quad 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^1(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\alpha_1]{} & \check{C}^1(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\beta_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^1) & \xrightarrow[0]{} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^2(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
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 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^0(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\alpha_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\beta_0]{} & \cdots \quad 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^1(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\alpha_1]{} & \check{C}^1(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\beta_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^1) & \xrightarrow[0]{} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^2(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^2) & \xrightarrow[\beta_2]{} & \check{C}^1(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
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 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^0(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\alpha_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\beta_0]{} & \cdots \quad 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^1(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\alpha_1]{} & \check{C}^1(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\beta_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^1) & \xrightarrow[0]{} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^2(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^2) & \xrightarrow[\beta_2]{} & \check{C}^1(\mathcal{U},\mathcal{F}^2) & \xrightarrow[0]{} & \check{C}^2(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & \check{C}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}) & \xrightarrow[\alpha_0]{} & \cdots \quad 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^0(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\alpha_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow[\beta_0]{} & \cdots \quad 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^1(M) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\alpha_1]{} & \check{C}^1(\mathcal{U},\mathcal{F}^1) & \xrightarrow[\beta_1]{} & \check{C}^2(\mathcal{U},\mathcal{F}^1) & \xrightarrow[0]{} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^2(M) & \xrightarrow[\alpha_2]{} & \check{C}^0(\mathcal{U},\mathcal{F}^2) & \xrightarrow[\beta_2]{} & \check{C}^1(\mathcal{U},\mathcal{F}^2) & \xrightarrow[0]{} & \check{C}^2(\mathcal{U},\mathcal{F}^2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$