Non-standard analysis and hyperreal numbers

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QGM Retreat

October 29, 2016

Differentiation

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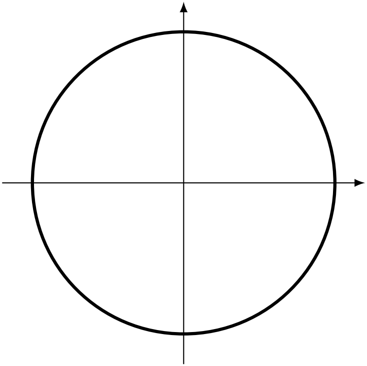

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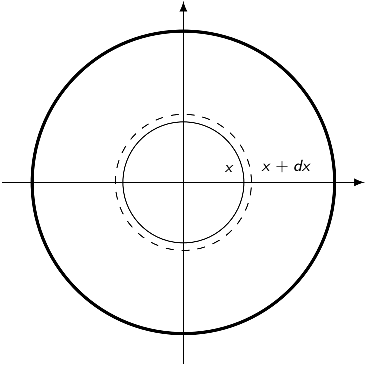

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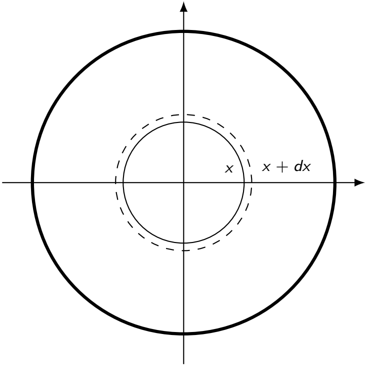

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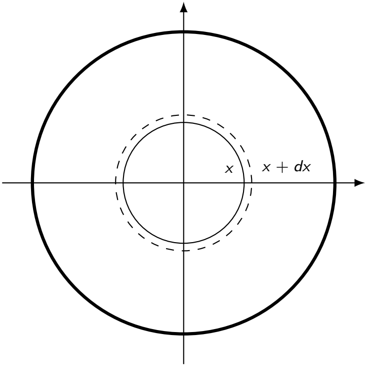

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$$A = \int dA = 2\pi \int_0^r x\,dx = \pi r^2. \qquad \square$$

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They can only be constructed abstractly using Zorn's Lemma. Uniqueness of ${}^*\mathbb{R}$ is equivalent to the Continuum Hypothesis.

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All relations and statements from $\mathbb{R}$ have a natural extension to ${}^*\mathbb{R}$. The new statement is *equivalent* to the old one.

Non-standard calculus

A function $u(x)$ is **differentiable** at $x_0 \in \mathbb{R}$ if

$$u'(x_0) = \mathrm{st}\left(\frac{\Delta u}{\Delta x}\right) = \mathrm{st}\left(\frac{u(x_0+\Delta x)-u(x_0)}{\Delta x}\right)$$

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for $dx > 0$ infinitesmal. Call f **integrabel** if $\int_a^b f(x)\,dx$ exists and is independent of dx.

Theorem 3 (Fundamental Theorem of Calculus)

If $f : [a, b] \to \mathbb{R}$ is continuous, then $F(t) = \int_a^t f(x)\, dx$ is differentiable for $t \in (a, b)$ with $F' = f$.

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$$u \approx 0: \qquad f(c) \approx f(m) \le \frac{F(c + u) - F(c)}{u} \le f(M) \approx f(c).$$

□

Hyperintegers

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A sequence $a\colon \mathbb{N} \to \mathbb{R}$ has a natural extension $a\colon {}^*\mathbb{N} \to {}^*\mathbb{R}$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ **converges** to a if we have $a_H \approx a$ for all infinitely great $H \in {}^*\mathbb{N}$.

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$$f(c) \leq s \leq f(c),$$

since f is continuous. □

A **filter** on $\mathbb{N}$ is a non-empty subset $F \subset \mathcal{P}(\mathbb{N})$ satisfying

- If $U \in F$ and $U \subset V$ then $V \in F$.
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Call F an **ultrafilter** if we also have

- For all $U \in \mathcal{P}(\mathbb{N})$, either $U \in F$ or $\mathbb{N} \setminus U \in F$.

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Call F an **ultrafilter** if we also have

- For all $U \in \mathcal{P}(\mathbb{N})$, either $U \in F$ or $\mathbb{N} \setminus U \in F$.

Call it **free** if

- F contains no finite sets.

A **filter** on $\mathbb{N}$ is a non-empty subset $F \subset \mathcal{P}(\mathbb{N})$ satisfying

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F is not of the form $F = \{ U \subset \mathbb{N} \mid u \in U \}$ for any u.