Representation theory for algebraic groups and quantum groups

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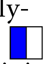
October 10, 2016

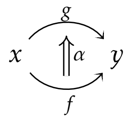
Traditionally, semisimple Lie algebras have been regarded as 'rigid' objects, impossible to deform continuously—a process known as *quantization*. In other words, a semisimple Lie algebra $\mathfrak{g}$ cannot be identified with the limit $q \to 1$ of a non-trivial family $\mathfrak{g}_q$ of Lie algebras, parametrized by a formal variable q. Therefore, it came as a surprise that this is possible once we replace $\mathfrak{g}$ by its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, whose representation theory is known to be entirely equivalent to that of $\mathfrak{g}$. The quantization $\mathfrak{U}_q(\mathfrak{g})$ is called the *quantum group* of $\mathfrak{g}$, and it is most easily defined through generators and relations, which are shown on the right in the simplest case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (following Jantzen 1996, p. 9). In the limit $q \to 1$, we recover the classical, finite-dimensional representation theory for the corresponding algebraic group.

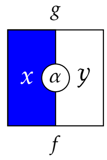
$$\mathbb{C}(q)[E, F, K^{\pm 1}]$$

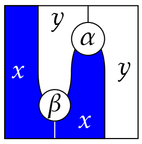
$$KEK^{-1} = q^2 E$$
$$KFK^{-1} = q^{-2} F$$
$$[E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

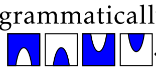
The discovery of quantum groups has had great influence on a wide range of mathematical disciplines in the past three decades. In representation theory, one topic of interest is *tilting modules* over quantum groups. Tilting modules may be regarded as a generalization of projective modules and constitute, in many cases, a richer class of modules. The study of tilting modules over quantum groups in prime characteristic was initiated by Donkin (1993), who proves, among other things, that indecomposable tilting modules are parametrized by their highest weight. He gives an explicit construction of these tilting modules, which however gives no clue about how to determine their *characters*. The existing solutions to the character problem, due to Soergel (1997) and based on deep results from algebraic geometry and representation theory, only cover the case where the base field is the complex numbers.

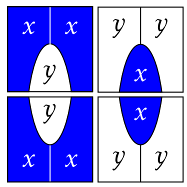
A recent breakthrough in the theory came when Riche and Williamson (2015) introduced an algebraic proof for Lie algebras of type A_n in the $q = 1$ case when the base field has prime characteristic $p > n$. The proof follows the modern trend of *categorifying* the problem, reformulating the question entirely by replacing $\mathfrak{U}_q(\mathfrak{g})$ by a category and using the language of morphisms and functors. Once again, in the spirit of abstract algebra, forgetting the concrete objects in order to make the underlying structure more transparent turned out to be surprisingly fruitful. The proof uses the modern methods of *diagrammatic algebra*, which is based on a graphical approach to the theory of *2-categories*. Recall that such a category consists of the same data as an ordinary (1-)category, but with an additional collection of 2-morphisms, maps between (1-)morphisms, which have their own composition rules and are subject to appropriate associativity-like axioms. A 2-morphism is usually notated $\alpha\colon f \Rightarrow g$ and illustrated with a diagram as on the upper right. In diagrammatic algebra, such a diagram is replaced by a picture as on the lower right, where the 'dimensions' of the entries in the diagram are inverted. The objects (zero-dimensional) become two-dimensional areas of a plane, the 1-morphisms stay one-dimensional, while the 2-morphisms (two-dimensional) become points. By convention, identity morphisms and 2-morphisms may be omitted, and often symbols are left out in order to focus on the underlying *shape*, so that the picture from before becomes simply . Vertical and horizontal composition of morphisms is notated simply by joining together different pictures in order to create more elaborate shapes as the one on the left, which with our conventions can be simplified to . The merit of this seemingly arbitrary graphical trickery becomes more evident once we realize that familiar statements about equality of compositions can be reformulated in terms of homotopies.

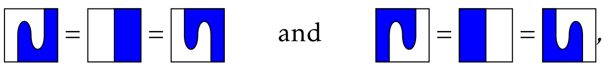




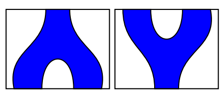


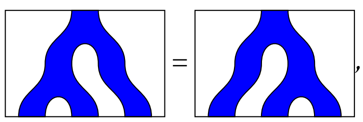
We shall now sketch the not at all obvious process of how diagrammatics becomes a natural language for quantum groups upon categorification and, more generally, for algebras in general. It turns out to be useful to add an additional level of structure and work with *Frobenius algebras*, bialgebras satisfying an additional number of assumptions—despite the fact that quantum groups do not admit such a structure themselves. Frobenius algebras arise naturally in the world of 2-categories: Suppose such a category $\mathscr{C}$ has a *biadjunction*, that is, a pair of maps $x \underset{g}{\overset{f}{\rightleftarrows}} y$ between objects along with a collection of 2-morphisms $1_x \Rightarrow gf \Rightarrow 1_x$ and $1_y \Rightarrow fg \Rightarrow 1_y$, subject to a number of axioms. In classical categorical language, these axioms are notationally heavy, but become beautifully simple once we represent the maps diagrammatically as on the right and abbreviate the diagrams . Then the axioms become

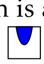
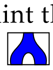




which demonstrates for the first time the striking similarity to homotopies. We aim to equip the composition fg with a multiplication and comultiplication turning it into a Frobenius algebra living inside the monoidal category $\mathrm{Hom}_{\mathscr{C}}(y,y)$. These maps are obtained by appropriate compositions of the maps from before and are represented by the diagrams on the right. Associativity of multiplication, for instance, now simply becomes a matter of equality of the diagrams





which once again is a hint that homotopies might be involved. As the diagrams suggest, joining  to  produces a shape homotopic to , the identity, hence  is a unit for multiplication. Similarly,  is counit for comultiplication.

Another advantage of the categorical viewpoint is that it leads naturally to *crystal bases* or *canonical bases*, a general phenomenon that has helped shape representation theory for the last three decades. The universal concept of canonical bases is informal, being in general just a principle that has led to good bases in many concrete algebraic settings. They were first introduced in a fundamental paper by Kazhdan and Lusztig (1979) into the theory of Hecke algebras, where an appropriate choice af basis leads to structure constants that can be described in terms of *Kazhdan–Lusztig polynomials* (see Humphreys 1990, Theorem 7.9, p. 158). The concept has since then spread into many branches of representation theory, notably the study of quantum groups, where it is an important tool in the diagrammatic Hecke category studied in details by Elias and Williamson (2014) and others during the last decade. In the case where the underlying field has prime characteristic p, the bases are called p-canonical. One of the main ideas behind the breakthrough by Riche and Williamson (2015) is that the elements in the p-canonical basis in the diagrammatic category correspond exactly to the indecomposable tilting modules in the representation category.

I aim for a relatively open project in which I can explore some of the many unsolved problems that still remain in the theory of tilting modules over quantum groups. There are current, quite promising attempts to generalize the proof by Riche and Williamson to other types than just A_n. This will provide the community with new perspectives on the field as a whole and give plenty of opportunities to expand our knowledge in several possible directions. In particular, there are many gaps to be filled in our understanding of the relation between canonical and p-canonical bases, of how to produce concrete character formulas in the diagrammatic category, and how the recent developments will have consequences for the vast amount of other applications of quantum groups, notably in fields like algebraic geometry.

To educate myself for the above mentioned project, I have started looking into the theory of Soergel bimodules and diagrammatics, partially in the form of an informal seminar together with my supervisor as well as Sergey Arkhipov and Jan Christensen. We focus on the theory of the BGG Category $\mathcal{O}$, its collection of various kinds of modules, including standard and tilting modules, and initiate its generalization to prime characteristic. Simultaneously, we are working ourselves towards the theory of diagrammatic categories, in which none of us are currently educated. My other activities include participation in Arkhipov's course on homological algebra as well as a continuation of our existing collaboration on the Schur–Weyl correspondence, a topic very closely related to the above mentioned one. Finally, I participate in a course on topological K-theory, both for the sake of my general education and as an introduction to the algebraic side of the subject.

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