Simplicial Methods in Homological Algebra

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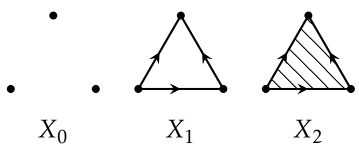
The study of simplicial sets (and simplicial objects in general) is a way to extract the purely combinatorial and algebraic considerations from the study of topological simplices, fundamental to algebraic topology. Surprisingly, this allows us to carry out significant parts of the well-known topological constructions like simplicial (co)homology without directly involving topology. At the same time, standard constructions like homotopy theory become much broader algebraic and categorical techniques, applicable far outside the realm of topological spaces.

We consider the **simplicial category $\mathbf{\Delta}$** defined in the following way: The objects in $\mathbf{\Delta}$ are finite ordered sets of the form $[n] = \{0, 1, \dots, n\}$ for all nonnegative integers n, and the morphisms $[m] \to [n]$ are order-preserving maps. Given a category $\mathscr{C}$, a **simplicial object** in $\mathscr{C}$ is a contravariant functor from $\mathbf{\Delta}$ to $\mathscr{C}$, that is, a functor $X\colon \mathbf{\Delta}^{\mathbf{op}} \to \mathscr{C}$. To ease notations, we usually write X_n instead of $X([n])$ and consequently denote a simplicial object by $X_{\bullet}$. The map $X_n \to X_m$ induced by a map $\alpha\colon [m] \to [n]$ is denoted α^*. The simplicial objects in a category $\mathscr{C}$ form a category $\mathbf{\Delta}^{\mathbf{op}}\mathscr{C}$, the morphisms being simply natural transformations of functors.



One case of particular interest is the category $\mathbf{\Delta}^{\mathbf{op}}\mathbf{Set}$ of **simplicial sets**. If $X_{\bullet}$ is one such, we usually think of X_n at the set of cells in the n-skeleton of some simplicial complex with a fixed orientation of the cells. We shall see later that there are two distinguished classes of simplicial maps, *face* and *degeneracy* maps. The degeneracy maps are embeddings $X_{n-1} \hookrightarrow X_n$ of one skeleton into the skeleton one dimension higher. The lower-dimensional cells of X_n obtained from these embeddings are not counted as "true" n-cells and are hence called *degenerate*, in contrast to the *non-degenerate* n-cells. The face maps are degree-decreasing maps $X_n \to X_{n-1}$ that associate to each cell its faces. For instance, there are two simplicial maps $\alpha^*\colon X_1 \to X_0$, mapping each non-degenerate 1-cell (i.e., edge) to its beginning and end points. All maps turn out to be compositions of face and degeneracy maps.

The **standard n-simplex** is the simplicial set $\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$. With the interpretation from the last paragraph, this corresponds to the topological n-simplex regarded as a simplicial complex, hence the notation. Indeed, the example drawn above is just Δ^2. The maps $\Delta^m \to \Delta^n$ induced from maps $\alpha\colon [m] \to [n]$ will be denoted by α_*. The standard n-simplices are universal in the sense that the Yoneda Lemma provides for each simplicial $X_{\bullet}$ an isomorphism of sets

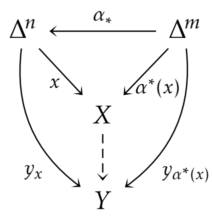
$$X_n \cong \mathrm{Hom}_{\mathbf{\Delta}^{\mathbf{op}}\mathbf{Set}}(\Delta^n, X) \tag{1.1}$$

given by $\varphi(\mathrm{id}_{[n]}) \leftarrow\!\shortmid \varphi$, which is natural in $[n]$ so that the simplicial set $X_\bullet$ is isomorphic to the contravariant functor $[n] \mapsto \mathrm{Hom}_{\mathbf{\Delta^{op}Set}}(\Delta^n, X)$.

1.1. Proposition. *The simplicial set $X_\bullet$ is a colimit*

$$X = \varinjlim_{\Delta^n \to X} \Delta^n$$

taken over the slice category consisting of all maps of simplicial sets $\Delta^n \to X$ with the obvious morphisms.



Proof. This is a special case of the categorical statement that any functor from a small category to sets is a colimit of representable functors, but we stay concrete. To keep notations clean, we ignore the distinction between elements of X_n and maps $\Delta^n \to X$. Then we must prove that for any collection of maps $y_x \colon \Delta^n \to Y$ indexed by maps $x \colon \Delta^n \to X$ (for varying n), there exists a map $X \to Y$ of simplicial sets making all diagrams like the one on the right commutative. Define this map by $x \mapsto y_x$ and notice that commutativity of the outer diagram yields $y_{\alpha^*(x)} = y_x \alpha_* = \alpha^* y_x$, which is exactly the statement that $X \to Y$ respects the simplicial structure. Uniqueness follows by applying x and y_x to $\mathrm{id}_{[n]} \in (\Delta^n)_n$ and using commutativity of the left triangle. □

There is a dual notion of **cosimplicial objects**, covariant functors $\mathbf{\Delta} \to \mathscr{C}$, where upper indices and lower stars are used. Parallel to the case of simplicial sets, we denote this category by $\mathbf{\Delta}\mathscr{C}$.

1.2. Geometric realization. The connection between the theory of topological and combinatorial simplices is the **geometric realization**. This is a functor associating to each simplicial set $X_\bullet$ a topological space $|X|$ (in fact, a simplicial complex) which formalizes much of the geometric intuition we used in the introduction. We initially define it on standard simplices, letting

$$|\Delta^n| = \{ (t_i) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \textstyle\sum t_i = 1 \}$$

be the traditional topological n-simplex. We extend it inspired by the identity from Proposition 1.1 by defining $|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|$, noting that this limit makes sense because **Top** is cocomplete. To provide a concrete construction of $|X|$, regard each X_n as discrete and put $|X| = (\coprod X_n \times |\Delta^n|)/\!\sim$, the quotient space by the equivalence relation where $(x,s) \in X_m \times |\Delta^m|$ is equivalent to $(y,t) \in X_n \times |\Delta^n|$ if there exists a map $\alpha \colon [m] \to [n]$ such that $\alpha^*(y) = x$ and $\alpha_*(s) = t$. In other words, we identify degeneracies of the same simplex. Note that this description also shows that $|X|$ is a simplicial complex.

1.3. Singular simplicial sets. In order not to limit ourselves to simplicial complexes, we need to develop methods that describe arbitrary topological spaces, ultimately leading to a reformulation of singular homology. Inspired by this very homology theory, we define for any topological space Y the **singular simplicial set** $\mathscr{S}(Y)_\bullet$ associated to Y by $\mathscr{S}(Y)_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)$. To obtain a simplicial structure on this, we simply pull back using the composed functor $\mathbf{\Delta} \to \mathbf{\Delta^{op}Set} \to \mathbf{Top}$ mapping $[n]$ to $|\Delta^n|$. Geometrically, this functor identifies elements of $[n]$ with vertices of $|\Delta^n|$ and extends maps linearly.

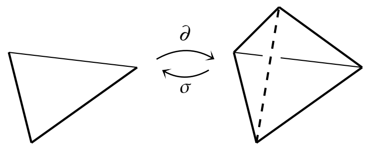
The functor $\mathscr{S}\colon \mathbf{Top} \to \mathbf{\Delta^{op}Set}$ is right adjoint to the geometric realization. This is obvious for standard simplices since the isomorphism of sets

$$\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \cong \mathrm{Hom}_{\mathbf{\Delta^{op}Set}}(\Delta^n, \mathscr{S}(Y))$$

is merely the Yoneda Lemma. The extension to arbitrary simplicial sets then follows by properties of limits:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{\Delta^{op}Set}}(X, \mathscr{S}(Y)) &= \mathrm{Hom}_{\mathbf{\Delta^{op}Set}}(\varinjlim \Delta^n, \mathscr{S}(Y))\\ &= \varprojlim \mathrm{Hom}_{\mathbf{\Delta^{op}Set}}(\Delta^n, \mathscr{S}(Y)) = \varprojlim \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)\\ &= \mathrm{Hom}_{\mathbf{Top}}(\varinjlim |\Delta^n|, Y) = \mathrm{Hom}_{\mathbf{Top}}(|X|, Y).\end{aligned}$$

1.4. Face and degeneracy maps.
Visualizing $\mathbf{\Delta}$ as a category of geometric simplices, there are two distinguished collections of maps that catch our immediate attention: Consider the geometric operations $\partial^i\colon |\Delta^{n-1}| \to |\Delta^n|$ and $\sigma^i\colon |\Delta^{n+1}| \to |\Delta^n|$ whose images double resp. miss the ith face. Thinking of these, we obtain maps $\partial^i\colon [n-1] \to [n]$ and $\sigma^i\colon [n+1] \to [n]$, called the ith **face** resp. **degeneracy** maps, given by



$$\partial^i(j) = \begin{cases} j & \text{if } j < i,\\ j+1 & \text{if } j \geq i,\end{cases} \qquad \sigma^i(j) = \begin{cases} j & \text{if } j \leq i,\\ j-1 & \text{if } j > i.\end{cases}$$

It may be shown that these generate all morphisms in $\mathbf{\Delta}$ with relations

$$\begin{aligned}\partial^j \partial^i &= \partial^i \partial^{j-1} && \text{if } i < j,\\ \sigma^j \sigma^i &= \sigma^i \sigma^{j+1} && \text{if } i \leq j,\\ \sigma^j \partial^i &= \begin{cases} \partial^i \sigma^{j-1} \\ \mathrm{id} \\ \partial^{i-1} \sigma^j \end{cases} && \begin{aligned}&\text{if } i < j,\\ &\text{if } i = j, j+1,\\ &\text{if } i > j+1.\end{aligned}\end{aligned}$$

We note the following result without proof.

1.5. Proposition. *Any map $\alpha\colon [m] \to [n]$ in $\mathbf{\Delta}$ has a unique epi-monic factorization $\alpha = \partial\sigma$. Furthermore, there exist unique decompositions $\partial = \partial^{i_1} \cdots \partial^{i_k}$ for $0 \leq i_k < i_{k-1} < \cdots < i_1 \leq n$ and $\sigma = \sigma^{j_1} \cdots \sigma^{j_l}$ for $0 \leq j_1 < j_2 < \cdots < j_l < m$.*

Thus providing a simplicial set $X_\bullet$ is equivalent to providing for each n a set X_n along with maps $\partial_i\colon X_n \to X_{n-1}$ and $\sigma_i\colon X_n \to X_{n+1}$ satisfying a similar set of relations with the order of the factors reversed:

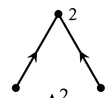
$$\begin{aligned}\partial_i \partial_j &= \partial_{j-1} \partial_i && \text{if } i < j,\\ \sigma_i \sigma_j &= \sigma_{j+1} \sigma_i && \text{if } i \leq j,\\ \partial_i \sigma_j &= \begin{cases} \sigma_{j-1} \partial_i \\ \mathrm{id} \\ \sigma_j \partial_{i-1} \end{cases} && \begin{aligned}&\text{if } i < j,\\ &\text{if } i = j, j+1,\\ &\text{if } i > j+1.\end{aligned}\end{aligned}$$

By abuse of terminology, we also call ∂_i and σ_i face resp. degeneracy maps. The elements of X_n that are images of the σ_i are called **degenerate**, and the rest are called **non-degenerate**. For instance, the standard n-simplex Δ^n has exactly one non-degenerate element in degree n, the identity map $[n] \to [n]$.

2 Homotopy theory

We aim to develop a simplicial homotopy theory which agrees with the traditional topological theory when taking the geometric realization.

2.1. Boundary, horns, and fibrations.
In preparation for simplicial homotopy theory, we need to identify some distinguished simplicial subsets of Δ^n and describe extension conditions of maps between them. The kth **face** is the simplicial subset $\partial_k\Delta^n \subset \Delta^n$ generated by $\partial_k(\mathrm{id}_{[n]})$. Equivalently, it is the image of the embedding $\partial_k(\mathrm{id}_{[n]})\colon \Delta^{n-1} \hookrightarrow \Delta^n$. Note that the face map $\partial_k\colon X_n \to X_{n-1}$, considered by (1.1) as a map $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Delta^{n-1}, X)$, is simply restriction to $\partial_k\Delta^n \cong \Delta^{n-1}$. The union of all faces is the **boundary** $\partial\Delta^n$. Notice that $\partial\Delta^0 = \varnothing_\bullet$, the empty simplicial set. Inside the boundary are the $n+1$ **horns**: The kth horn Λ^n_k is the union $\bigcup_{j\neq k}\partial_j\Delta^n$ of all faces except the kth one. Taking the geometric realization, we recover the topological boundary, face, and horn. One may to formulate the definition of a Serre fibration is as a map $p\colon E \to B$ of topological spaces satisfying the **horn filling condition** (the central right diagram): Given a map $|\Delta^n| \to B$, any partial lift $|\Lambda^n_k| \to E$ may be extended to a lift $|\Delta^n| \to E$, corresponding to "filling the horn". Inspired by this, we define a **Kan fibration** (in the context of simplicial sets often called simply a **fibration**) to be a map $p\colon E \to B$ of simplicial sets satisfying the combinatorial horn filling condition: Given a map $\Delta^n \to B$, any partial lift $\Lambda^n_k \to E$ may be extended to a lift $\Delta^n \to E$ (the lower right diagram).



Λ^2_2

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ |\Delta^n| & \longrightarrow & B \end{array}$$

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & B. \end{array}$$

A simplicial set X where $X \to *$ is a Kan fibration is called **fibrant** or a **Kan complex**. This simply means that any map $\Lambda^n_k \to X$ may be extended to a map $\Delta^n \to X$. Equivalently, it means that given any collection of elements $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in X_n$ satisfying the compatibility conditions $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$, $i, j \neq k$, there exists $y \in X_{n+1}$ such that $\partial_i y = x_i$ for all $i \neq k$. For instance, this is the case for singular simplicial sets $\mathscr{S}(Y)_\bullet$; this follows by adjunction and because $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$.

2.2. Simplicial homotopies.
Let X and Y be simplicial sets with Y fibrant. We use suggestive notation and define $I = \Delta^1$ and (as earlier) $* = \Delta^0$. Given $f, g\colon X \to Y$, a **homotopy** between f and g is a map $H\colon X \times I \to Y$ satisfying $H|_{X\times 0} = f$ and $H|_{X\times 1} = g$, that is, the diagram on the upper right is commutative. As usual, we write $f \simeq g$ and call f and g homotopic if a homotopy exists between them. If $A \subset X$ is a simplicial subset, a **relative homotopy** or a **homotopy rel** A between two maps f and g satisfying $f|_A = g|_A =: \alpha$ is a homotopy as above which furthermore is constant on A (hence agrees with α). In other words, it makes the diagram on the lower right commutative. Both absolute and relative homotopy are equivalence relations according to Goerss and Jardine (1999), Corollary 6.2 (it depends heavily on Y being fibrant).

$$\begin{array}{ccc} X \times 1 = X & & \\ \downarrow & \searrow^{f} & \\ X \times I & \xrightarrow{H} & Y \\ \uparrow & \nearrow_{g} & \\ X \times 0 = X & & \end{array}$$

$$\begin{array}{ccc} X \times I & \xrightarrow{H} & Y \\ \uparrow & & \uparrow \alpha \\ A \times I & \longrightarrow & A \end{array}$$

$$\begin{array}{ccc} \Delta^n & \xrightarrow{f} & X \\ \big\uparrow & & \big\uparrow{\scriptstyle x_0} \\ \partial\Delta^n & \longrightarrow & * \end{array}$$

This allows us to define *simplicial homotopy groups*. If X is a fibrant simplicial set with a basepoint $x_0 \in X_0$, we may think of x_0 as an element of all X_n, namely as the image in degree n of the map $\Delta^0 \to X$ corresponding to x_0. We now define the nth **homotopy group** $\pi_n(X, x_0)$ as the set of homotopy classes (rel $\partial\Delta^n$ if you will) of maps $(\Delta^n, \partial\Delta^n) \to (X, x_0)$, which as in classical homotopy theory means that $\partial\Delta^n$ gets mapped to the point x_0. An equivalent definition mimicking the classical formulation in terms of spheres is as the set of homotopy classes of maps $(\partial\Delta^{n+1}, *) \to (X, x_0)$. The identity (1.1) means that we may as well define $\pi_n(X, x_0)$ as homotopy classes of elements of X_n, more specifically, elements $x \in X_n$ satisfying the condition $\partial_i(x) = x_0$ for all i. Two such elements $x, x' \in X_n$ are homotopic exactly if there exists an element $y \in X_{n+1}$ (the homotopy) satisfying

$$\partial_i(y) = \begin{cases} x_0 & \text{if } i < n, \\ x & \text{if } i = n, \\ x' & \text{if } i = n+1. \end{cases}$$

The proof of the equivalence of these definitions is technical and is covered in Friedman (2016), Proposition 9.7.

Notice that in the case where $n = 0$, the boundary is empty, and the definition becomes basepoint-independent. Hence we write simply $\pi_0(X)$ and refer to its elements as **path-components**. Notice that two points $x_0, x_0' \in X_0$ belong to the same path-component exactly if there exists a map $\gamma\colon I \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

To define a group structure on $\pi_n(X, x_0)$ for $n \geq 1$, let $x, y \in X_n$ represent two classes, so that $\partial_i(x) = x_0 = \partial_i(y)$ for all i. Then for all $i \neq n$ we define

$$z_i = \begin{cases} x_0, & \text{if } 0 \leq i \leq n-2, \\ x, & \text{if } i = n-1, \\ y, & \text{if } i = n+1. \end{cases}$$

This easily satisfies the compatibility conditions since $\partial_j z_i = x_0$ for all j and all $i \neq n$. Since X is fibrant, there exists $z \in X_{n+1}$ with $z_i = \partial_i(z)$ for $i \neq n$. It follows from the simplicial identities that $\partial_n(z)$ has $\partial_i(\partial_n(z)) = x_0$ for all i and hence represents a class in π_n. We therefore define the group operation in $\pi_n(X, x_0)$ by $[x][y] = [\partial_n(z)]$. It may be shown (see, for instance May 1967, pp. 9–11) that this product is well-defined and endows $\pi_n(X, x_0)$ with the structure of a group (for $n \geq 1$) which is Abelian for $n \geq 2$, in accordance with classical homotopy theory. One may also define relative homotopy groups and show how they fit into the usual long exact sequence, but we shall not expand upon this. It is shown in May (ibid.), Theorem 16.6, that the simplicial homotopy groups (including the "group" π_0) agree with the topological homotopy groups of the geometric realization $|X|$.

3 Homology theory

In this chapter, we shall concern ourselves with the category $\mathbf{\Delta^{op}Ab}$ of simplicial Abelian groups (though there are generalizations of our results to arbitrary Abelian categories). We initially note that it is possible to turn any simplicial Abelian group $A_{\bullet}$ into a chain complex: We simply think of A_n as the nth term of the complex and define the differential map $d\colon A_n \to A_{n-1}$ by

$$d = \sum_{i=0}^{n} (-1)^n \partial_i,$$

the alternating sum of the face operators $\partial_i\colon A_n \to A_{n-1}$. This allows us to define the **homology** $H_{\bullet}(A)$ of the simplicial Abelian group $A_{\bullet}$ as the homology of this complex.

We use these considerations to associate to any simplicial set $X_{\bullet}$ a homology theory with coefficients in a ring R. Define $(R^X)_{\bullet}$ to be the simplicial R-module given by $(R^X)_n = R^{X_n}$, the free R-module on the set X_n. The homology groups (with coefficients in R) associated to this complex are called **simplicial homology groups** with coefficients in R and are denoted $H_{\bullet}(X;R)$. This complex (and hence its homology) coincides with the one used to define the simplicial homology $H_{\bullet}(|X|;R)$ of the simplicial complex $|X|$.

Returning to our arbitrary simplicial Abelian group $A_{\bullet}$, there is a distinguished subcomplex of $A_{\bullet}$, namely the **normalized complex** $NA_{\bullet}$ given by

$$NA_n = \bigcap_{i=0}^{n-1} \operatorname{Ker}(\partial_i\colon A_n \to A_{n-1}),$$

the differential map being $d = (-1)^n \partial_n$. We regard the empty intersection as the whole group, hence $NA_0 = A_0$. Its homology groups are called **homotopy groups** and are denoted $\pi_{\bullet}(A) = H_{\bullet}(NA_{\bullet})$. It is not hard to see that this agrees with the homotopy groups of the underlying simplicial set as defined earlier (taking zero to be the basepoint). Somewhat surprisingly, we have

3.1. Theorem. *The inclusion $NA_{\bullet} \hookrightarrow A_{\bullet}$ is a quasi-isomorphism, hence we have $\pi_{\bullet}(A) \cong H_{\bullet}(A)$.*

The proof of this relies on a lemma which will come in handy later on as well. We denote by $DA_{\bullet}$ the subcomplex of $A_{\bullet}$ consisting of degenerate elements, which in this case consists of sums of images of the degeneracy maps σ_i.

3.2. Lemma. *The complex $A_{\bullet}$ splits as $A_{\bullet} = NA_{\bullet} \oplus DA_{\bullet}$, hence $NA_{\bullet} \cong A_{\bullet}/DA_{\bullet}$.*

Proof. Since $NA_{\bullet}$ and $DA_{\bullet}$ are already known to be subcomplexes, it amounts to proving $A_n = NA_n \oplus DA_n$ for some fixed n. To prove $NA_n \cap DA_n = 0$, suppose there exists a non-zero y in this intersection. Since y is degenerate, we may write

$$y = \sigma_i(x_i) + \sigma_{i+1}(x_{i+1}) + \cdots + \sigma_{n-1}(x_{n-1})$$

for some $x_i \in A_{n-1}$ and choose this expression with $i < n$ maximal. Then applying $\sigma_i\partial_i$ yields

$$\begin{aligned} 0 = \sigma_i\partial_i(y) &= \sigma_i\Big(x_i + \partial_i\sigma_{i+1}(x_{i+1}) + \cdots + \partial_i\sigma_{n-1}(x_{n-1})\Big) \\ &= \sigma_i\Big(x_i + \sigma_i\partial_i(x_{i+1}) + \cdots + \sigma_{n-2}\partial_i(x_{n-1})\Big) \\ &= \sigma_i(x_i) + \sigma_{i+1}\sigma_i\partial_i(x_{i+1}) + \cdots + \sigma_{n-1}\sigma_i\partial_i(x_{n-1}) \end{aligned}$$

using the simplicial identities. Thus $y = y - \sigma_i \partial_i(y)$ yields an expression with a greater choice of i, a contradiction.

To prove $A_n = NA_n + DA_n$, we proceed somewhat like before, letting $y \in A_n$ and picking the smallest integer i such that $\partial_i(y) \neq 0$. Then we consider $y' = y - \sigma_i \partial_i(y)$ which satisfies $\partial_i(y') = 0$, while for $j < i$ we have

$$\partial_j(y') = \partial_j(y) - \partial_j \sigma_i \partial_i(y) = \partial_j(y) - \sigma_{i-1} \partial_{i-1} \partial_j(y) = 0$$

using the simplicial identities. By downward induction on the integer i we have $y' \in NA_n + DA_n$ and hence by construction $y \in NA_n + DA_n$ as well. □

Proof of Theorem 3.1. Due to the lemma, it suffices to prove that the subcomplex $DA_.$ is acyclic. We therefore filter $DA_.$ by

$$F_i DA_n = \begin{cases} 0, & \text{if } i = 0, \\ \sigma_0(A_{n-1}) + \cdots + \sigma_p(A_{n-1}) & \text{if } 0 < i < n, \\ DA_n & \text{if } i \geq n. \end{cases}$$

Each $F_i DA_.$ is a subcomplex according to the simplicial identities. This canonically bounded filtration gives rise to a first quadrant spectral sequence with

$$E^1_{pq} = H_{p+q}(F_p DA_./F_{p-1} DA_.) \implies H_{p+q}(DA_.).$$

The claim will thus follow by proving that each complex $F_p DA_./F_{p-1} DA_.$ is acyclic. Notice that $(F_p DA_./F_{p-1} DA_.)_n$ is zero unless $0 < p < n$, so assume this is the case and note that it is a quotient of $\sigma_p(A_{n-1})$. Letting $x \in A_{n-1}$, we calculate in $F_p DA_.$ mod $F_{p-1} DA_.$ the identities

$$\begin{aligned} d\sigma_p(x) &= \sum_{i=0}^{p-1} (-1)^i \sigma_{p-1} \partial_i + (-1)^p x + (-1)^{p+1} x + \sum_{i=p+2}^{n} (-1)^i \sigma_p \partial_{i-1}(x) \\ &= \sum_{i=p+2}^{n} (-1)^i \sigma_p \partial_{i-1}(x) \end{aligned}$$

and, similarly,

$$\begin{aligned} d\sigma_p^2(x) + \sigma_p d\sigma_p(x) &= \sum_{i=p+2}^{n+1} (-1)^i \sigma_p \partial_{i-1} \sigma_p(x) + \sum_{i=p+2}^{n} (-1)^i \sigma_p^2 \partial_{i-1}(x) \\ &= (-1)^{p+2} \sigma_p(x) + \sum_{i=p+3}^{n+1} (-1)^i \sigma_p^2 \partial_{i-2}(x) + \sum_{i=p+2}^{n} (-1)^i \sigma_p^2 \partial_{i-1}(x) \\ &= (-1)^p \sigma_p(x). \end{aligned}$$

Multiplying by $(-1)^p$ we see that the identity map on $F_p DA_./F_{p-1} DA_.$ is nullhomotopic, so that $F_p DA_./F_{p-1} DA_.$ is nullhomotopic and therefore acyclic. □

4 Dold–Kan correspondence

THE SOLE PURPOSE of this chapter is to prove a classical result which essentially classifies all simplicial Abelian groups.

4.1. Dold–Kan Theorem. *There exists an equivalence of categories*

$$\mathbf{\Delta^{op}Ab} \underset{\Gamma}{\overset{N}{\rightleftarrows}} \mathbf{Ch}_+(\mathbf{Ab})$$

between $\mathbf{\Delta^{op}Ab}$ *and the category of all chain complexes of Abelian groups which are zero outside the nonnegative gradings.*

As the notation suggests, the functor $N\colon \mathbf{\Delta^{op}Ab} \to \mathbf{Ch}_+(\mathbf{Ab})$ maps $A_\bullet$ to the normalized complex $NA_\bullet$. The functor in the opposite direction requires more work:

To motivate and justify the functor Γ, we construct from $A \in \mathbf{\Delta^{op}Ab}$ a new simplicial Abelian group $\Gamma_\bullet = \Gamma(NA_\bullet)_\bullet$ by letting

$$\Gamma_n = \bigoplus_{[n]\twoheadrightarrow[k]} NA_k,$$

the sum running over all surjections in $\mathbf{\Delta}$ of $[n]$ onto some other simplex $[k]$. It is not immediately obvious how to give this the structure of a simplicial object since $NA_\bullet$ is *not* a simplicial subgroup of $A_\bullet$; it is notably not preserved by degeneracy maps. However, it easily follows from the definition of $NA_\bullet$ that it is closed under *face* maps ∂_i and hence (from Proposition 1.5) under all maps induced from *injective* maps $[l] \hookrightarrow [k]$ (in fact, all of them will be zero except the identity and ∂_n). To define the induced map $\alpha^*\colon \Gamma_n \to \Gamma_m$, pick an arbitrary summand $NA_k \subset \Gamma_n$ corresponding to some $\theta\colon [n] \twoheadrightarrow [k]$. Now find the unique epi-monic factorization $\iota\pi$ of $\theta\alpha$ as on the upper right. Then $\alpha^*\theta^* = \pi^*\iota^*$ forces the square on the lower right to be commutative. Note that $\iota^*\colon NA_k \to NA_l$ makes sense precisely because it is induced from an injection. We therefore define $\alpha^*\colon \Gamma_n \to \Gamma_m$ by mapping the summand $NA_k \subset \Gamma_n$ (corresponding to θ) to the summand $NA_l \subset \Gamma_m$ (corresponding to π) using the map ι^*. To see that this indeed defines a functor, we need to show that it respects composition, so let us show $(\alpha\beta)^* = \beta^*\alpha^*$. But this more or less boils down to choosing two epi-monic factorizations

$$\begin{array}{ccc} [k] & \overset{\iota}{\hookleftarrow} & [l] \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\pi} \\ [n] & \overset{\alpha}{\longleftarrow} & [m] \end{array}$$

$$\begin{array}{ccc} NA_k & \overset{\iota^*}{\longrightarrow} & NA_l \\ {\scriptstyle\theta^*}\downarrow & & \downarrow{\scriptstyle\pi^*} \\ A_n & \overset{\alpha^*}{\longrightarrow} & A_m \end{array}$$

$$\begin{array}{ccccc} [k] & \overset{\iota}{\hookleftarrow} & [l] & \overset{\kappa}{\hookleftarrow} & [q] \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\pi} & & \uparrow{\scriptstyle\rho} \\ [n] & \overset{\alpha}{\longleftarrow} & [m] & \overset{\beta}{\longleftarrow} & [p] \end{array}$$

and noting that $(\iota\kappa)\rho$ must be the unique epi-monic factorization of $\theta\alpha\beta$. Notice that we have defined the simplicial structure exactly so that we obtain a map $\Phi_\cdot\colon \Gamma_\cdot \to A_\cdot$ of simplicial Abelian groups: In the nth degree, the map

$$\Phi_n\colon \bigoplus_{[n]\twoheadrightarrow[k]} NA_k \to A_n$$

is obtained by pulling each summand NA_k back to A_n by the map $[n] \twoheadrightarrow [k]$.

This construction can be generalized to turn any complex $C_\cdot \in \mathbf{Ch}_+(\mathbf{Ab})$ into a simplicial Abelian group $\Gamma(C)_\cdot$ by letting

$$\Gamma(C)_n = \bigoplus_{[n]\twoheadrightarrow[k]} C_k.$$

As above, in order for this to make sense, we only need to define how to induce maps from *injective* maps on C_k. We define this by letting $\partial_i\colon C_k \to C_{k-1}$ be zero for $i < k$ and the $(-1)^k$ times the differential for $i = k$, noting that this easily respects the relations in $\mathbf{\Delta}$ and extends to all injective maps by Proposition 1.5.

The starting point of the proof of the equivalence $N\Gamma \cong \mathrm{Id}$ is the observation that we may write

$$\Gamma(C)_n = C_n \oplus \bigoplus_{[n]\twoheadrightarrow[k]\neq[n]} C_k \tag{4.1}$$

as a direct sum of the Abelian group C_n and summands corresponding to surjections $[n] \twoheadrightarrow [k]$ for $k < n$, all of which are images of degeneracy maps. It is part of the definition that the C_n summand is annihilated by face maps ∂_i for $i < n$, so that $C_n \subset N(\Gamma(C))_n$. To prove that equality holds, we appeal to Lemma 3.2 once again, noticing that the degenerate subcomplex is just the summand on the right in the decomposition (4.1). This completes one half of the proof of Dold–Kan, the isomorphism of functors $N\Gamma \cong \mathrm{Id}$. The other half, $\Gamma N \cong \mathrm{Id}$, boils down to the following:

4.2. Theorem. *The above map $\Phi_\cdot$ is an isomorphism of simplicial Abelian groups.*

Proof. We proceed by induction on n, noting that the $n = 0$ case simply reads $NA_0 \cong A_0$, which is clear. For $n > 0$, surjectivity follows from Lemma 3.2: We can easily generate the NA_n summand because of the construction of Φ_n, while the DA_n summand is in the image under degeneracy maps of elements of A_{n-1}, which lie in the image of Φ_{n-1} by induction.

To prove injectivity, suppose that (x_θ) is an element in the kernel of Φ, with x_θ lying in the summand corresponding to $\theta\colon [n] \twoheadrightarrow [k]$. Then $0 = \Phi((x_\theta)) = \sum \theta^*(x_\theta)$ shows immediately that $(\mathrm{id})^* x_{\mathrm{id}} = x_{\mathrm{id}} = 0$ because this is the only summand lying in NA_n. Thus suppose that $x_{\theta_0} \neq 0$ is a summand corresponding to some $\theta_0\colon [n] \twoheadrightarrow [k]$ with $n \neq k$, and define a partial ordering on the maps $[n] \twoheadrightarrow [k]$ by $\theta \leq \eta$ if $\theta(x) \leq \eta(x)$ for all x. We may as well assume that θ_0 is minimal with respect to this ordering. Now we may chose a section $s\colon [k] \hookrightarrow [n]$ of θ_0 by mapping each $x \in [k]$ to the maximal number in the preimage $\theta_0^{-1}(x)$. This ensures that s is not a section of any $\eta > \theta_0$. Now the fact that Φ is a simplicial map means that it commutes with s^*, meaning that $\Phi(s^*(x_\theta)) = 0$. By the induction hypothesis this implies $s^*(x_\theta) = 0$. Now the identity component of $s^*(x_\theta)$ is zero as above and consists of the summands x_θ with $\theta s = \mathrm{id}$. By the choice of θ_0 and s this is just x_{θ_0}, and we have a contradiction. This shows that Φ_n is indeed injective. □

5 Classifying space

THERE EXISTS AN INTERPRETATION of the classifying space of a group as a simplicial set. We first let $\mathscr{C}$ be an arbitrary (small) category. For any n, the object $[n] = \{0, 1, \ldots, n\}$ is a partially ordered set and hence may be regarded as a category in the usual way. The **nerve** $N\mathscr{C}_\bullet$ is the simplicial set with $N\mathscr{C}_n = \mathrm{Fun}([n], \mathscr{C})$ the set of functors $[n] \to \mathscr{C}$, or, equivalently, the set of sequences of composable maps $c_0 \to c_1 \to \cdots \to c_n$ in $\mathscr{C}$. This becomes a simplicial set in the obvious way, with face maps ∂_i composing these maps at the ith spot for $0 < i < n$ and discarding them at the end points for $i = 0, n$, and with degeneracy maps σ_i adding identities at the ith spot. Now given a (discrete) group G, we may regard it as a one-object category as usual and define the **classifying space** BG as the simplicial set $BG_\bullet = NG_\bullet$. We shall see that its geometric realization $|BG|$ recovers the classical topological notion of a classifying space.

It is not hard to see that we have $BG_n = G^n$, the nth Cartesian power of G (where we interpret G^0 as the trivial group), and that face and degeneracy maps are given by

$$\partial_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n, \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n, \end{cases}$$
$$\sigma_i(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n).$$

5.1. Proposition. *The simplicial set $BG_\bullet$ is fibrant.*

Proof. Suppose $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in BG_n$ satisfy the compatibility conditions. The proof is quite straightforward, but includes some case checking depending on the value of k, so to keep things clean, we concentrate on the case where $k \neq 0, n+1$. Then $\partial_0 x_{n+1} = \partial_n x_0$ shows that we may write $x_0 = (g_2, \ldots, g_{n+1})$ and $x_{n+1} = (g_1, \ldots, g_n)$ for suitable $g_i \in G$. We therefore put $y = (g_1, \ldots, g_{n+1})$. Then for $0 < i < n+1$, $i \neq k$, we have $\partial_0(x_i) = \partial_{i-1}(x_0)$ and $\partial_n(x_i) = \partial_i(x_{n+1})$, which shows that $x_i = (g_0, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) = \partial_i(y)$. Thus y satisfies the desired conditions. □

Let us verify that BG (or rather $|BG|$) faithfully recovers the traditional definition of a classifying space for a discrete group, that is, it is a $K(G, 1)$ space: There is only one basepoint $1 \in BG_0$, so we simply write $\pi_n(BG)$. For the purpose of calculating this, we regard $\pi_n(BG)$ as homotopy classes of elements in X_n with boundary 1. But if $n \neq 1$, the only such element of X_n is $1 = (1, \ldots, 1)$, while all elements of $BG_1 = G$ have boundary 1. Calculations show that each homotopy class in BG_1 only contains one element, so as sets, we have $\pi_1(BG) = G$ while $\pi_n(BG) = 1$ if $n \neq 1$. To see that π_1 is also isomorphic to G as a group, pick $x, y \in \pi_1(BG)$. The product xy in π_1 is by definition $\partial_1 h$, where $h \in BG_2 = G^2$ satisfies $\partial_0 h = x$ and $\partial_2 h = y$. But $\partial_0 h$ is the second coordinate of h and $\partial_2 h$ is the first one, so $h = (y, x)$, and we have $\partial_1 h = yx$. Hence π_1 is in fact the opposite group of G, which is isomorphic to G by the inversion map in G. A possible workaround for this issue is to replace G by the opposite category of G, but this makes the rest of the formulas strange.

With regard to its homology, we note that $H_\cdot(BG;\mathbb{Z})$ as defined above is the homology of the same chain complex used to define the simplicial homology of the simplicial complex $|BG|$. We thus have $H_\cdot(BG;\mathbb{Z}) = H_\cdot(|BG|;\mathbb{Z}) = H_\cdot(G;\mathbb{Z})$.

6 Čech cohomology

It is possible to formulate Čech cohomology in terms of simplicial sets: Let X be any space (we omit all assumptions of being paracompact or having manifold structure) and $\mathscr{U} = \{U_i\}_{i\in I}$ an open cover; for this formulation, we do not require an ordering on the index set I. As is common, we shall write $U_{i_0\cdots i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$ for any combination of index elements i_j. The **nerve** $N(\mathscr{U})_\cdot$ of the covering is the simplicial set with $N(\mathscr{U})_n$ the set of all $(n+1)$-tuples $\boldsymbol{i} = (i_0, i_1, \dots, i_n)$ of elements in I such that the intersection $U_{\boldsymbol{i}}$ is non-empty (the same kind of construction occurs in Example 8.1.8 in Weibel 1994). The map $\alpha^*\colon N(\mathscr{U})_n \to N(\mathscr{U})_m$ induced from $\alpha\colon [m] \to [n]$ is given by

$$\alpha^*(i_0, i_1, \dots, i_n) = (i_{\alpha(0)}, i_{\alpha(1)}, \dots, i_{\alpha(m)}),$$

noting that this preserves the non-emptiness condition because the new intersection contains the old one.

Now suppose $\mathscr{F}$ is a (pre)sheaf of Abelian groups on X. We then obtain a cosimplicial Abelian group $\check{C}^\cdot = \check{C}^\cdot(X, \mathscr{U}; \mathscr{F})$ given by

$$\check{C}^n = \prod_{\boldsymbol{i}\in N(\mathscr{U})_n} \mathscr{F}(U_{\boldsymbol{i}}).$$

The map induced from $\alpha\colon [m] \to [n]$ maps $s \in \check{C}^m$ to $\alpha_*(s) \in \check{C}^n$ given on the $\boldsymbol{i}$th coordinate by

$$\alpha_*(s)_{\boldsymbol{i}} = \operatorname{res}^{U_{\alpha^*(\boldsymbol{i})}}_{U_{\boldsymbol{i}}}(s_{\alpha^*(\boldsymbol{i})}).$$

We recover the usual Čech cocomplex (and thus Čech cohomology) by defining the differential $\partial\colon \check{C}^n \to \check{C}^{n+1}$ by $\partial = \sum(-1)^i \partial^i$.

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