TORSION GEOMETRY, SUPERCONFORMAL SYMMETRY AND T-DUALITY

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OUTLINE

1 TORSION GEOMETRY

- Metric geometry with torsion
- KT Geometry
- HKT Geometry
Outline

1. Torsion Geometry
   - Metric geometry with torsion
   - KT Geometry
   - HKT Geometry

2. Superconformal Symmetry
   - Superconformal Quantum Mechanics
   - The Superalgebras $D(2, 1; \alpha)$
   - Geometric Structure
1. **Torsion Geometry**
   - Metric geometry with torsion
   - KT Geometry
   - HKT Geometry

2. **Superconformal Symmetry**
   - Superconformal Quantum Mechanics
   - The Superalgebras $D(2, 1; \alpha)$
   - Geometric Structure

3. **T-duality**
   - T-duality as a Twist Construction
   - HKT Examples
   - General HKT with Circle Symmetry
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   - General HKT with Circle Symmetry
**Torsion Geometry**

**Metric geometry with torsion**

- Metric $g$, connection $\nabla$, torsion $T^\nabla (X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$
- $\nabla g = 0$
Metric geometry with torsion

- Metric $g$, connection $\nabla$, torsion
  $$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$
- $\nabla g = 0$
- $c(X,Y,Z) = g(T^\nabla(X,Y),Z)$ a three-form
**Metric geometry with torsion**

- metric $g$, connection $\nabla$, torsion
  
  \[ T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \]
- $\nabla g = 0$
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\[ \nabla = \nabla^{LC} + \frac{1}{2} c \]

- any $c \in \Omega^3(M)$ will do
- $\nabla, \nabla^{LC}$ same geodesics/dynamics
- *strong* if $dc = 0$
**Torsion Geometry**

**Metric geometry with torsion**
- Metric $g$, connection $\nabla$, torsion $T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$
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- Any $c \in \Omega^3(M)$ will do
- $\nabla$, $\nabla^{LC}$ same geodesics/dynamics
- Strong if $dc = 0$

Study *compact* simply-connected torsion geometries with
- Compatible complex structures
- Small symmetry group
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**KT Geometry**

\[ g, \nabla = \nabla^{LC} + \frac{1}{2} c, \quad c \in \Lambda^3 T^* M \]

**KT Geometry**

Additionally

- \( I \) integrable complex structure
- \( g(IX, IY) = g(X, Y) \)
- \( \nabla I = 0 \)

Two form \( F_I(X, Y) = g(IX, Y) \)
**KT Geometry**

\[ g, \nabla = \nabla^{LC} + \frac{1}{2} c, \quad c \in \Lambda^3 T^* M \]

**KT Geometry**

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- $I$ integrable complex structure
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Two form $F_I(X, Y) = g(IX, Y)$

$\nabla$ is unique

\[ c = -\text{Id} F_I \]

The Bismut connection
**KT Geometry**

\[ g, \nabla = \nabla^{LC} + \frac{1}{2} c, \quad c \in \Lambda^3 T^* M \]

### KT Geometry

- KT geometry = Hermitian geometry + Bismut connection
- \( c = 0 \) is Kähler geometry
- Strong KT is \( \partial \bar{\partial} F_I = 0 \)

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the Bismut connection

- KT geometry = Hermitian geometry + Bismut connection
- \( c = 0 \) is Kähler geometry
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**Example**

\[ M^6 = S^3 \times S^3 = SU(2) \times SU(2) \]

**Gauduchon (1991)**

every compact Hermitian \( M^4 \) is conformal to strong KT
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HKT Geometry

HKT structure

\( (g, \nabla, I, J, K) \) with
- \( (g, \nabla, A) \) KT, \( A = I, J, K \)
- \( IJ = K = -JI \)

\[ c = -AdF_A \] is independent of \( A \)
HKT Geometry

HKT Structure

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- \((g, \nabla, A)\) KT, \(A = I, J, K\)
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Martín Cabrera and Swann (2007)

\[IdF_I = JdF_J = KdF_K\]

implies \(I, J, K\) integrable, so HKT.
### HKT Geometry

**HKT structure**

\[(g, \nabla, I, J, K) \text{ with} \]

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**Examples**

**Dim 4** \(T^4, K3, S^3 \times S^1\) (Boyer, 1988)

**Dim 8** Hilbert schemes, \(SU(3)\), nilmanifolds, vector bundles over discrete groups (Verbitsky, 2003; Barberis and Fino, 2008)

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Compact, simply-connected examples which are neither hyperKähler nor homogeneous?
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Superconformal Quantum Mechanics

$N$ particles in 1 dimension

$$H = \frac{1}{2} P^*_a g^{ab} P_b + V(x)$$

Standard quantisation

$$P_a \sim -i \frac{\partial}{\partial x^a}, \quad a = 1, \ldots, N$$
Superconformal Quantum Mechanics

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Michelson and Strominger (2000); Papadopoulos (2000)

- Operator $D$ with $[D, H] = 2iH \iff$ vector field $X$ with $L_X g = 2g$ & $L_X V = -2V$
- $K$ so span${iH, iD, iK} \cong \mathfrak{sl}(2, \mathbb{R}) \iff X^b = g(X, \cdot)$ is closed
- then $K = \frac{1}{2} g(X, X)$.

Choose a superalgebra containing $\mathfrak{sl}(2, \mathbb{R})$ in its even part.
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The Superalgebras $D(2, 1; \alpha)$

The classification of simple Lie superalgebras contains one continuous family

$D(2, 1; \alpha)$
The Superalgebras $D(2, 1; \alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

\[ D(2, 1; \alpha) \]

- \( \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \)
The Superalgebras $D(2, 1; \alpha)$

The classification of simple Lie superalgebras contains one continuous family

$D(2, 1; \alpha)$

- $g = g_0 + g_1$
- $g_0 = \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{sl}(2, \mathbb{C})_+ + \mathfrak{sl}(2, \mathbb{C})_-$
- $g_1 = \mathbb{C}^2 \otimes \mathbb{C}_+^2 \otimes \mathbb{C}_-^2 = \mathbb{C}_Q^4 + \mathbb{C}_S^4$
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- $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}_+^2 \otimes \mathbb{C}_-^2 = \mathbb{C}_Q^4 + \mathbb{C}_S^4$
- $[S^a, Q^a] = D$
- $[S^1, Q^2] = -\frac{4\alpha}{1+\alpha} R^3_+ - \frac{4}{1+\alpha} R^3_-$
The Superalgebras $D(2, 1; \alpha)$

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Simple for $\alpha \neq -1, 0, \infty$. 

The Superalgebras $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains one continuous family

$D(2,1;\alpha)$

- $g = g_0 + g_1$
- $g_0 = sl(2,\mathbb{C}) + sl(2,\mathbb{C})_+ + sl(2,\mathbb{C})_-$
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Simple for $\alpha \neq -1, 0, \infty$.

Over $\mathbb{C}$, isomorphisms between the cases $\alpha^{\pm 1}, -(1 + \alpha)^{\pm 1}, -(\alpha/(1 + \alpha))^{\pm 1}$.
The Superalgebras $D(2, 1; \alpha)$

The classification of simple Lie superalgebras contains one continuous family

$D(2, 1; \alpha)$

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$
- $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{sl}(2, \mathbb{C})_+ + \mathfrak{sl}(2, \mathbb{C})_-$
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- $[S^a, Q^a] = D,$
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Simple for $\alpha \neq -1, 0, \infty$.

- Over $\mathbb{C}$, isomorphisms between the cases $\alpha^\pm 1, -(1 + \alpha)^\pm 1$, $-(\alpha / (1 + \alpha))^\pm 1$.
- Real form
  $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R}) + \mathfrak{su}(2)_+ + \mathfrak{su}(2)_-$.
- Over $\mathbb{R}$, isomorphisms for $\alpha^\pm 1$.
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\[ N = 4B \text{ quantum mechanics} \]

with \( D(2, 1; \alpha) \)
superconformal symmetry

\[ \leftrightarrow \]

\[ \text{HKT manifold } M \]

with \( X \) a special homothety of type \((a, b)\)

- \( L_X g = ag \),
- \( L_{IX} J = bK \),
- \( L_X I = 0, L_{IX} I = 0, \ldots \)
Superconformal Geometry

\( \mathcal{N} = 4B \) Quantum Mechanics

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\[ \alpha = \frac{a}{b} - 1 \]

Action of \( \mathbb{R} \times SU(2) \)

rotating \( I, J, K \)

↔

HKT manifold \( M \)

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**Superconformal Geometry**

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**HKT manifold $M$**

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- $L_X g = ag$,
- $L_{IX} J = bK$,
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For $a \neq 0$

- $M$ is non-compact
- $\mu = \frac{2}{a(a-b)} \|X\|^2$ is an HKT potential

$$F_I = \frac{1}{2} (dd_I + d_J d_K) \mu = \frac{1}{2} (1 - J) dId\mu.$$
Example

\[ M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{H}P(n) \]
\[ a = 2, \ b = -2, \ \alpha = -2. \]
**Supers conformal Geometry II**

**Example**

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**Poon and Swann (2003)**

\( a \neq 0 \) corresponds to

\[ Q = M / (\mathbb{R} \times SU(2)) = \mu^{-1}(1) / SU(2) \] a QKT orbifold

(of special type).
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E.g. \[ Q = k\mathbb{C}P(2). \]
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For \(S\) 3-Sasaki, \[M = S \times \mathbb{R}\]

warped product, is

hyperKähler with special homothety \(\alpha = -2\)
Superconformal Geometry II

**Example**

\[ M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n) \]
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If \( a \neq 0 \) corresponds to \( Q = M/(\mathbb{R} \times SU(2)) = \mu^{-1}(1)/SU(2) \) a QKT orbifold (of special type).

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For S 3-Sasaki, \( M = S \times \mathbb{R} \) warped product, is hyperKähler with special homothety \( \alpha = -2 \)

Get to \( a = 0 \), special isometry, by potential change

\[ g_1 = \frac{1}{\mu} g - \frac{1}{2\mu^2} (d^H \mu)^2 \]
**Superconformal Geometry II**

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Discrete quotient

\[ M = (\mu^{-1}(1) \times \mathbb{R})/\mathbb{Z}(\varphi, 2) \]

with \( g_1 \) is HKT with special isometry \( X \)
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E.g. \( Q = k\mathbb{CP}(2) \).

For 3-Sasaki, \( M = S \times \mathbb{R} \) warped product, is
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Get to \( a = 0 \), special isometry, by potential change
\[
\mathfrak{g}_1 = \frac{1}{\mu} \mathfrak{g} - \frac{1}{2\mu^2} (d\mathbb{H}\mu)^2
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In this case
- \( dX^b = 0 \)
- \( b_1(M) \geq 1 \)
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**T-duality as a Twist**

- $X$ generating a circle action on $M$
- $(P, \theta, Y) \xrightarrow{\pi} M$ an invariant principal $S^1$-bundle
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- $X$ generating a circle action on $M$
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- $X' = \tilde{X} + aY$ a lift of $X$ generating a free circle action, $da = -X \lrcorner F_\theta$
T-duality as a Twist

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**Definition**

A *twist* $W$ of $M$ with respect to $X$ is

$$W := P / \langle X' \rangle$$

Transverse locally free lifts always exist for $X \downarrow F\theta$ exact.
**T-duality as a Twist**

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**Dually**

$M$ is a twist of $W$ with respect to $X_W = (\pi_W)_* Y, \theta_W = \frac{1}{a} \theta$

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**Dually**

\(M\) is a twist of \(W\) with respect to \(X_W = (\pi_W)_* \gamma, \theta_W = \frac{1}{a} \theta\)

**Definition**

Tensors \(\alpha\) on \(\alpha_W\) on \(M\) and \(W\) are \(\mathcal{H}\)-related, \(\alpha_W \sim_{\mathcal{H}} \alpha\) if their pull-backs agree on \(\mathcal{H} = \ker \theta\)

\[
d\alpha_W \sim_{\mathcal{H}} d\alpha - F_\theta \wedge \frac{1}{a} X \downarrow \alpha
\]

if invariant
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Twisting HKT

Twist by

\[ g_W \sim_{\mathcal{H}} g, \quad F^W_I \sim_{\mathcal{H}} F_I, \text{etc.} \]
Twisting HKT

Twist by

$$g_w \sim_{\mathcal{H}} g, \quad F^W_I \sim_{\mathcal{H}} F_I, \text{etc.}$$

Then

$$\text{Id} F^W_I \sim_{\mathcal{H}} \text{Id} F_I + \frac{1}{a} X^b \wedge IF_\theta$$

For HKT need

$$c = -\text{Id} F_I = -JdF_I = -KdF_K$$
**Twisting HKT**

Twist by

$$g_W \sim_H g, \quad F^W_I \sim_H F_I, \text{ etc.}$$

Then

$$IdF^W_I \sim_H IdF_I + \frac{1}{a} X^b \wedge IF_\theta$$

For HKT need

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**Proposition**

*HKT twists to HKT via a circle if and only if* \( F_\theta \in S^2E = \bigcap_I \Lambda_I^{1,1} \), i.e., an instanton
**Twisting HKT**

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For HKT need

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**Proposition**

HKT twists to HKT via a circle if and only if \( F_\theta \in S^2 E = \bigcap I \Lambda^{1,1}_I \), i.e., an instanton

\[ X \text{ a special isometry, } X \perp F_\theta = 0 \text{ twists to } X_W \text{ a special isometry} \]
Twisting HKT

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\[ g_W \sim_H g, \quad F^W_I \sim_H F_I, \text{etc.} \]

Then
\[ \text{Id} F^W_I \sim_H \text{Id} F_I + \frac{1}{a} X^b \wedge IF_\theta \]

For HKT need
\[ c = -\text{Id} F_I = -JdF_I = -KdF_K \]

**Proposition**

HKT twists to HKT via a circle if and only if \( F_\theta \in S^2 E = \bigcap I \Lambda^1_1 \), i.e., an instanton

X a special isometry, \( X \perp F_\theta = 0 \) twists to \( X_W \) a special isometry

**Theorem**

M HKT with special isometry \((\alpha = -1)\). Can
- untwist locally to \( dX^b = 0 \) on \( S \times S^1 \)
- change potential on \( S \times \mathbb{R} \) to \( a \neq 0, (\alpha = -2) \)
Twisting HKT

Twist by

\[ g_W \sim_{\mathcal{H}} g, \quad F_I^W \sim_{\mathcal{H}} F_I, \text{etc.} \]

Then

\[ IdF_I^W \sim_{\mathcal{H}} IdF_I + \frac{1}{a} X^b \wedge IF_\theta \]

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**Theorem**

\( M \text{ HKT with special isometry} \)

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Twists to \( X_W \) a special isometry

**Theorem**

\( M \) HKT with special isometry \( (\alpha = -1) \). Can

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  on \( S \times S^1 \)
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\[ F_\theta = d X^b \text{ is an instanton} \]

Many simply-connected examples when \( b_2(S) \geq 1 \)
E.g., \( Q = k \mathbb{C}P(2) \)
Outline

1 Torsion Geometry
   - Metric geometry with torsion
   - KT Geometry
   - HKT Geometry

2 Superconformal Symmetry
   - Superconformal Quantum Mechanics
   - The Superalgebras $D(2, 1; \alpha)$
   - Geometric Structure

3 T-duality
   - T-duality as a Twist Construction
   - HKT Examples
   - General HKT with Circle Symmetry
General HKT with Circle Symmetry

- \( M = N_1 \times N_2 \)
- \( N_2 \) with an HKT circle symmetry \( X \)
- \([F_\theta] \in H^2(N_1, \mathbb{Z}), F_\theta \in S^2E\)
General HKT with Circle Symmetry

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Twists to $N_2 \rightarrow W \rightarrow N_1$ HKT with circle symmetry
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Generate simply-connected examples

**Example**

$N_1$ a K3 surface
$F_\theta$ self-dual, primitive

Generalises to torus actions
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Twists to \( N_2 \to W \to N_1 \) HKT with circle symmetry

Generate simply-connected examples

**Example**

- \( N_1 \) a K3 surface
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HKT nilmanifold \( M = G/\Gamma \)

\( g^* \) basis \( e_1, \ldots, e_n \) with

\[ de_{i+1} \in \Lambda^2 \text{span}\{e_1, \ldots, e_i\} \]

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**Barberis, Dotti Miatello, and Verbitsky (2007)**

\( I, J, K \) are Abelian

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Generalises to torus actions
**General HKT with Circle Symmetry**

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Twists to $N_2 \rightarrow W \rightarrow N_1$ HKT with circle symmetry

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**Proposition**

Every HKT nilmanifold may be obtained by successive twists of a torus $T^{4n}$.

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$I, J, K$ are Abelian

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Summary

- $D(2, 1; \alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action
**Summary**

- $D(2, 1; \alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action
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- $D(2,1;\alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action
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Summary

- $D(2,1;\alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action
- $\alpha \neq -1$ comes from $\mathbb{R} \times SO(3)$ bundles over certain QKT orbifolds
- $\alpha = -1$ comes from previous examples via change of potential and twist
- construct non-homogeneous compact simply-connected examples with $\alpha = -1$
- construct other compact HKT manifolds by further twists
- non-instanton twists by tori

\[ \sum_{i,j} (a^{-1})_{ij} X_i \wedge IF_j \] independent of $I$

gives further non-compact HKT examples


